Handling Handles: Non-Planar AdS/CFT Integrability

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General Idea

String amplitudes in AdS_5 can be cut into basic patches (rectangles, pentagons, or hexagons), which can be *bootstrapped* using *integrability* at *any value of the 't Hooft coupling*.

- Amplitudes are given as infinite sums and integrals over intermediate states from *gluing together* these integrable patches.
- This holds at the planar level as well as for *non-planar processes* suppressed by $1/N_{\mathrm{c}}$.

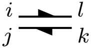
$\mathcal{N} = 4$ SYM & The Planar Limit

$\mathcal{N} = 4$ super Yang–Mills: Gauge field A_μ, scalars Φ_I, fermions $\psi_{\alpha A}$.

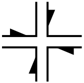
Gauge group: $\mathrm{U}(N_\mathrm{c}) \,/\, \mathrm{SU}(N_\mathrm{c})$.
Adjoint representation: All fields are $N_\mathrm{c} \times N_\mathrm{c}$ matrices.

Double-line notation:

Propagators:

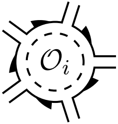
$\langle \Phi_I{}^i{}_j \Phi_J{}^k{}_l \rangle \sim g_{\mathrm{YM}}^2 \delta^i{}_l \delta^k{}_j =$ 

Vertices:

$\mathrm{Tr}(\Phi\Phi\Phi\Phi) \sim \frac{1}{g_{\mathrm{YM}}^2}$ 

- Diagrams consist of color index loops $\simeq$ oriented disks $\sim \delta^i{}_i = N_\mathrm{c}$
- Disks are glued along propagators $\rightarrow$ oriented compact surfaces

Local operators:

$\mathcal{O}_i = \mathrm{Tr}(\Phi \ldots) \sim$ 

- One fewer color loop $\rightarrow$ factor $1/N_\mathrm{c}$
- Surface: Hole $\sim$ boundary component

Planar Limit & Genus Expansion

Every diagram is associated to an oriented compact surface.

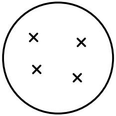
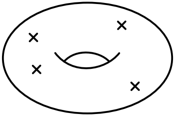
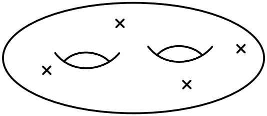
Genus Expansion: ['t Hooft 1974]

Absorb one factor of N_c in the 't Hooft coupling $\lambda = g_\mathrm{YM}^2 N_\mathrm{c}$

Use Euler formula $V - E + F = 2 - 2g$

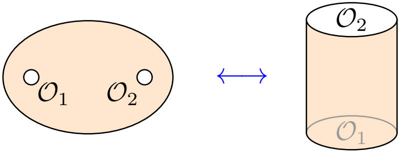
$\Rightarrow$ **Correlators** of single trace operators $\mathcal{O}_i = \mathrm{Tr}(\Phi_1 \Phi_2 \dots)$:
't Hoof genus expansion

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \frac{1}{N_\mathrm{c}^{n-2}} \sum_{g=0}^{\infty} \frac{1}{N_\mathrm{c}^{2g}} G_g(\lambda)$$

$\sim \frac{1}{N_\mathrm{c}^2}$  $+ \frac{1}{N_\mathrm{c}^4}$  $+ \frac{1}{N_\mathrm{c}^6}$  $+ \dots$

Planar Spectrum: Features of Integrability

Simplest observables: Planar two-point functions $\simeq$ scaling dimensions



$\mathcal{O}_i = \mathrm{Tr}(\Phi \dots)$
single-trace

Perturbatively: Degeneracies in the spectrum $\rightarrow$ Higher charges

Spin chain picture: Organize operators around vacuum operators

$$\mathrm{Tr}\, Z^L\,, \quad Z = \alpha^I \Phi_I\,, \quad \alpha^I \alpha_I = 0 \qquad \text{(half-BPS, protected)}\,.$$

Other operators: Insert impurities $\{\Phi_I, \psi_{\alpha A}, D_\mu\}$ into $\mathrm{Tr}\, Z^L$.

Dilatation operator acts *locally* in color space (neighboring fields)
$\rightarrow$ Impurities are *magnons*, with *rapidity* (momentum) u
and $\mathfrak{su}(2|2)^2 \subset \mathfrak{psu}(2,2|4)$ *flavor index*.

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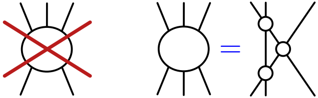
Dynamics of magnons: Integrability:
$\rightarrow$ No particle production
$\rightarrow$ Individual momenta preserved
$\rightarrow$ Factorized scattering



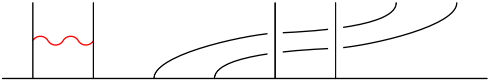
Two-body **S-matrix** completely fixed *to all loops* [Beisert 2005] [Janik 2006] [Beisert, Hernandez Lopez 2006]

> Planar spectrum (asymptotic) solved exactly by **Bethe ansatz**.

Non-Planar Corrections: (Past) Status

Degeneracies are lifted at subleading orders in $1/N_c$.

Interactions are long-ranged, non-local from the start:



$\rightarrow$ Hilbert space much bigger
$\rightarrow$ Spin chain picture?
$\rightarrow$ Fate of local S-matrix? Definition?
$\rightarrow$ No integrable spin chain!

No **dual superconformal symmetry**

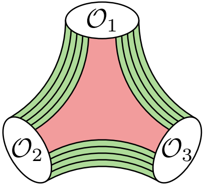
Classical integrability of σ model (strong coupling) not clear

$\Rightarrow$ **"Integrability is lost".**

Three-Point Functions: Hexagons

Differences: Topology: Pair of pants instead of cylinder
Non-vanishing for three generic operators (two-point: diagonal)
$\Rightarrow$ Previous techniques not directly applicable

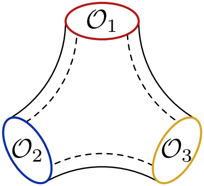
Observation:



The green parts are similar to two-point functions: Two segments of physical operators joined by parallel propagators ("bridges", $\ell_{ij} = (L_i + L_j - L_k)/2$).

The red part is new: "Worldsheet splitting", "three-point vertex" (open strings)

Take this serious $\rightarrow$ cut worldsheet along "bridges": [Basso,Komatsu Vieira '15]



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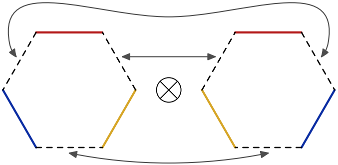
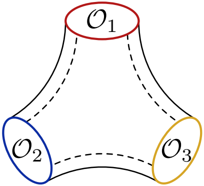


Hexagons & Gluing



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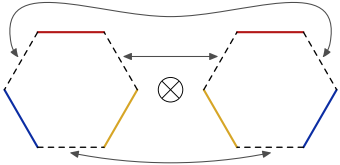


$\otimes$ **Glue** hexagons along three *mirror channels*: [Basso, Komatsu Vieira '15] [Basso, Goncalves Komatsu, Vieira '15]

- Sum over complete state basis (magnons) in the mirror theory
- Mirror magnons: Boltzmann weight $\exp(-\tilde{E}_{ij}\ell_{ij})$, $\tilde{E}_{ij} = \mathcal{O}(g^2)$
  $\to$ mirror excitations are strongly suppressed.

Hexagonal worldsheet patches (form factors):

- Function of rapidities u and $\mathfrak{su}(2|2)^2$ labels $(A, \dot{A})$ of all magnons.
- Conjectured exact expression, based on diagonal $\mathfrak{su}(2|2)$ symmetry, form factor axioms, and integrability assumptions. [Basso, Komatsu Vieira '15]

Hexagon proposal supported by very non-trivial matches.

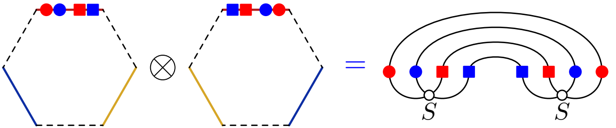
The Hexagon Form Factor

All excitations on the same physical edge (canonical frame): [Basso,Komatsu Vieira '15]

$$\mathcal{H}(\chi^{A_1}\chi^{\dot{A}_1}\chi^{A_2}\chi^{\dot{A}_2}\dots\chi^{A_n}\chi^{\dot{A}_n})$$
$$= (-1)^{\mathfrak{F}}\left(\prod_{i<j} h_{ij}\right)\langle\chi^{A_1}\chi^{A_2}\dots\chi^{A_n}|S|\chi^{\dot{A}_n}\dots\chi^{\dot{A}_2}\chi^{\dot{A}_1}\rangle$$

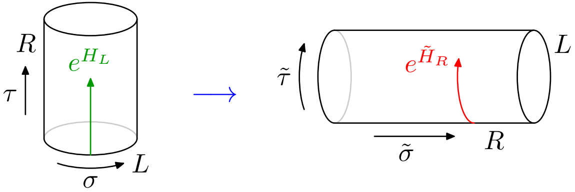
- $\chi^A = \phi^a|\psi^\alpha$: Left $\mathfrak{su}(2|2)$ fundamental magnon
- $\chi^{\dot{A}} = \phi^{\dot{a}}|\psi^{\dot{\alpha}}$: Right $\mathfrak{su}(2|2)$ fundamental magnon
- $\mathfrak{F}$: Fermion number operator
- S: Beisert S-matrix
- $h_{ij} = \dfrac{x_i^- - x_j^-}{x_i^- - x_j^+}\dfrac{x_j^+ - 1/x_i^-}{x_2^+ - 1/x_1^+}\dfrac{1}{\sigma_{ij}}$, $\quad x^\pm(u) = x(u \pm \tfrac{i}{2})$, $\quad \frac{u}{g} = x + \frac{1}{x}$
  σ_{ij}: BES dressing phase

Example:
Two magnons
(●■, ●■)



Mirror Map

Double Wick rotation: $(\sigma, \tau) \to (i\tilde{\tau}, i\tilde{\sigma})$ — exchanges space and time

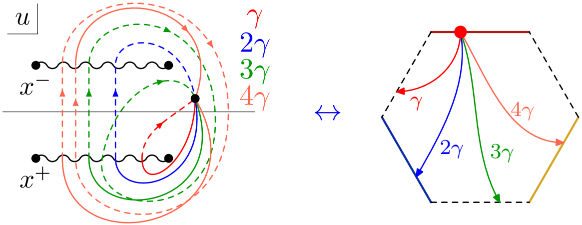


Magnon states: Energy and momentum interchange:

$$p\,\sigma \to p^\gamma\, i\tilde{\tau} \equiv \tilde{E}\tilde{\tau}\,, \quad E\,\tau \to E^\gamma\, i\tilde{\sigma} \equiv \tilde{p}\tilde{\sigma} \quad \Rightarrow \quad (\tilde{E}, \tilde{p}) = (ip^\gamma, iE^\gamma)\,.$$

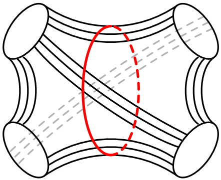
Continuations: $u \to u^\gamma$. All quantities given in terms of $x^\pm(u)$.

Four sheets:



Planar Four-Point Functions: Hexagonalization

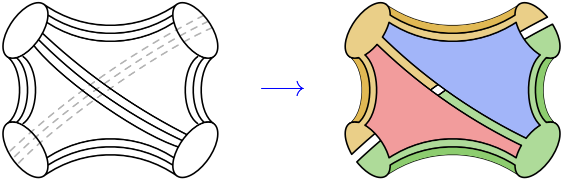
Move on to planar four-point functions:
One way to cut (now that three-point is understood): **OPE cut**



Problem: Sum over physical states!

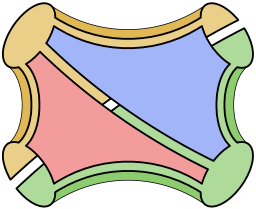
- No loop suppression, all states contrib.
- Double-trace operators.

Instead: Cut along propagator bridges [Fleury '16 Komatsu] [Eden '16 Sfondrini]

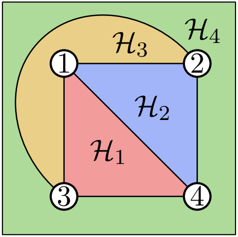


Benefits:
- Mirror states highly suppressed in g.
- No double traces.

Hexagonalization: Formula



$\longleftrightarrow$



$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \Bigg[\prod_{\substack{\text{channels} \\ c \in \{1,2,3\}}} d_c^{\ell_c} \sum_{\psi_c} \mu(\psi_c) \Bigg] \mathcal{H}_1(\psi_1, \psi_2, \psi_3)\, \mathcal{H}_2(\psi_1, \psi_2, \psi_3)$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle = \sum_{\substack{\text{planar} \\ \text{prop. graphs}}} \Bigg[\prod_{\substack{\text{channels} \\ c \in \{1,\ldots,6\}}} d_c^{\ell_c} \sum_{\psi_c} \mu(\psi_c) \Bigg] \mathcal{H}_1\, \mathcal{H}_2\, \mathcal{H}_3\, \mathcal{H}_4$$

New Features:

[Fleury '16 Komatsu]

- Bridge lengths vary, may go to zero $\Rightarrow$ Mirror corrections at one loop
- Hexagons are in different "frames" $\Rightarrow$ Weight factors

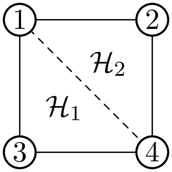
Hexagonalization: Frames

Hexagon depends on positions x_i and polarizations α_i of the three half-BPS "vacuum" operators $\mathcal{O}_i = \mathrm{Tr}[(\alpha_i \cdot \Phi(x_i))^k]$.

Any three x_i and α_i preserve a diagonal $\mathfrak{su}(2|2)$ that defines the state basis and S-matrix of excitations on the hexagon.

Three-point function: Both hexagons connect to the same three operators, so their frames ($\mathfrak{su}(2|2)$ and state basis) are identical.

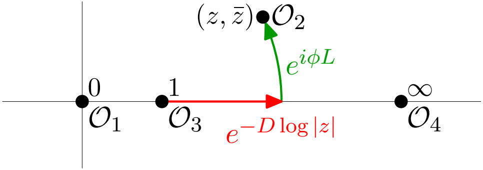
Higher-point function: Two neighboring hexagons always share two operators, but the third/fourth operator may not be identical.
$\Rightarrow$ The two hexagon frames are misaligned.

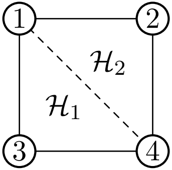


In order to consistently sum over mirror states, need to align the two frames by a $\mathrm{PSU}(2,2|4)$ transformation that maps $\mathcal{O}_3$ onto $\mathcal{O}_2$.

Hexagonalization: Weight Factors

By conformal and R-symmetry transformation, bring $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_4$ to canonical configuration: [Fleury '16, Komatsu]





Transformation that maps $\mathcal{O}_3$ to $\mathcal{O}_2$: $g = e^{-D\log|z|} e^{i\phi L} e^{J\log|\alpha|} e^{i\theta R}$, where $e^{2i\phi} = z/\bar{z}$, $e^{2i\theta} = \alpha/\bar{\alpha}$, and $(\alpha,\bar{\alpha})$ is the R-coordinate of $\mathcal{O}_3$.

Hexagon $\mathcal{H}_1 = \hat{\mathcal{H}}$ is canonical, and $\mathcal{H}_2 = g^{-1}\hat{\mathcal{H}}g$.

Sum over states in mirror channel: [Fleury '16, Komatsu]

$$\sum_{\psi} \mu(\psi) \langle \mathcal{H}_2|\psi\rangle\langle\psi|\mathcal{H}_1\rangle = \sum_{\psi} \mu(\psi)\langle g^{-1}\hat{\mathcal{H}}|\psi\rangle\langle\psi|g|\psi\rangle\langle\psi|\hat{\mathcal{H}}\rangle$$

Weight factor: $\langle\psi|g|\psi\rangle = e^{-2i\tilde{p}_\psi \log|z|} e^{J_\psi \varphi} e^{i\phi L_\psi} e^{i\theta R_\psi}$, $i\tilde{p} = (D-J)/2$.

$\rightarrow$ Contains all non-trivial dependence on cross ratios $z,\bar{z}$ and $\alpha,\bar{\alpha}$.

Non-Planar Processes: Idea

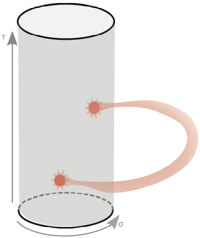
Hexagonalization: Works for planar (4,5)-point functions [Fleury '16 Komatsu] [Fleury '17 Komatsu]

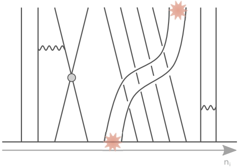
Extend to non-planar processes?

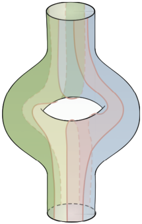
- Fix worldsheet topology
- Dissect into planar hexagons
- Glue hexagons (mirror states)

Simple Proposal:

$$\big\langle \mathcal{O}_1 \dots \mathcal{O}_n \big\rangle^{\text{full}} = \frac{1}{N_\text{c}^{n-2}} \sum_g \frac{1}{N_\text{c}^{2g}} \sum_{\substack{\text{graphs} \\ (\text{genus } g)}} \prod_c d_c^{\ell_c} \sum_{\substack{\text{mirror} \\ \text{states}}} \mathcal{H}_1\, \mathcal{H}_2\, \mathcal{H}_3 \dots \mathcal{H}_F$$







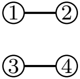
The Data: Kinematics

Half-BPS operators:

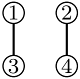
$$\mathcal{Q}_i^k \equiv \mathrm{Tr}\big[(\alpha_i \cdot \Phi(x_i))^k\big]\,, \quad \Phi = (\phi_1, \dots, \phi_6)\,, \quad \alpha_i^2 = 0\,.$$

For equal weights (k,k,k,k): Expand in X, Y, Z:

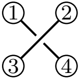
$$X \equiv \frac{\alpha_1 \cdot \alpha_2\, \alpha_3 \cdot \alpha_4}{x_{12}^2 x_{34}^2} =$$



$$, \quad Y \equiv$$



$$, \quad Z \equiv$$



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Focus on $Z = 0$ (polarizations): [Arutyunov Sokatchev '03] [Arutyunov, Penati '03 Santambrogio, Sokatchev]

$$G_k \equiv \langle \mathcal{Q}_1^k \mathcal{Q}_2^k \mathcal{Q}_3^k \mathcal{Q}_4^k \rangle^{\text{loops}} = R \sum_{m=0}^{k-2} \mathcal{F}_{k,m}\, X^m Y^{k-2-m}$$

Supersymmetry factor: $R = z\bar{z}X^2 - (z+\bar{z})XY + Y^2$

Main data: Coefficients $\mathcal{F}_{k,m} = \mathcal{F}_{k,m}(g; z, \bar{z})$

Cross ratios: $z\bar{z} = s = \dfrac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad (1-z)(1-\bar{z}) = t = \dfrac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}\,.$

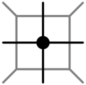
The Data: Quantum Coefficients

Data Functions: Correlator coefficients:

$$\mathcal{F}_{k,m} = \sum_{\ell=1}^{\infty} g^{2\ell} \mathcal{F}_{k,m}^{(\ell)}(z,\bar{z})\,, \quad \text{'t Hooft coupling: } g^2 = \frac{g_{\mathrm{YM}}^2 N_{\mathrm{c}}}{16\pi^2}\,.$$

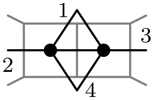
One and two loops: Two ingredients: Box integrals

$$F^{(1)}(z,\bar{z}) = \frac{x_{13}^2 x_{24}^2}{\pi^2} \int \frac{\mathrm{d}^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} =$$



,

$$\frac{F^{(2)}(z,\bar{z})}{x_{14}^2} = \frac{x_{13}^2 x_{24}^2}{(\pi^2)^2} \int \frac{\mathrm{d}^4 x_5\, \mathrm{d}^4 x_6}{x_{15}^2 x_{25}^2 x_{45}^2 x_{56}^2 x_{16}^2 x_{36}^2 x_{46}^2} =$$



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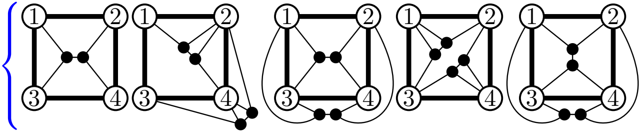
& Color factors: $C_{k,m}^{i} \in \Big\{$



$\Big\}$

① $= \mathrm{Tr}(T^{(a_1} \ldots T^{a_k)})$,

$= f_{ab}{}^{c}$

The Data: Color Factors

To obtain non-planar corrections: Need to expand color factors.

$$C^i_{k,m} = N_{\mathrm{c}}^{2k} k^4 \left({}^{\bullet}C^i_{k,m} + {}^{\circ}C^i_{k,m} N_{\mathrm{c}}^{-2} + \mathcal{O}(N_{\mathrm{c}}^{-4}) \right) , \quad i \in \{\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d}\} ,$$

Compute by brute force:

k	m	$\frac{1}{2}{}^{\circ}C^{1,\mathrm{U}}_{k,m}$	$\frac{1}{2}{}^{\circ}C^{1,\mathrm{SU}}_{k,m}$	${}^{\circ}C^{\mathrm{a},\mathrm{U}}_{k,m}$	$2{}^{\circ}C^{\mathrm{b},\mathrm{U}}_{k,m}$	$\frac{1}{2}{}^{\circ}C^{\mathrm{c},\mathrm{U}}_{k,m}$	${}^{\circ}C^{\mathrm{d},\mathrm{U}}_{k,m}$	${}^{\circ}C^{\mathrm{a},\mathrm{SU}}_{k,m}$	$2{}^{\circ}C^{\mathrm{b},\mathrm{SU}}_{k,m}$	$\frac{1}{2}{}^{\circ}C^{\mathrm{c},\mathrm{SU}}_{k,m}$	${}^{\circ}C^{\mathrm{d},\mathrm{SU}}_{k,m}$
2	0	1	1	0	−2	−1	−1	0	−2	−1	−1
3	0	1	9	−5	−2	−1	−1	−9	−18	−9	−9
3	1	1	9	0	3	−1	−1	0	−5	−9	−9
4	0	−5	13	−7	10	5	5	−25	−26	−13	−13
4	1	−12	24	4	15	13	14	−23	−21	−23	−22
4	2	−5	13	0	21	5	5	0	3	−13	−13
5	0	−23	9	−1	46	23	23	−33	−18	−9	−9
5	1	−51	13	31	47	55	59	−33	−17	−9	−5
5	2	−51	13	39	76	55	59	−9	12	−9	−5
5	3	−23	9	0	63	23	23	0	31	−9	−9
6	0	−61	−11	20	122	61	61	−30	22	11	11
6	1	−126	−26	92	107	135	144	−8	7	35	44
6	2	−159	−59	139	187	175	191	39	87	75	91
6	3	−126	−26	110	201	135	144	35	101	35	44
6	4	−61	−11	0	139	61	61	0	89	11	11

also: $k = 7, 8, 9$. All color factors are quartic polynomials in m and k.

The Data: Result

$$\mathcal{F}^{(1),\mathrm{U}}_{k,m}(z,\bar z) =$$

$$-\frac{2k^2}{N_c^2}\left\{1+\frac{1}{N_c^2}\left[\left[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\right]k^4+\left[\tfrac{9}{2}r^2-\tfrac{13}{8}\right]k^3+\left[\tfrac{1}{6}r^2+\tfrac{15}{8}\right]k^2-\tfrac{1}{2}k\right]\right\}F^{(1)}\,,$$

$$\mathcal{F}^{(2),\mathrm{U}}_{k,m}(z,\bar z) =$$

$$\begin{aligned}
&\frac{4k^2}{N_c^2}\Bigg[\left\{1+\frac{1}{N_c^2}\left[\left[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\right]k^4+\left[\tfrac{9}{2}r^2-\tfrac{13}{8}\right]k^3+\left[\tfrac{1}{6}r^2+\tfrac{15}{8}\right]k^2-\tfrac{1}{2}k\right]\right\}F^{(2)}\\
&\quad+\Bigg\{\frac{t}{4}+\frac{1}{N_c^2}\Bigg[\left(\left[\tfrac{7}{2}r^2-\tfrac{1}{8}\right]k^2+\tfrac{5}{8}k-\tfrac{1}{4}\right)s_+-r\left(\left[\tfrac{17}{6}r^2-\tfrac{7}{8}\right]k^3+3k^2-\tfrac{13}{12}k\right)s_-\\
&\qquad+\left(\left[\tfrac{29}{24}r^4-\tfrac{11}{16}r^2+\tfrac{15}{128}\right]k^4+\left[\tfrac{17}{8}r^2-\tfrac{21}{32}\right]k^3-\left[\tfrac{23}{24}r^2-\tfrac{39}{32}\right]k^2-\tfrac{9}{8}k+\tfrac{1}{2}\right)t\Bigg]\Bigg\}\left(F^{(1)}\right)^2\\
&\quad-\frac{1}{N_c^2}\Bigg[r\left\{\left[\tfrac{7}{6}r^2-\tfrac{1}{8}\right]k^3+\tfrac{3}{2}k^2+\tfrac{10}{3}k\right\}F^{(2)}_{\mathrm{C},-}\\
&\qquad\quad+\left\{\left[\tfrac{5}{4}r^2-\tfrac{19}{48}\right]k^3+\left[\tfrac{3}{2}r^2+\tfrac{7}{8}\right]k^2+\tfrac{1}{3}k\right\}F^{(2)}_{\mathrm{C},+}\Bigg]\\
&\quad+\frac{1}{4}\left\{1+\frac{(k-1)(k^3+3k^2-46k+36)}{12N_c^2}\right\}\left(s\delta_{m,0}+\delta_{m,k-2}\right)\left(F^{(1)}\right)^2\\
&\quad+\left\{1+\frac{(k-2)_4}{12N_c^2}\right\}\left(\delta_{m,0}F^{(2)}_{z-1}+\delta_{m,k-2}F^{(2)}_{1-z}\right)\Bigg]\,,
\end{aligned}$$

where $r=(m+1)/k-1/2$. $\mathcal{F}_{k,m}$: Coefficient of X^mY^{k-2-m}.

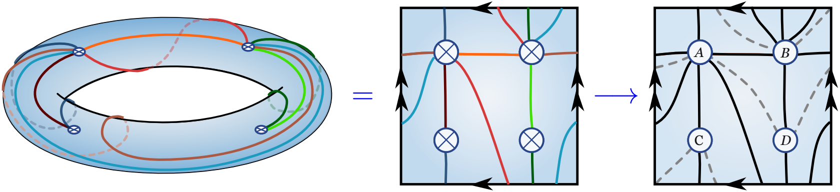
Sum over Graphs: Cutting the Torus

Sum over propagator graphs: Split into

- Sum over "skeleton graphs" with non-parallel edges ($\equiv$ "bridges")
- Sum over distributions of parallel propagators on bridges

Torus with four punctures: *How many hexagons/bridges?*
Euler: $F + V - E = 2 - 2g$.
Our case: $g = 1$, $V = 4$, $E = \frac{3}{2}F \quad \Rightarrow \quad F = 8$, $E = 12$.

$\to$ Construct all genus-one graphs with 4 punctures and up to 12 edges.



Propagators may populate < 12 bridges and still form a genus-one graph.
Such graphs will contain higher polygons besides hexagons.
$\to$ Subdivide into hexagons by inserting zero-length bridges (ZLBs)

Maximal Graphs

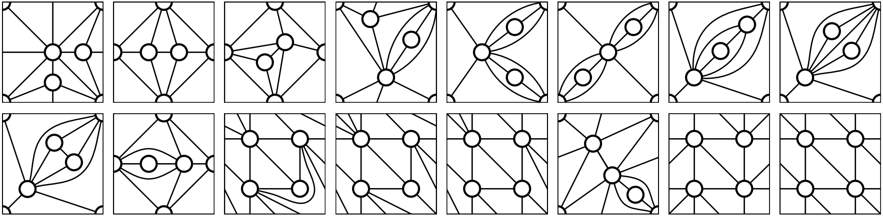
Focus on Maximal Graphs: Graphs with a maximal number of edges.

- Adding any further edge would increase the genus
- Maximal graphs $\Leftrightarrow$ triangulations of the torus.

Construction:

- Manually: Add one operator at a time, in all possible ways.
- Computer algorithm: Start with the empty graph, add one bridge in all possible ways, iterate. $\rightarrow$ Systematic.

Complete list of maximal graphs:



Submaximal Graphs

Submaximal graphs: Graphs with a non-maximal number of edges.

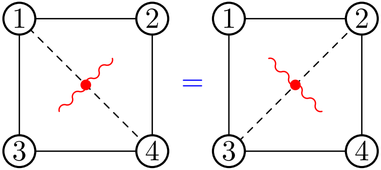
- Obtained from maximal graphs by deleting bridges.
- Number of genus-one graphs by number of bridges:

#bridges:	12	11	10	9	8	7	6	5	$\leq$4
#graphs:	7	28	117	254	323	222	79	11	0

Hexagonalization:

Submaximal graphs contain higher polygons (octagons, decagons, . . .).

- Must be subdivided into hexagons by zero-length bridges.
- Subdivision is not physical: Can pick any (flip invariance): [Fleury '16 Komatsu]



First Test: Large k: Data and Graphs

Focus on leading order in large k $\to$ several simplifications:

Data:

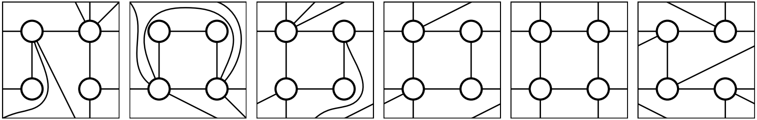
$$\mathcal{F}^{(1),\mathrm{U}}_{k,m}(z,\bar{z}) = -\frac{2k^2}{N_c^2}\left\{1+\frac{1}{N_c^2}\left[\left[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\right]k^4+\mathcal{O}(k^3)\right]\right\}F^{(1)}\,,$$

$$\mathcal{F}^{(2),\mathrm{U}}_{k,m}(z,\bar{z}) = \frac{4k^2}{N_c^2}\Bigg[\left\{1+\frac{1}{N_c^2}\left[\left[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\right]k^4+\mathcal{O}(k^3)\right]\right\}F^{(2)}$$
$$+\left\{1+\frac{1}{N_c^2}\left[\left[\tfrac{29}{6}r^4-\tfrac{11}{4}r^2+\tfrac{15}{32}\right]k^4+\mathcal{O}(k^3)\right]\right\}\frac{t}{4}\left(F^{(1)}\right)^2\Bigg]\,.$$

Combinatorics of distributing propagators on bridges:
Sum over distributions of m propagators on $j+1$ bridges $\to$ $m^j/j!$

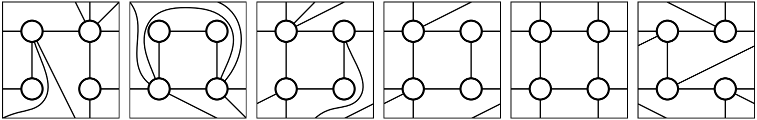
- $\Rightarrow$ Only graphs with maximum bridge number contribute.
- $\Rightarrow$ All bridges carry a large number of propagators.

Graphs:
$(Z=0)$



First Test: Large k: Graphs and Labelings

Graphs:



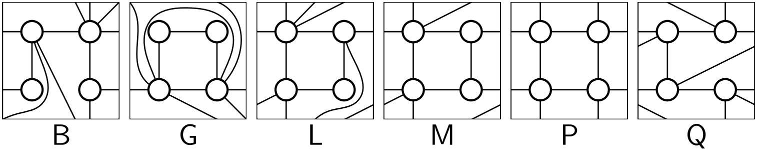
B G L M P Q

Sum over labelings:

Case	Inequivalent Labelings (clockwise)	Combinatorial Factor
B	$(1,2,4,3),(2,1,3,4),(3,4,2,1),(4,3,1,2)$	$m^3(k-m)/6$
B	$(1,3,4,2),(3,1,2,4),(2,4,3,1),(4,2,1,3)$	$m(k-m)^3/6$
G	$(1,2,4,3),(3,4,2,1)$	$m^4/24$
G	$(1,3,4,2),(2,4,3,1)$	$(k-m)^4/24$
L	$(1,2,4,3),(3,4,2,1),(2,1,3,4),(4,3,1,2)$	$m^2/2\cdot(k-m)^2/2$
M	$(1,2,4,3),(2,1,3,4),(1,3,4,2),(3,1,2,4)$	$m^2(k-m)^2/2$
P	$(1,2,4,3)$	$m^2(k-m)^2/2$
Q	$(1,2,4,3)$	$m^2(k-m)^2$

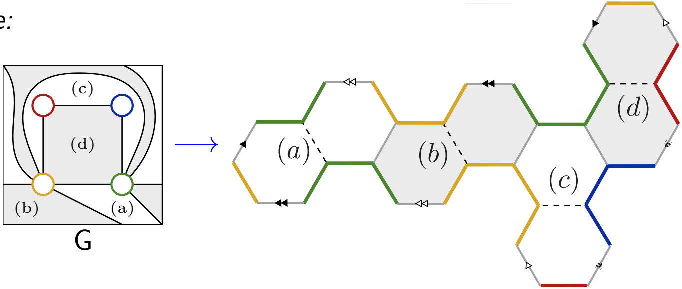
First Test: Large k: Octagons

Graphs:



All graphs consist of only octagons!
Split each octagon into two hexagons with a zero-length bridge.

Example:



First Test: Large k: Mirror Particles

Loop Counting:

Expand mirror propagation $\mu(u)\, e^{-\ell \tilde{E}(u)}$ and hexagons $\mathcal{H}$ in coupling g.

$\to$ n particles on bridge of size ℓ: $\mathcal{O}(g^{2(n\ell+n^2)})$

All graphs consist of octagons framed by parametrically large bridges.

$\to$ Only excitations on zero-length bridges inside octagons survive.

Excited Octagons:

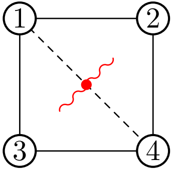
n particles on a zero-length bridge $\to$ $\mathcal{O}(g^{2n^2})$

$\to$ Octagons with $1/2/3/4$ particles start at $1/4/9/16$ loops.

Octagon 1–2–4–3 with 1 particle:

[Fleury '16 Komatsu] [TB,Caetano,Fleury Komatsu,Vieira '18]

$$\mathcal{M}(z,\alpha) = \left[z + \bar{z} - (\alpha + \bar{\alpha})\frac{\alpha\bar{\alpha} + z\bar{z}}{2\alpha\bar{\alpha}}\right] \cdot \left(g^2 F^{(1)}(z) - 2g^4 F^{(2)}(z) + 3g^6 F^{(3)}(z) + \dots\right)$$



For $Z = 0$: R-charge cross ratios

$\alpha = z\bar{z}\, X/Y$ and $\bar{\alpha} = 1$.

First Test: Large k: Match and Prediction

We are Done:

Sum over graph topologies and labelings (with bridge sum factors),
Sum over one-particle excitations of all octagons.
$\Rightarrow$ Result matches data and produces prediction for higher loops!

Summing all octagons gives:

$$\mathcal{F}^{\mathrm{U}}_{k,m}(z,\bar{z})\Big|_{\text{torus}} = -\frac{2k^6}{N_c^4}\Bigg\{$$

$$g^2\left[\tfrac{17}{6}r^4 - \tfrac{7}{4}r^2 + \tfrac{11}{32}\right]F^{(1)} \qquad \checkmark \text{ match}$$

$$-\,2g^4\left[\left[\tfrac{17}{6}r^4 - \tfrac{7}{4}r^2 + \tfrac{11}{32}\right]F^{(2)} + \left[\tfrac{29}{6}r^4 - \tfrac{11}{4}r^2 + \tfrac{15}{32}\right]\frac{t}{4}\left(F^{(1)}\right)^2\right] \qquad \checkmark \text{ match}$$

$$+\,g^6\left[\left[\ldots\right]F^{(3)} + \left[\ldots\right]\left(F^{(2)}\right)\left(F^{(1)}\right) + \left[\ldots\right]\left(F^{(1)}\right)^3\right] \qquad \text{prediction!}$$

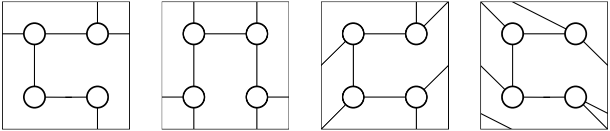
$$+\,\mathcal{O}(g^8) + \mathcal{O}(1/k)\Bigg\}\,.$$

More Tests: $k = 2, 3, 4, 5, \ldots$

Small and finite k:

Few propagators $\rightarrow$ Fewer bridges $\rightarrow$ Graphs with fewer edges
$\Rightarrow$ Graphs composed of not only octagons, but bigger polygons

Example: Graphs for $k = 3$:



Hexagonalization:

Each $2n$-gon: Split into $n-2$ hexagons by $n-3$ zero-length briges.

Loop Expansion: Much more complicated!

All kinds of excitation patterns already at low loop orders

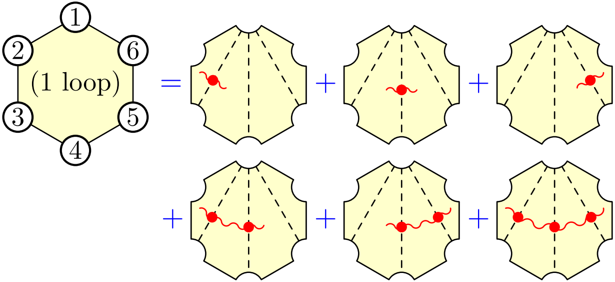
- Single particles on several adjacent zero-length (or $\ell = 1$) bridges
- Strings of excitations wrapping around operators

Finite k: One Loop: Sum over ZLB-Strings

Restrict to one loop: Only single particles on one or more adjacent zero-length bridges contribute.

$\Rightarrow$ Excitations confined to single polygons bounded by propagators.

For each polygon: Sum over all possible one-loop strings:



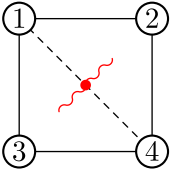
One-strings: understood ✓

Longer strings: need to compute!

Two-String: Result

One-String: Can be written as

$$\mathcal{M}^{(1)}(z,\alpha) = m(z) + m(z^{-1})\,,$$

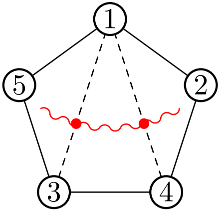


with building block

$$m(z) = m(z,\alpha) = g^2\,\frac{(z+\bar{z}) - (\alpha+\bar{\alpha})}{2}\,F^{(1)}(z,\bar{z})$$

Two-string: Despite complicated computation, simplifies to [Fleury '17 Komatsu]

$$\begin{aligned}
&\mathcal{M}^{(2)}(z_1,z_2,\alpha_1,\alpha_2)\\
&= m\left(\frac{z_1-1}{z_1 z_2}\right) + m\left(\frac{1-z_1+z_1 z_2}{z_2}\right)\\
&\quad + m(z_1(1-z_2)) - m(z_1) - m(z_2^{-1})\,,
\end{aligned}$$

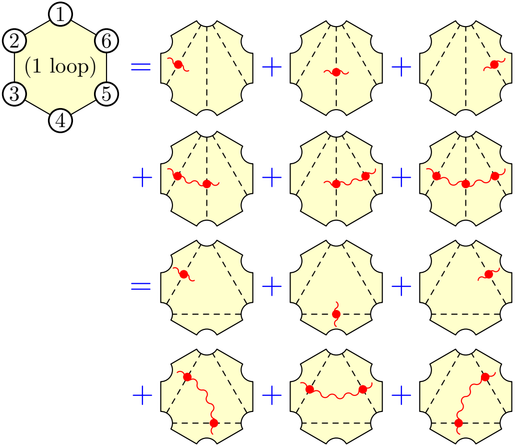


with the same building block $m(z)$!

Finite k: Larger Strings

Larger strings: Computation will be even more complicated!

But: Can in fact bootstrap all of them by using flip invariance!



Apply recursively:

- 3-string $\simeq$ 1-strings & 2-strings
- . . . iterate . . .
- n-string $\simeq$ 1-strings & 2-strings

$\Rightarrow$ Can write all polygons in terms of only 1-strings & 2-strings.

$\Rightarrow$ All n-strings can be written as linear combinations of one-string building blocks $m(z)$.

Finite k: Results

Done! Sum over all graphs, expand all polygons to their one-loop values.

Numbers of labeled graphs with assigned bridge sizes:

k:	2	3	4	5
$g=0$:	3	8	15	24
$g=1$:	0	32	441	2760

Result: For $k = 2, 3, 4, 5, \ldots$:
Matches the $\mathrm{U}(N_\mathrm{c})$ data $\mathcal{F}_{k,m}$, up to a copy of the planar term!

$$\mathcal{F}_{k,m}: \quad \texttt{Result} = \underbrace{(\texttt{torus data})}_{\checkmark\checkmark\checkmark} + \frac{1}{N_\mathrm{c}^2}\underbrace{(\texttt{planar data})}_{???}$$

What does this mean?? $\Rightarrow$ Puzzle.

Difference between $\mathrm{U}(N_\mathrm{c})$ and $\mathrm{SU}(N_\mathrm{c})$? $\to$ No
Operator normalizations? $\to$ No
Need to include planar graphs on the torus? If yes, how?

Finite k: Stratification

We are computing a worldsheet process.
The string amplitude involves integration over moduli space $\mathcal{M}_{g,n}$.

Sum over graphs: Reminiscent of moduli space integration.
This can be made more precise:
Moduli space $\Leftrightarrow$ space of *metric ribbon graphs* $\mathrm{RGB}^{\mathrm{met}}_{g,n}$.

Metric Ribbon Graphs with labeled Boundary:
Regular graphs, but edges at each vertex have definite ordering.
Double-line notation defines n oriented boundary components (faces).
Faces define compact oriented surface of definite genus g.
Assign length $\ell_j \in \mathbb{R}_+$ to each edge.

Bijection: Via Strebel theory:

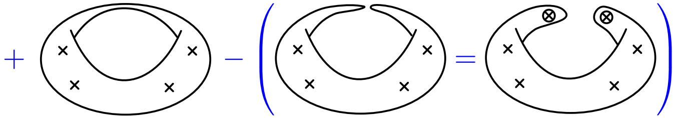
$$\mathcal{M}_{g,n} \times \mathbb{R}^n_+ \quad \longleftrightarrow \quad \mathrm{RGB}^{\mathrm{met}}_{g,n} = \coprod_{\Gamma \in \mathrm{RG}_{g,n}} \frac{\mathbb{R}^{e(\Gamma)}_+}{\mathrm{Aut}_\partial(\Gamma)}$$

Finite k: Stratification

Discretization: Need to be careful at the boundaries of the space. Do not overcount/undercount. Boundary of torus moduli space: All bridges traversing a handle reduce to zero size $\longrightarrow$ handle gets pinched.

This problem has been considered before in the context of matrix models. [Deligne Mumford '69] [Chekhov 1995]

Resolution: In the sum over graphs, include planar graphs drawn on the torus. This leads to some overcounting. Compensate by subtracting planar graphs with two extra fictitious zero-size operators. *Stratification*.

$\Rightarrow$ 

Including these contributions indeed accounts for the $(\text{planar})/N_c^2$ term!

$\Rightarrow$ Now have a complete match for $k = 2, 3, 4, 5$.

Summary & Outlook

Summary: Method to compute higher-genus terms in $1/N_c$ expansion.

- Sum over free graphs, decompose into planar hexagons, integrate over mirror states.
- Large k: Only octagons, match at two loops, three-loop prediction
- Match for various finite $k \to$ stratification

Outlook: There are many things to do that we currently explore:

- Study more examples: Higher loops / genus, more general operators
- Understand details/implications of stratification beyond one loop
- Evaluation of mirror particles at higher loops
- Connect to recent supergravity loop computations at strong coupling? [Aharony, Alday '16 Bissi, Perlmutter] [Alday, Bissi Perlmutter '17] [Alday Bissi '17] [Aprile, Drummond, Heslop Paul '17, '17, '17, '18]
- Promising: Large k at higher genus: Only octagons.