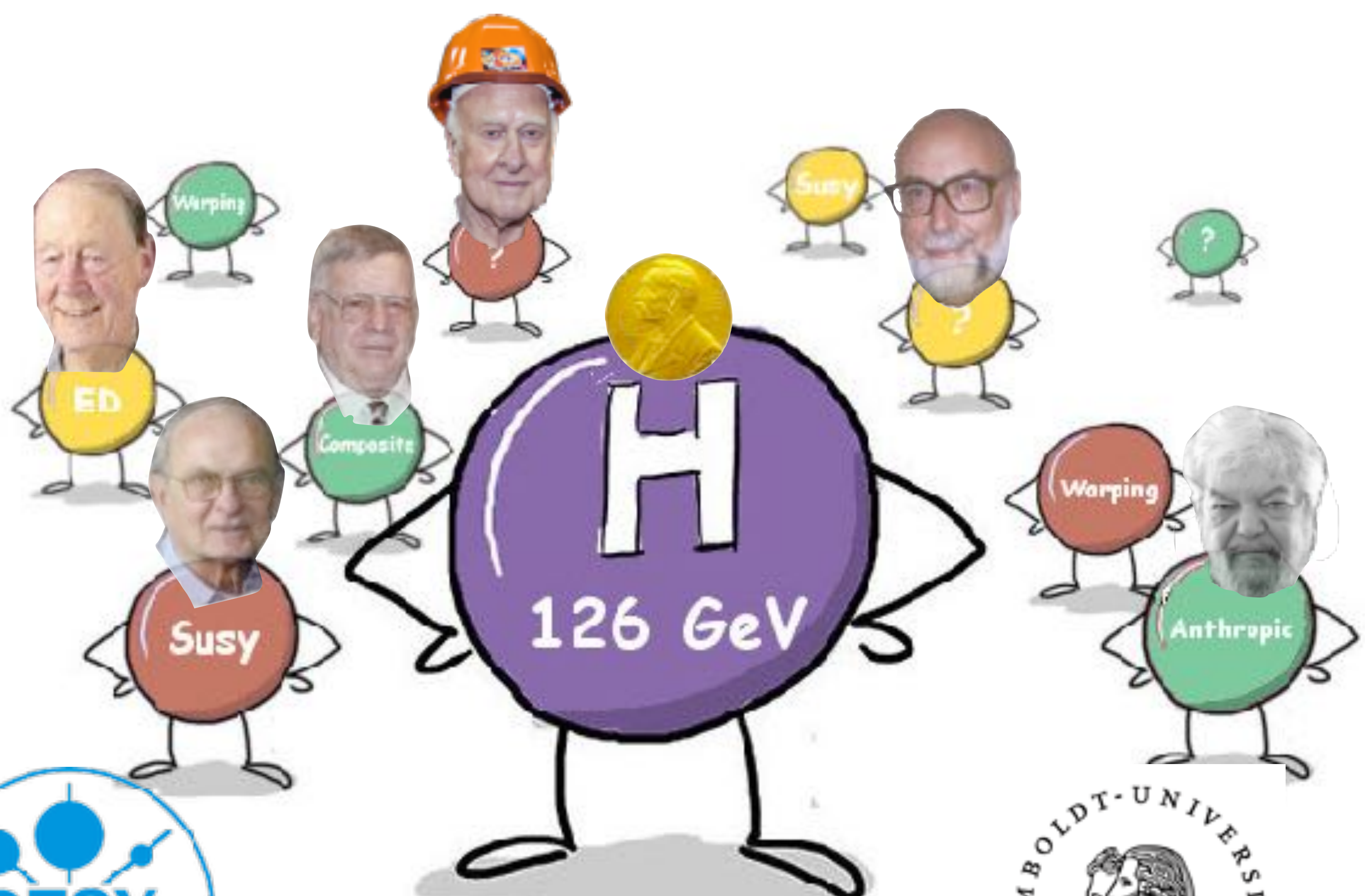
Introduction HEP Theory

DESY summer student lectures 2018



Lectures 3+4/6

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Outline

1. Friday 27.07

   Quantum field theory, dimensional analysis

2. Friday 27.07

   elementary particles, different fundamental interactions

3. Thursday 02.08

   Noether theorem and Symmetries (space-time, internal gauge symmetries, continuous, global), Fermi theory, effective theory, gauge symmetry, QED, non-abelian gauge symmetries, Standard Model

4. Thursday 02.08

   Spontaneous symmetry breaking, Goldstone theorem, Higgs mechanism

5. Friday 03.08

   Electroweak precision test, stability of the EW vacuum, the hierarchy problem. Field quantization, S-matrix, Feynman rules, scattering, cross sections, decay rates, calculation tricks, sample calculation

6. Friday 03.08

   Non-abelian gauge theories, Standard Model Lagrangian and its phenomenological properties

the order will probably change

Recap

1. QM+Special relativity:

   i. uncertainty principle + mass $\leftrightarrow$energy == particle creation

   ii. antiparticle: logarithmic corrections to e^- mass: running of α

2. Finite number of elementary particles & 3 families

3. There are different fundamental forces among the elementary particles

4. Non-trivial (quantum) consistency of particle content

   $\leftrightarrow$ electric neutrality of the atoms

QFT: the need for particle creation

Heuristically, we saw why QM+Special Relativity cannot live with fixed number of particles
More rigorous proof by considering a QM description of a free relativistic particle

particle localized at the origin at t=0: $\langle\vec{r}|\psi(0)\rangle = \delta^3(\vec{r})$

(equivalently, $\langle\vec{k}|\psi(0)\rangle = 1$, as dictated by uncertainty principle)

its time evolution follows from Schrödinger eq.: $|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle$

In particular, the probability to find the particle at the time t at the position $\vec{r}$ is given by

$$\langle\vec{r}|e^{-i\hat{H}t}|\psi(0)\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\langle\vec{r}|e^{-i\hat{H}t}|\vec{k}\rangle\langle\vec{k}|\psi(0)\rangle$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\sqrt{\vec{k}^2+m^2}t} e^{i\vec{k}\cdot\vec{r}}$$

ex.
$$= \frac{1}{r}\int_m^\infty dz\, z\, e^{-zr}\sinh(\sqrt{z^2-m^2}t)$$ *>0* for *r>t>0*

non-vanishing probability to be outside the lightcone, i.e. violation of causality

need to consider **multiparticle states**: $\hat{H} \neq \sqrt{\hat{\vec{P}}^2+m^2}$ $\left(\sum_{\vec{k}}\sqrt{\vec{k}^2+m^2} \neq \sqrt{(\sum_{\vec{k}}\vec{k})^2+m^2}\right)$

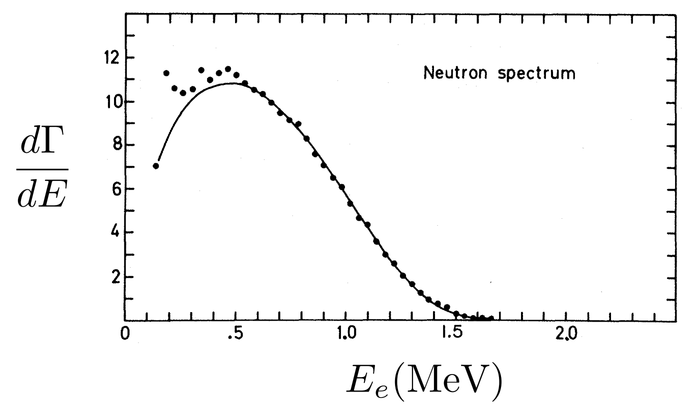
Beta decay

$${}^{40}_{19}\mathrm{K} \to {}^{40}_{20}\mathrm{Ca}^+ + e^- \qquad {}^{64}_{29}\mathrm{Cu} \to {}^{64}_{30}\mathrm{Zn}^+ + e^- \qquad {}^{3}_{1}\mathrm{H} \to {}^{3}_{2}\mathrm{He}^+ + e^-$$

- Two body decays: A→B+C

$$E_B = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A} c^2 \qquad p = \frac{\sqrt{\lambda(m_A, m_B, m_C)}}{2m_A} c$$

$$\lambda(m_A, m_B, m_C) = (m_A + m_B + m_C)(m_A + m_B - m_C)(m_A - m_B + m_C)(m_A - m_B - m_C)$$



fixed energy of daughter particles (pure SR kinematics, independent of the dynamics)

⇛ non-conservation of energy?

Pauli '30: ∃ neutrino, very light since end-point of spectrum is close to 2-body decay limit

ν first observed in '53 by Cowan and Reines

- N-body decays: A→B$_1$+B$_2$+...+B$_N$

$$E_{B_1}^{\min} = m_{B_1} c^2 \qquad E_{B_1}^{\max} = \frac{m_A^2 + m_{B_1}^2 - (m_{B_2} + \ldots + m_{B_N})^2}{2m_A} c^2$$

$$n \longrightarrow p + e^- + \bar{\nu}_e$$

Fermi theory '33

$$\mathcal{L} = G_{\mathcal{F}} (\bar{n} p)(\bar{\nu}_e e)$$

exp: G_F=1.166x10^{-5} GeV^{-2}

Gauge Theories

Fermi Theory

(paper rejected by Nature: declared too speculative !)

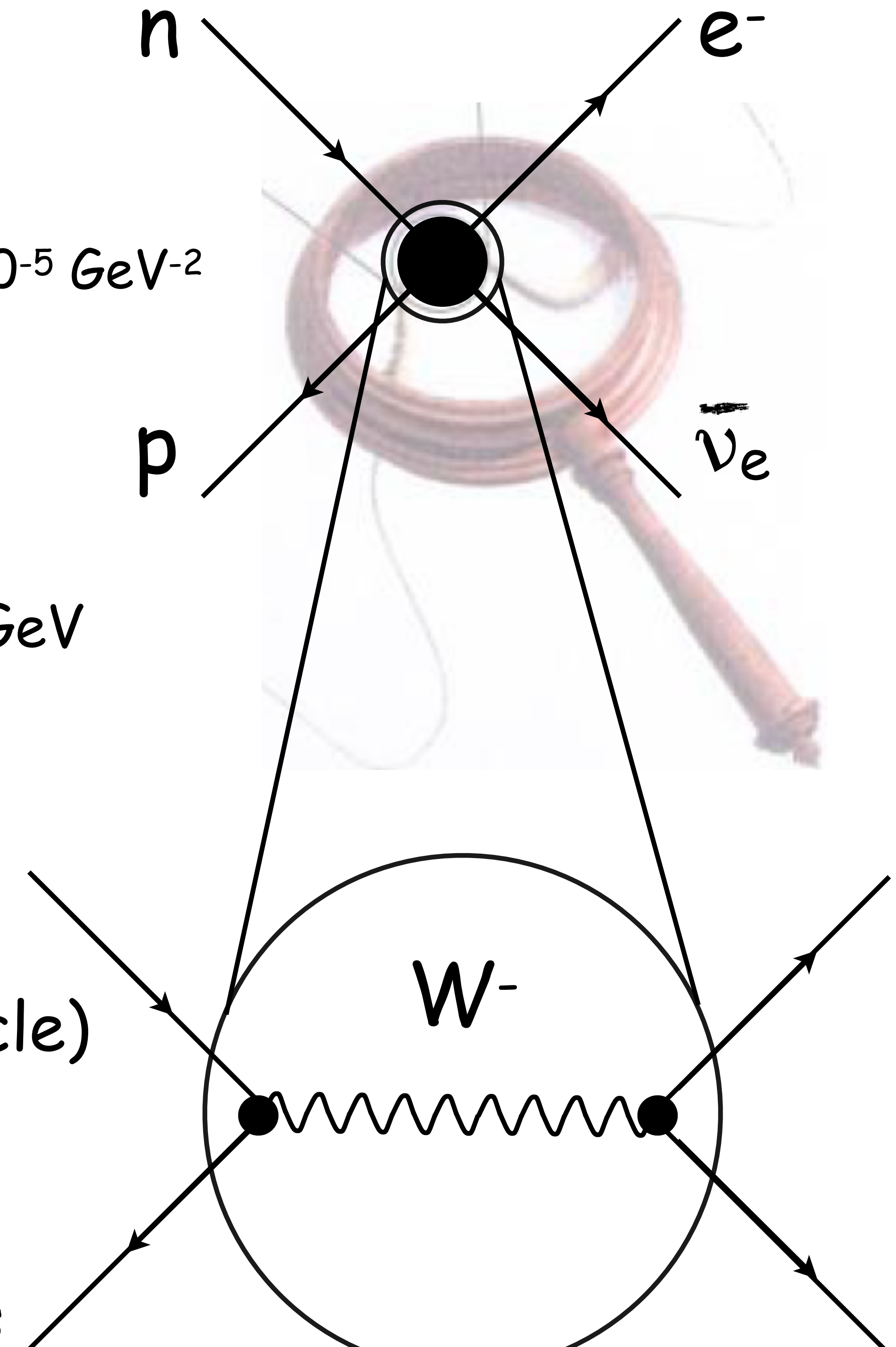
$$n \longrightarrow p + e^- + \bar{\nu}_e$$

exp: $G_F = 1.166 \times 10^{-5}$ GeV^{-2}

$$\mathcal{L} = G_{\mathcal{F}} (\bar{n} p)(\bar{\nu}_e e)$$

$$\mathcal{A} \propto G_{\mathcal{F}} E^2$$

- no continuous limit
- inconsistent above 300 GeV



Gauge theory

microscopic theory

(exchange of a massive spin 1 particle)

$$G_{\mathcal{F}} = \frac{\sqrt{2} g^2}{8 m_W^2}$$

exp: m_W=80.4 GeV

➡ g ≈0.6, ie, same order as e=0.3

unification EM & weak interactions

Why Gauge Theories?

How are we sure that muon and neutron decays proceed via the same interactions?

$\tau_\mu \approx 10^{-6}$s vs. $\tau_{neutron} \approx 900$s

$$\underbrace{\mathcal{L}}_{[\text{mass}]^4} = \underbrace{G_F}_{[\text{mass}]^{-2}} \underbrace{\psi^4}_{[\text{mass}]^{3/2\times 4}} \quad \Rightarrow \quad \underbrace{\Gamma}_{[\text{mass}]} \propto G_F^2 m^5$$

for the muon, the relevant mass scale is the muon mass m_μ=105MeV: $\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3} \sim 10^{-19}\,\text{GeV}$

for the neutron, the relevant mass scale is $(m_n - m_p) \approx 1.29$MeV: $\Gamma_n = \mathcal{O}(1)\frac{G_F^2 \Delta m^5}{\pi^3} \sim 10^{-28}\,\text{GeV}$

ex: what about $\pi^\pm$ decay $\tau_\pi \approx 10^{-8}$s? Why $\frac{\Gamma(\pi^- \to e^- \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)} \sim 10^{-4}$?

What about weak scattering process, e.g. $e\nu_e \to e\nu_e$?

$$\underbrace{\sigma}_{[\text{mass}]^{-2}} \propto \underbrace{G_F^2}_{[\text{mass}]^{-2\times 2}} \underbrace{E^2}_{[\text{mass}]^2} \quad \Rightarrow$$

non conservation of probability
(non-unitary theory)
inconsistent at energy above 300GeV

Why Gauge Theories?

What about weak scattering process, e.g. $e\nu_e \to e\nu_e$?

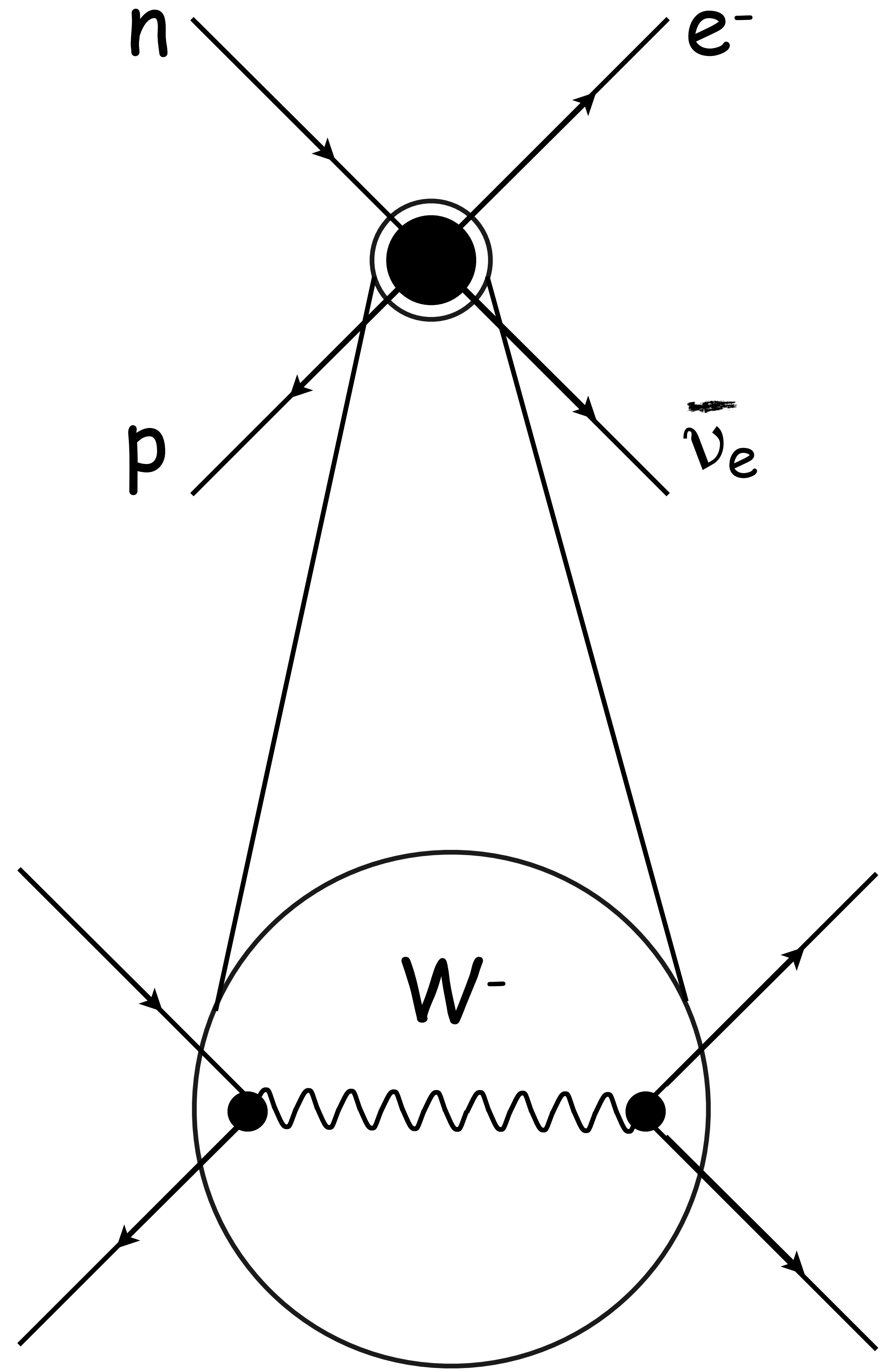
$$\sigma \propto G_F^2 E^2$$

$[\text{mass}]^{-2}$ $[\text{mass}]^{-2\times 2}$ $[\text{mass}]^2$

Gauge theory

$$\sigma \propto g^4 \frac{E^2}{m_W^2\,(E^2 + m_W^2)}$$

- match with Fermi theory at low energy $G_F \propto \dfrac{g^2}{m_W^2}$
  (we say that the Fermi theory is an **effective theory** of the weak gauge theory at low energy)
- good high energy behavior



From Gauge Theory to Fermi Theory

We can derive the Fermi current-current contact interactions by "integrating out" the gauge bosons, i.e., by replacing in the Lagrangian the W by their equation of motion. Here is a simple derivation (a better one taking into account the gauge kinetic term and the proper form of the fermionic current will be presented in the lecture, for the moment, take it as a heuristic derivation)

$$\mathcal{L} = -m_W^2 W_\mu^+ W_\nu^- \eta^{\mu\nu} + g W_\mu^+ J_\nu^- \eta^{\mu\nu} + g W_\nu^- J_\nu^+ \eta^{\mu\nu}$$

$$J^{+\mu} = \bar{n}\gamma^\mu p + \bar{e}\gamma^\mu \nu_e + \bar{\mu}\gamma^\mu \nu_\mu + \ldots \qquad \text{and} \qquad J^{-\mu} = \left(J^{+\mu}\right)^*$$

The equation of motion for the gauge fields: $\dfrac{\partial \mathcal{L}}{\partial W_\mu^+} = 0 \qquad \Rightarrow \qquad W_\mu^- = \dfrac{g}{m_W^2} J_\mu^-$

Plugging back in the original Lagrangian, we obtain an effective Lagrangian (valid below the mass of the gauge bosons):

$$\mathcal{L} = \frac{g^2}{m_W^2} J_\mu^+ J_\nu^- \eta^{\mu\nu}$$

Which is the Fermi current-current interaction. The Fermi constant is given by $G_F = \dfrac{g^2}{m_W^2}$

(the correct expression involves a different normalisation factor)

In the current-current product, the term $(\bar{n}\gamma^\mu p)(\bar{\nu}_e\gamma^\nu e)\eta_{\mu\nu}$ is responsible for beta decay, while the term $(\bar{\mu}\gamma^\mu \nu_\mu)(\bar{\nu}_e\gamma^\nu e)\eta_{\mu\nu}$ is responsible for muon decay. Both decays are controlled by the same coupling, as indicated by the measurements of the lifetimes of the muon and neutron.

Why non-abelian Gauge Theories?

EM = exchange of photon = U(1) gauge symmetry

EM U(1) $\quad \phi \to e^{i\alpha}\phi \quad$ but $\quad \partial_\mu\phi \to e^{i\alpha}\,(\partial_\mu\phi) + \underbrace{i(\partial_\mu\alpha)}_{\neq 0 \text{ if local transformations}}\,\phi$

EM field and covariant derivative $\quad \partial_\mu\phi + ieA_\mu\phi \to e^{i\alpha}(\partial_\mu\phi + ieA_\mu\phi)$

if $\quad A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha$

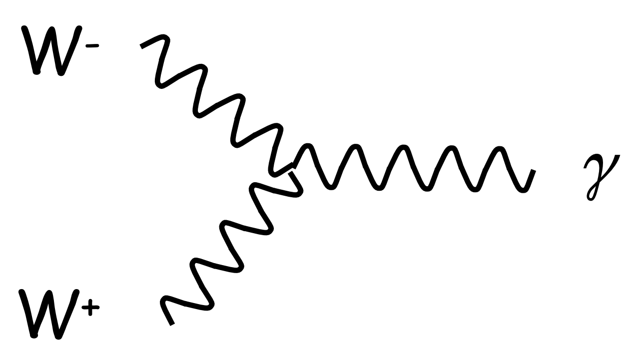
the EM field keeps track of the phase in different points of the space-time

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \to F_{\mu\nu}$$

photon do not interact with itself because it doesn't carry an electric charge

W carries an electric charge since it mediates charged current interactions

W interacts with the photon ☛ non-abelian interactions



Gauge Theories: EM & Yang-Mills

EM U(1) $\quad \phi \to e^{i\alpha}\phi \quad$ but $\quad \partial_\mu \phi \to e^{i\alpha}\left(\partial_\mu \phi\right) + \underbrace{i(\partial_\mu \alpha)}_{\neq 0 \text{ if local transformations}}\phi$

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$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to F_{\mu\nu}$$

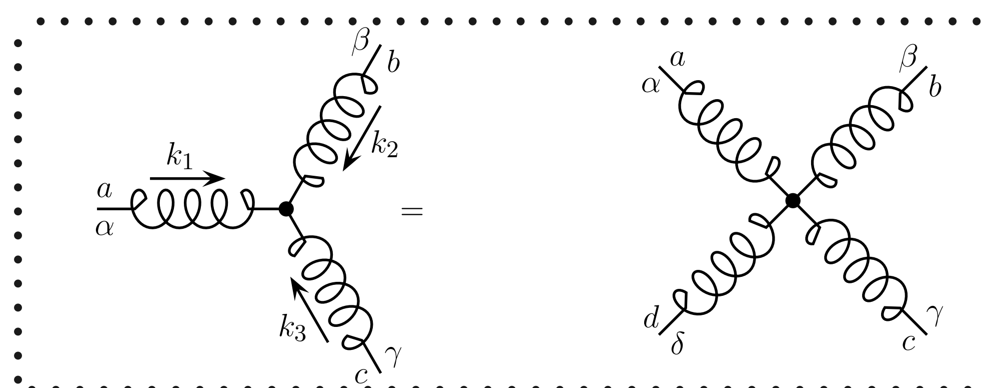
Yang-Mills : non-abelian transformations $\quad \phi \to U\phi$

$$\partial_\mu\phi + igA_\mu\phi \to U(\partial_\mu\phi + igA_\mu\phi)$$

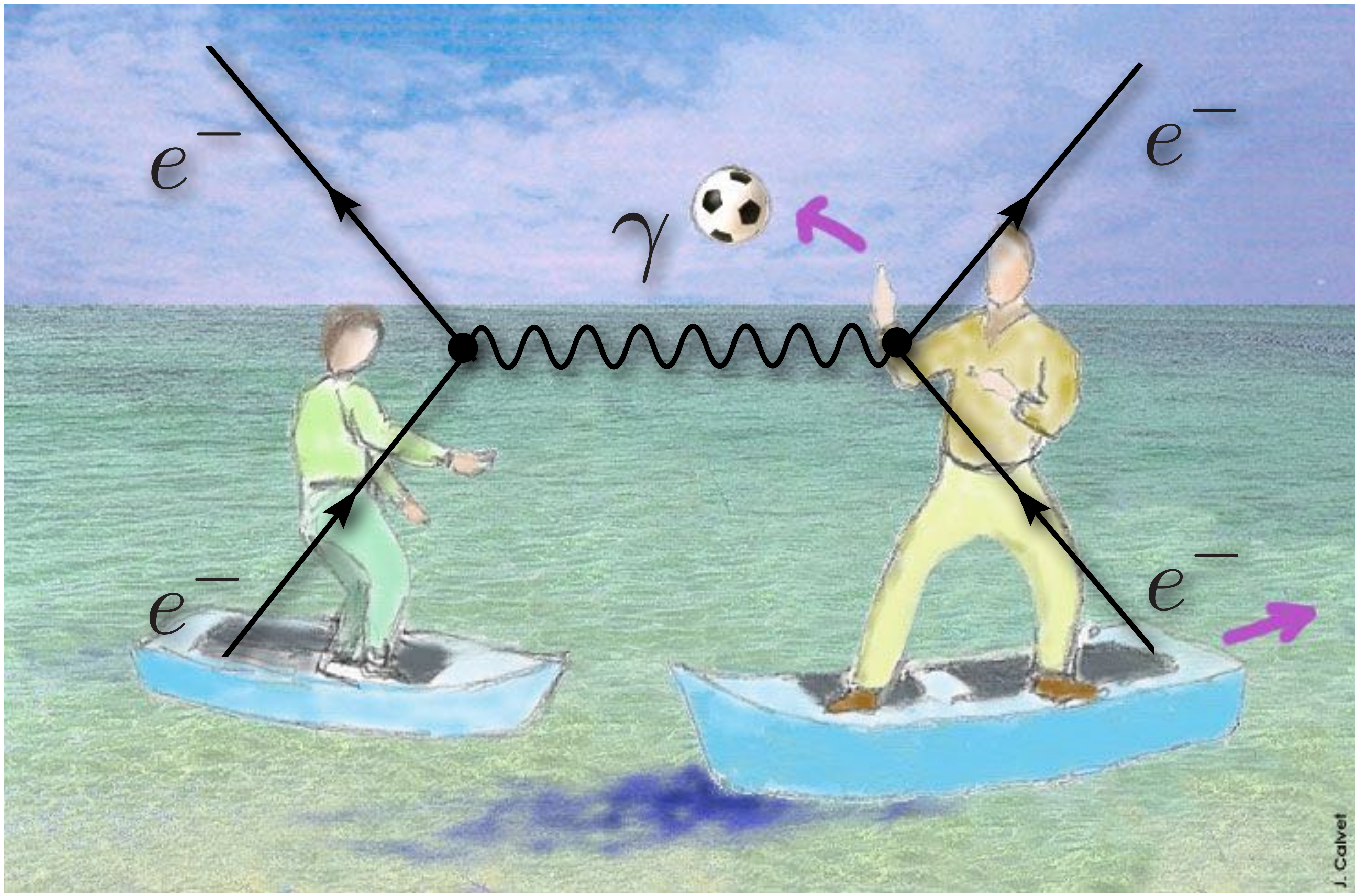
if $\quad A_\mu \to UA_\mu U^{-1} - \frac{i}{g}U\partial_\mu U^{-1}$

ex

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \underbrace{ig[A_\mu, A_\nu]}_{\text{non-abelian int.}} \to UF_{\mu\nu}U^{-1}$$



Interactions between Particles



Elementary particles interact on each other
by the exchange of gauge bosons

The Standard Model: Interactions

- $U(1)_Y$ ***electromagnetic interactions***

  ➘ Photon γ

  { light, atoms, molecules — 10^{-5}

- $SU(2)_L$ ***weak interactions***

  ➘ bosons $W^{\pm}, Z^0$

  { β decay

  $$n \xrightarrow{W^\pm} p + e^- + \bar{\nu}_e$$

  $$e^+ + e^- \xrightarrow{Z^0} D^+_{(c\bar{s})} + D^-_{(\bar{c}s)}$$

  — 10^{-2}

- $SU(3)_c$ ***strong interactions***

  ➘ gluons g^a

  { atomic nuclei, α decay

  $$^{238}_{92}\mathrm{U} \longrightarrow {}^{234}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He}$$

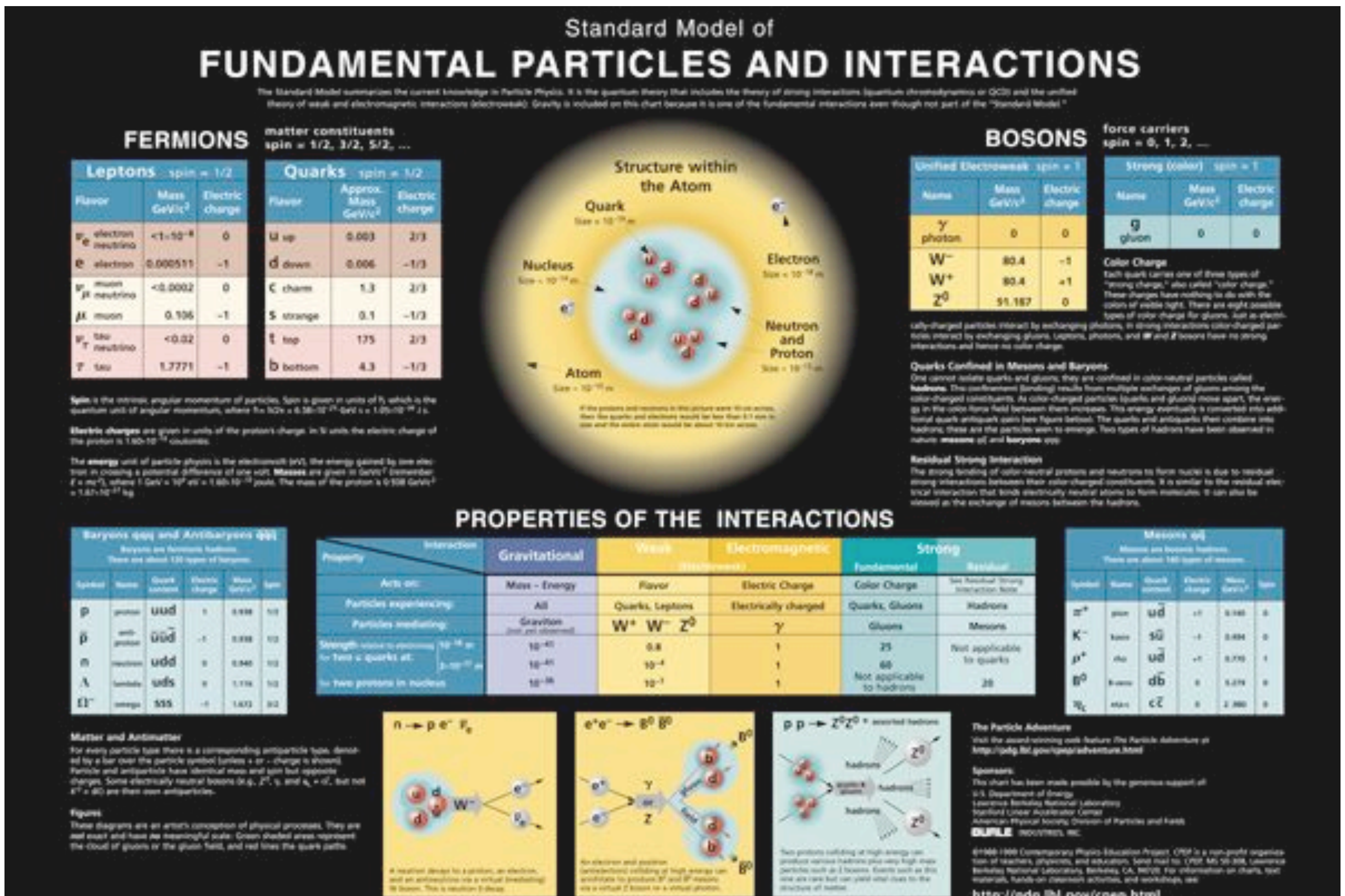
  — 1

strength

The Standard Model

Nobel Prize '79





description of all elementary particles and their interactions

The underlying principles of the SM

The beauty of the SM comes from the the identification of a unique dynamical principle describing the different interactions that seem so different from each others

$\boxed{\text{gauge theory = spin-1}}$

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at the same time a particular and predictive structure that still leaves room for a rich variety of phenomena

(long range interaction, spontaneous symmetry breaking, confinement)

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$\boxed{\text{gauge theory = spin-1}}$

at the same time a particular and predictive structure that still leaves room for a rich variety of phenomena
(long range interaction, spontaneous symmetry breaking, confinement)

$\boxed{\text{gravitation = general relativity= spin-2}}$

much more rigid theory = unique theory

Classical field theory

Classical mechanics & Lagrangian formalism

a system is described by $S = \int dt \mathcal{L}(q, \dot{q})$

position (q) momentum ($\dot{q}$)

action principle determines classical trajectory:

$\delta S = 0$ →→ Euler-Lagrange equations $\boxed{\dfrac{\partial \mathcal{L}}{\partial q_i} - \dfrac{\partial}{\partial t}\dfrac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0}$

conjugate momenta $p_i = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}$ hamiltonian $H(p, q) = \sum_i p_i \dot{q}_i - \mathcal{L}$

Extend Lagrangian formalism to dynamics of fields

$$S = \int d^4x \mathcal{L}(\varphi, \partial_\mu \varphi) \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}$$

$$\delta S = 0 \quad →→ \quad \boxed{\frac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} = 0}$$

conjugate momenta $\Pi_i = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \varphi_i)}$ hamiltonian $H(x) = \sum_i \Pi_i(x) \partial_0 \varphi_i(x) - \mathcal{L}$

Noether theorem

Invariance of action under ***continuous* & *global*** transformation → There is a conserved current/charge

$$\partial_\mu j^\mu = 0 \qquad Q = \int d^3x j^0(x,t)$$

example of transformation

$$\boxed{\varphi \to \varphi e^{i\alpha}} \quad (*)$$

Let us consider small transformation $\alpha \ll 1$

$$\varphi \to \varphi + i\alpha\varphi$$
$$\delta\varphi = i\alpha\varphi$$
$$\delta\partial_\mu\varphi = i\alpha\partial_\mu\varphi$$

1) invariance of $\mathcal{L}$ under (*): $\delta\mathcal{L} = 0 = i\alpha\left(\frac{\delta\mathcal{L}}{\delta\varphi}\varphi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\partial_\mu\varphi\right)$

2) Euler-Lagrange equations: $\frac{\delta\mathcal{L}}{\delta\varphi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi} = 0$

$$\Rightarrow \boxed{\partial_\mu\left(\underbrace{\varphi\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}}_{\equiv J_\mu}\right) = 0}$$

conserved current

Symmetries and conservation laws

Noether's theorem (from classical field theory) :
To each ***continuous*** symmetry of the system corresponds a conserved quantity

I- Continuous global space-time symmetries:

translation invariance in space ➔ momentum conservation
translation invariance in time ➔ energy conservation
rotational invariance ➔ angular momentum conservation

Fields are classified according to their transformation properties under Lorentz group:

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu \qquad \phi(x) \to \phi'(x')$$

$$\phi'(x) = \phi(x)$$ scalar

$$V^\mu \to \Lambda^\mu_\nu V^\nu$$ vector

The true meaning of spin arises in the context of a fully Lorentz-invariant theory (while it is introduced ad-hoc in non-relativistic quantum mechanics)

Symmetries and conservation laws

I- Continuous global space-time (Poincaré) symmetries all particles have (m, s)

➔ energy, momentum, angular momentum conserved

II- Global (continuous) internal symmetries ➔ B, L conserved

(accidental symmetries)

III- Local or gauge internal symmetries ➔ color, electric charge conserved

$$SU(3)_c \times SU(2)_L \times U(1)_Y$$

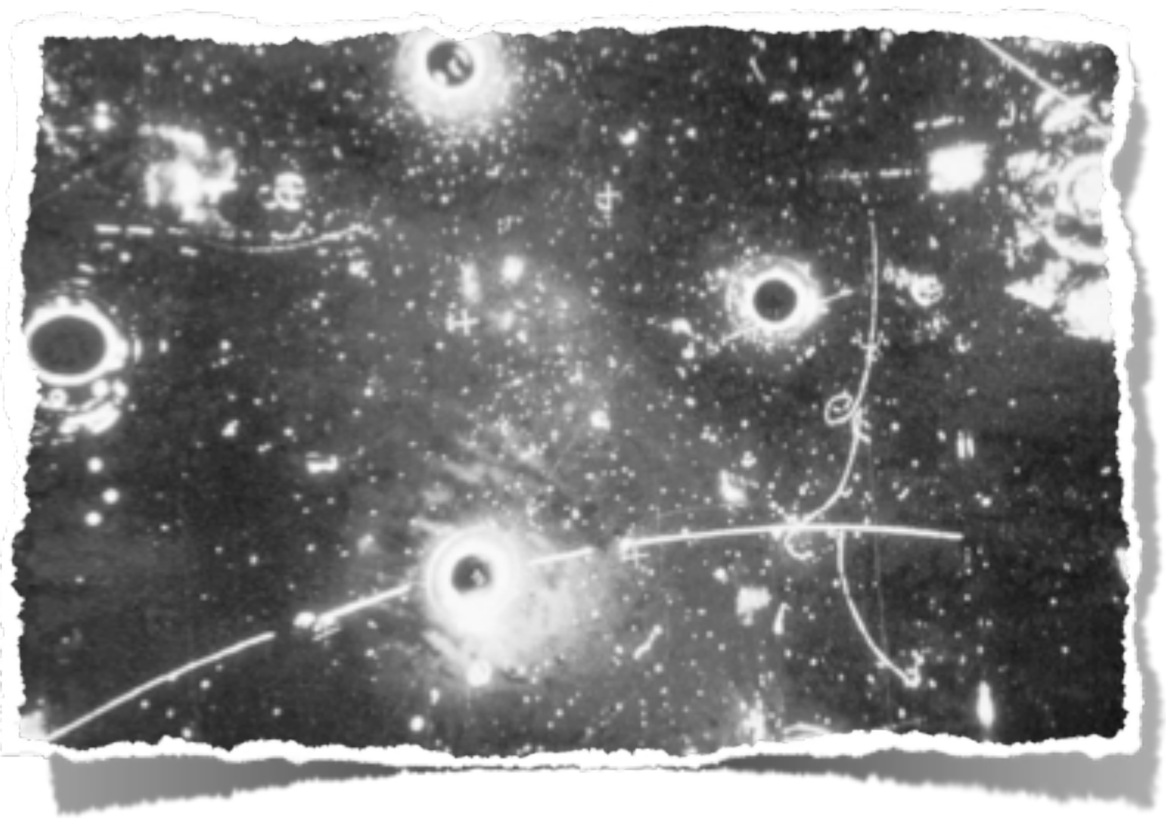
IV- Discrete symmetries ➔ CPT

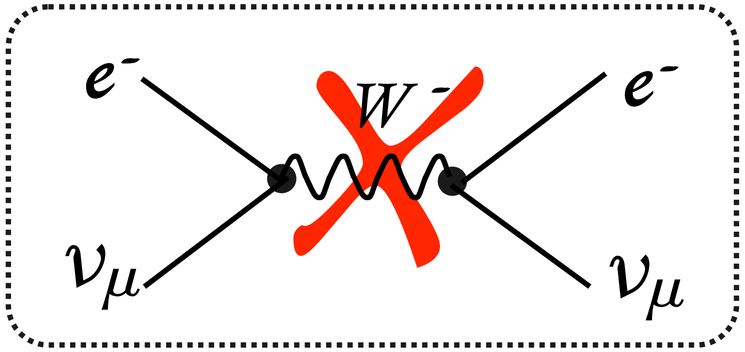
The Standard Model

the strong, weak and electromagnetic interactions of the elementary particles are described by gauge interactions

$SU(3)_C \times SU(2)_L \times U(1)_Y$

$\nu_\mu\, e^- \to \nu_\mu\, e^-$





[Gargamelle collaboration, ’73]

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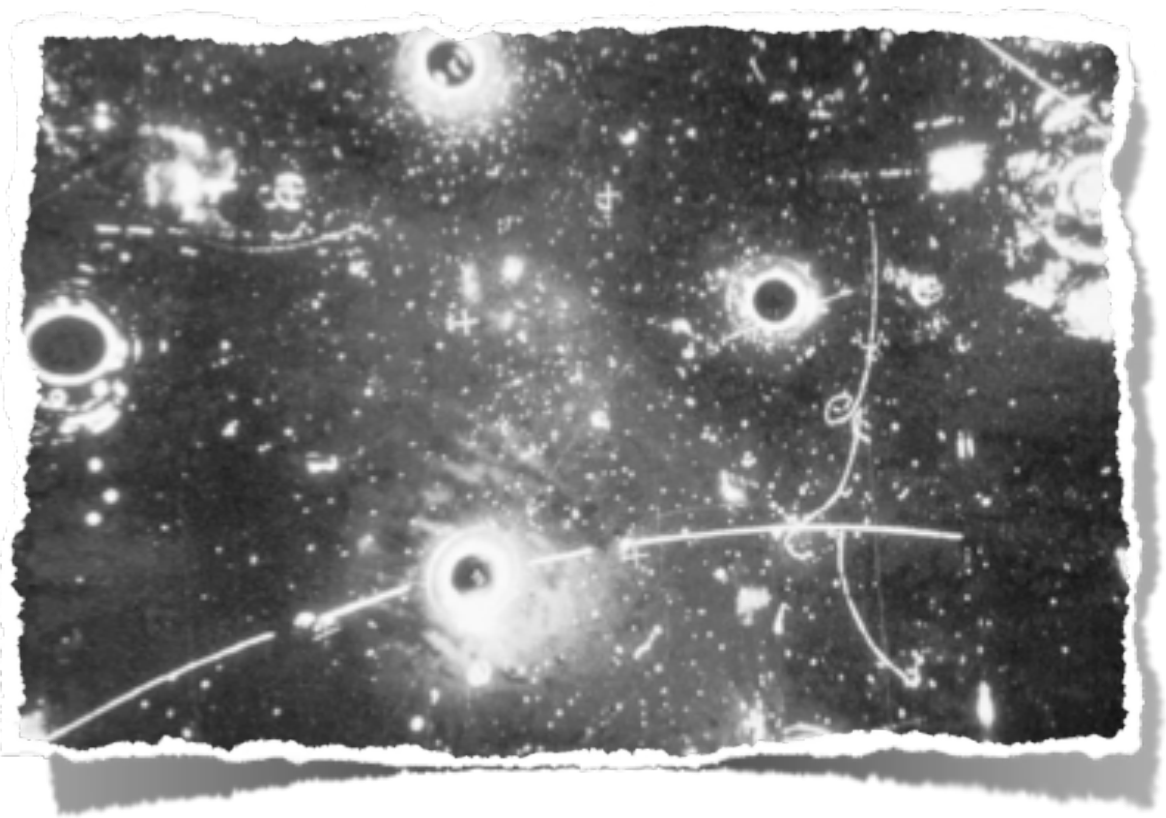

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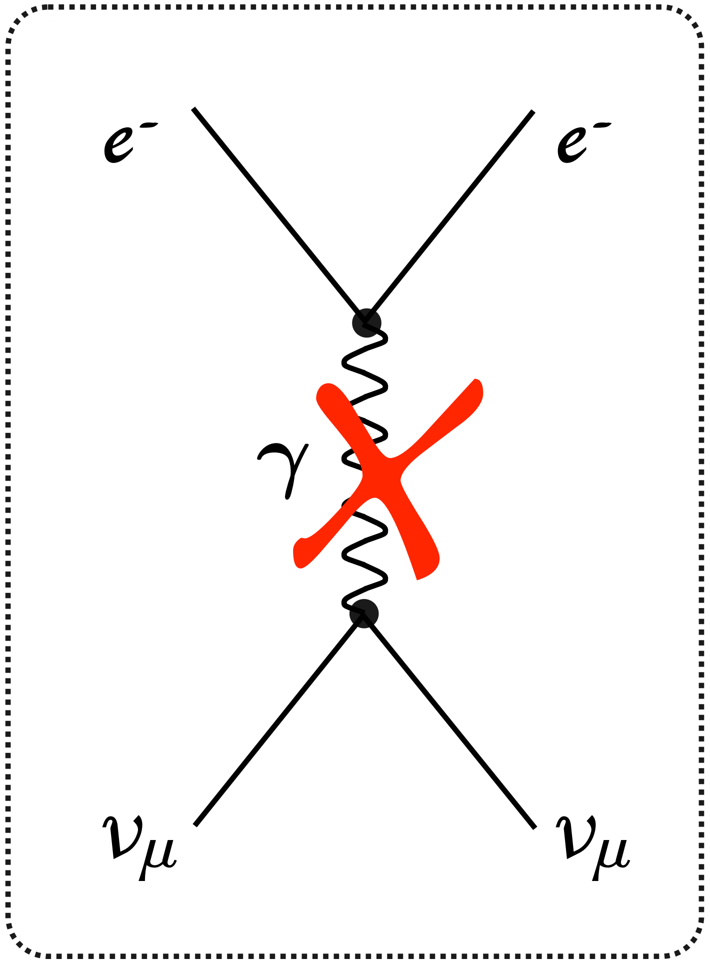
$\nu_\mu e^- \to \nu_\mu e^-$



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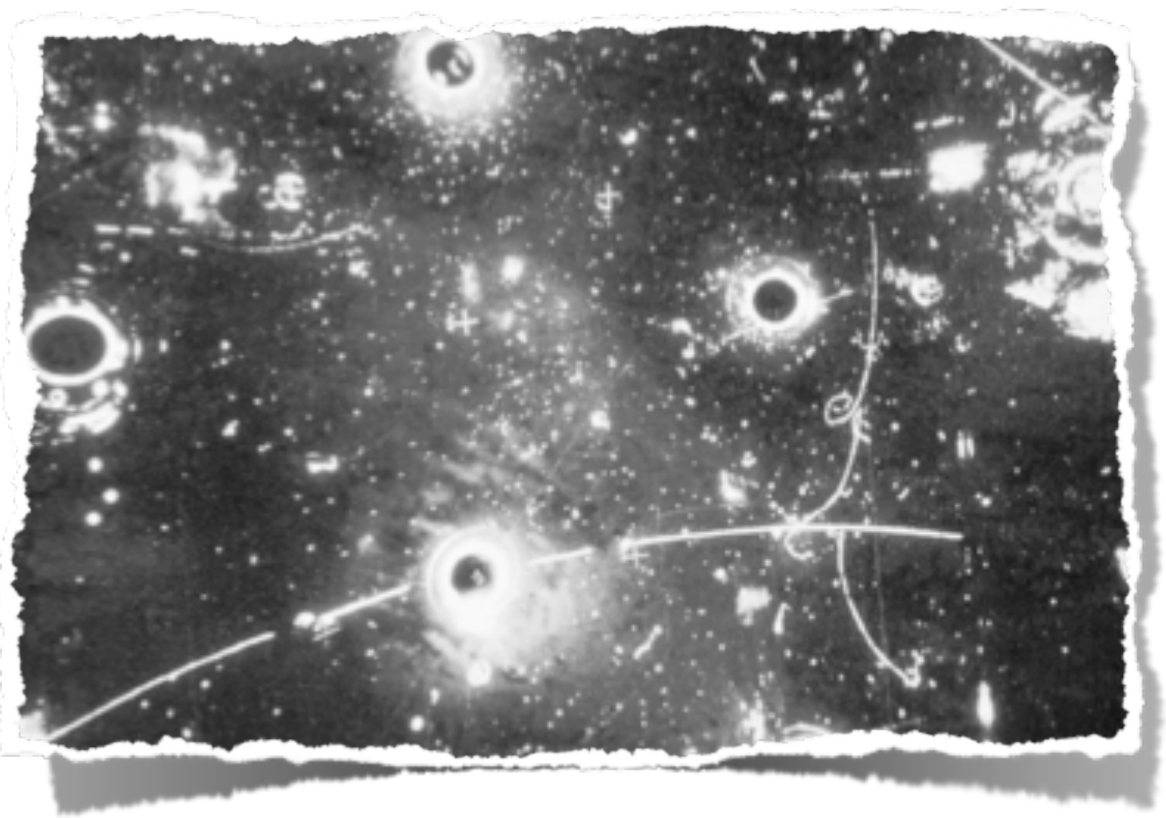

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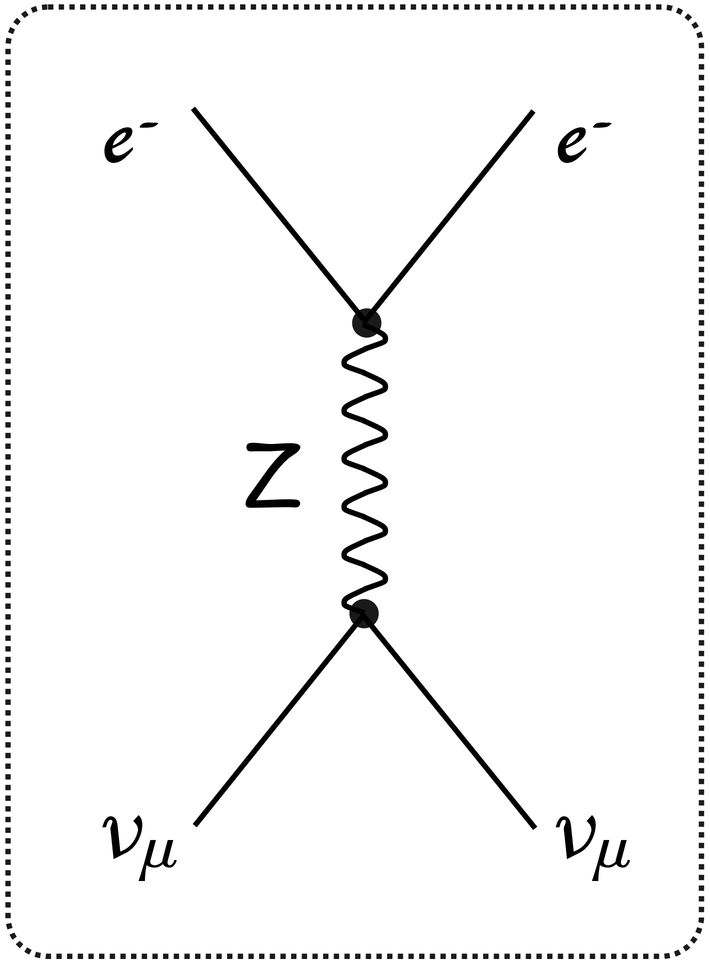
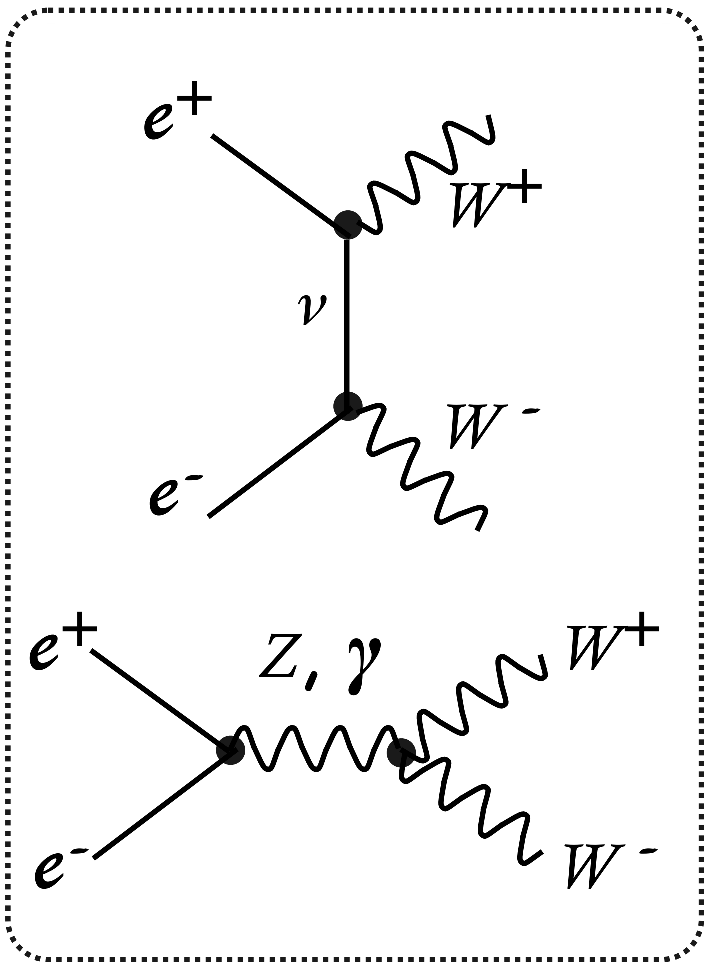


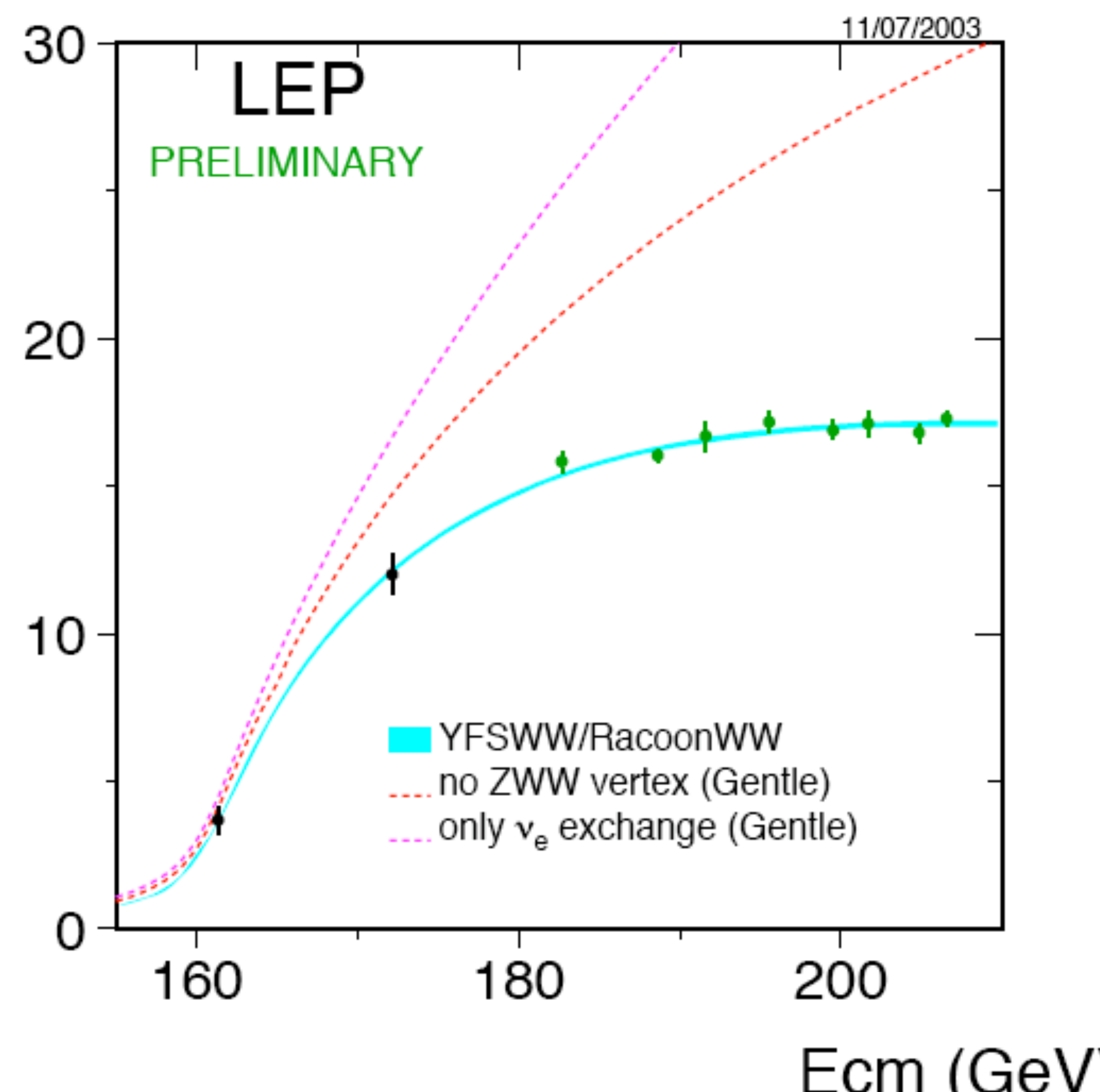
Gauge Theory as a Dynamical Principle

the strong, weak and electromagnetic interactions of the elementary particles are described by gauge interactions

$$SU(3)_C \times SU(2)_L \times U(1)_Y$$

$e^+e^- \to W^+W^-$





The Standard Model and the Mass Problem

the strong, weak and electromagnetic interactions of the elementary particles are described by gauge interactions

$SU(3)_C \times SU(2)_L \times U(1)_Y$

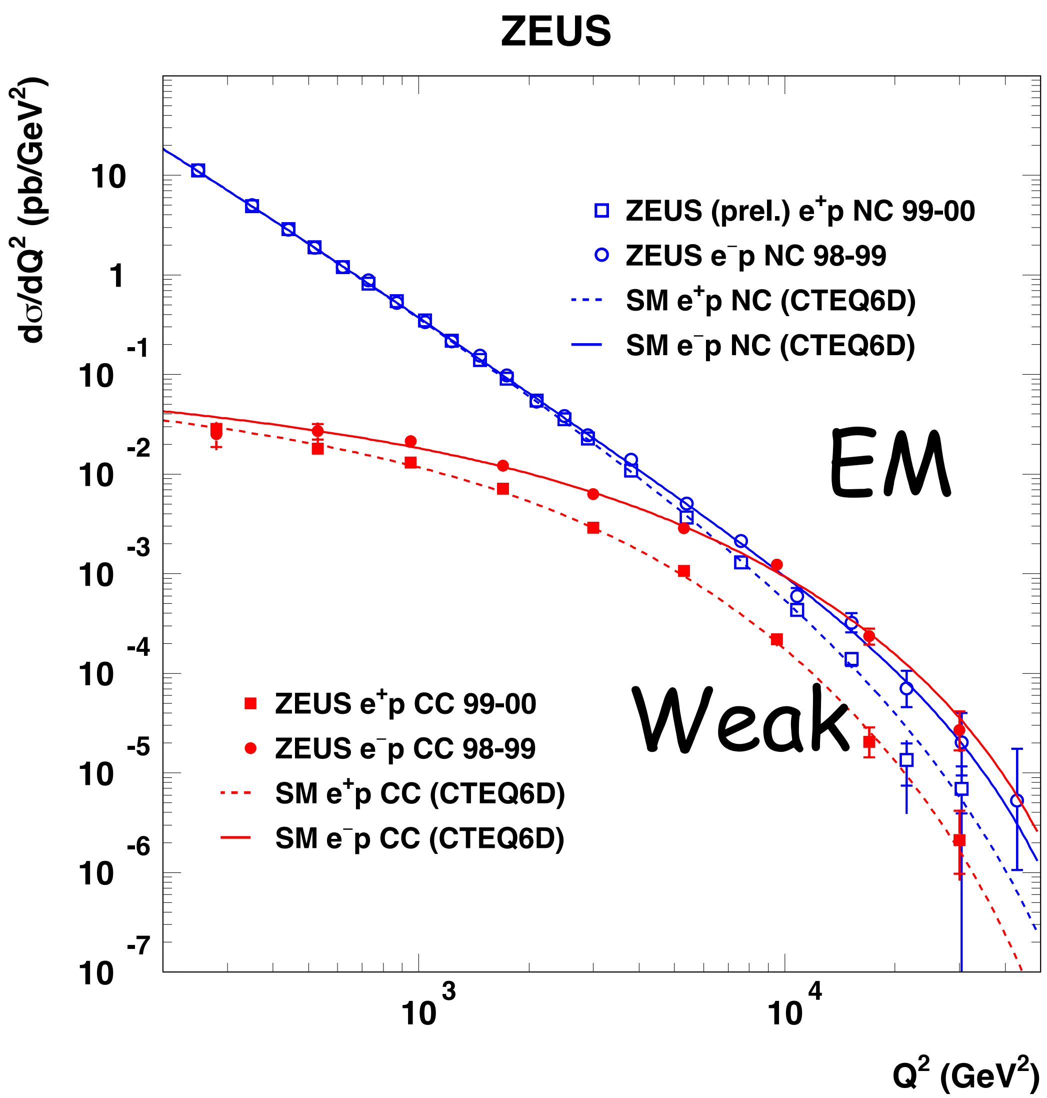
the masses of the quarks, leptons and gauge bosons don't obey the full gauge invariance

- $\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$ is a doublet of $SU(2)_L$ but $m_{\nu_e} \ll m_e$
- a mass term for the gauge field isn't invariant under gauge transformation $\delta A^a_\mu = \partial_\mu \epsilon^a + g f^{abc} A^b_\mu \epsilon^c$

spontaneous breaking of gauge symmetry

Electroweak Unification

High energy (~ 100 GeV)



Low energy

This room is full of photons
but no W/Z

The symmetry between W, Z and γ
is broken at large distances

EM forces ≈ long ranges

Weak forces ≈ short range

$$m_\gamma < 6 \times 10^{-17} \text{ eV}$$
$$m_{W^\pm} = 80.425 \pm 0.038 \text{ GeV}$$
$$m_{Z^0} = 91.1876 \pm 0.0021 \text{ GeV}$$

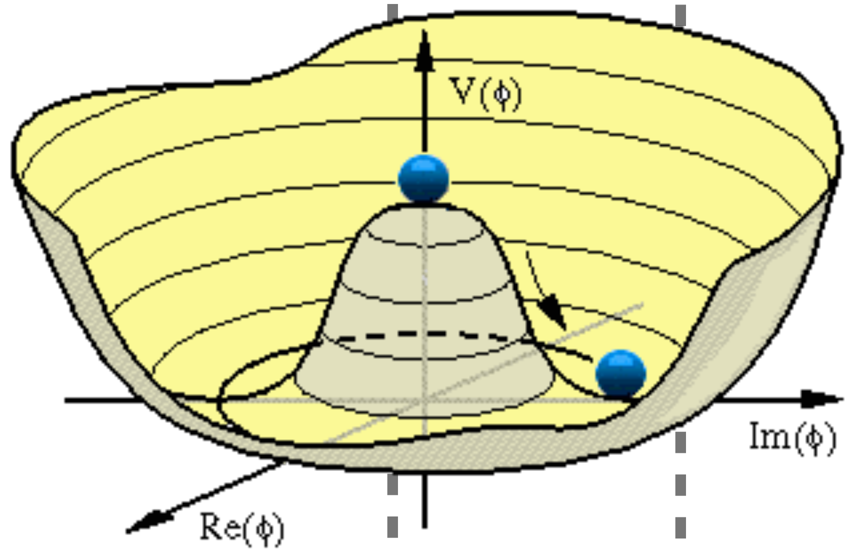
Higgs Mechanism

Symmetry of the Lagrangian

$$SU(2)_L \times U(1)_Y$$

Higgs Doublet

$$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$$



Symmetry of the Vacuum

$$U(1)_{e.m.}$$

Vacuum Expectation Value

$$\langle H \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \quad \text{with} \quad v \approx 246 \text{ GeV}$$

$$D_\mu H = \partial_\mu H - \frac{i}{2} \begin{pmatrix} g W^3_\mu + g' B_\mu & \sqrt{2} g W^+_\mu \\ \sqrt{2} g W^-_\mu & -g W^3_\mu + g' B_\mu \end{pmatrix} H \quad \text{with} \quad \mathrm{W}^\pm_\mu = \frac{1}{\sqrt{2}} \left(\mathrm{W}^1_\mu \mp \mathrm{W}^2_\mu \right)$$

$$|D_\mu H|^2 = \frac{1}{4} g^2 v^2 W^+_\mu W^{-\,\mu} + \frac{1}{8} \left(W^3_\mu \; B_\mu \right) \begin{pmatrix} g^2 v^2 & -g g' v^2 \\ -g g' v^2 & g'^2 v^2 \end{pmatrix} \begin{pmatrix} W^{3\,\mu} \\ B^\mu \end{pmatrix}$$

- Gauge boson spectrum
  - electrically charged bosons $\quad M_W^2 = \frac{1}{4} g^2 v^2$
  - electrically neutral bosons $\quad Z_\mu = c W^3_\mu - s B_\mu$, $\quad \gamma_\mu = s W^3_\mu + c B_\mu$

Weak mixing angle

$$c = \frac{g}{\sqrt{g^2+g'^2}} \qquad s = \frac{g'}{\sqrt{g^2+g'^2}}$$

$$M_Z^2 = \frac{1}{4}(g^2 + g'^2) v^2$$

$$M_\gamma = 0$$

Interactions Fermions-Gauge Bosons

Gauge invariance says:

$$\mathcal{L} = gW^3_\mu \left(\sum_i T_{3L\,i}\ \bar\psi_i \bar\sigma^\mu \psi_i \right) + g' B_\mu \left(\sum_i y_i\ \bar\psi_i \bar\sigma^\mu \psi_i \right)$$

Going to the mass eigenstate basis:

$$Z_\mu = cW^3_\mu - sB_\mu \qquad \qquad c = \frac{g}{\sqrt{g^2+g'^2}}$$

with

$$\gamma_\mu = sW^3_\mu + cB_\mu \qquad \qquad s = \frac{g'}{\sqrt{g^2+g'^2}}$$

$$Q = T_{3L} + Y$$

$$\boxed{\mathcal{L} = \sqrt{g^2+g'^2} Z_\mu \left(\sum_i (T_{3L\,i} - s^2 Q_i)\ \bar\psi_i \bar\sigma^\mu \psi_i \right) + \frac{gg'}{\sqrt{g^2+g'^2}} \gamma_\mu \left(\sum_i Q_i\ \bar\psi_i \bar\sigma^\mu \psi_i \right)}$$

not protected by gauge invariance
corrected by radiative corrections + new physics

protected by $U(1)_{em}$ gauge invariance
$\Rightarrow$ no correction

electric charge

$$e = \frac{gg'}{\sqrt{g^2+g'^2}} = sg = cg'$$

Custodial Symmetry

- Rho parameter

$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2\theta_w} = \frac{\frac{1}{4}g^2v^2}{\frac{1}{4}(g^2+g'^2)v^2\frac{g^2}{g^2+g'^2}} = 1$$

- Consequence of an approximate global symmetry of the Higgs sector

$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$ Higgs doublet = 4 real scalar fields

$V(H) = \lambda\left(H^\dagger H - \frac{v^2}{2}\right)^2$ is invariant under the rotation of the four real components

$$SO(4) \sim SU(2)_L \times SU(2)_R$$

$SU(2)_R$

$SU(2)_L$ $\left(i\sigma^2 H^\star \quad H\right) = \Phi$

2x2 matrix

$$\Phi^\dagger\Phi = H^\dagger H \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

$$V(H) = \frac{\lambda}{4}\left(\mathrm{tr}\Phi^\dagger\Phi - \mathrm{v}^2\right)^2$$

explicitly invariant under $SU(2)_L \times SU(2)_R$

Custodial Symmetry

Higgs vev

$$\langle H \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \qquad \langle \Phi \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

$$SU(2)_L \times SU(2)_R \to SU(2)_V$$

unbroken symmetry in the broken phase

$$(W^1_\mu, W^2_\mu, W^3_\mu) \quad \text{transforms as a triplet}$$

$$(Z_\mu \, \gamma_\mu) \begin{pmatrix} M_Z^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z^\mu \\ \gamma^\mu \end{pmatrix} = (W^3_\mu \, B_\mu) \begin{pmatrix} c^2 M_Z^2 & -csM_Z^2 \\ -csM_Z^2 & s^2 M_Z^2 \end{pmatrix} \begin{pmatrix} W^{3\,\mu} \\ B^\mu \end{pmatrix}$$

The $SU(2)_V$ symmetry imposes the same mass term for all W^i thus $c^2 M_Z^2 = M_W^2$

$\rho = 1$

The hypercharge gauge coupling and the Yukawa couplings break the custodial $SU(2)_V$, which will generate a (small) deviation to $\rho = 1$ at the quantum level.

Fermion Masses

SM is a chiral theory (≠ QED that is vector-like)

$m_e \bar{e}_L e_R + h.c.$ is not gauge invariant

The SM Lagrangian doesn't not contain fermion mass terms
fermion masses are emergent quantities
that originate from interactions with Higgs vev

$$y_{ij} \bar{f}_{L_i} H f_{R_j} = \frac{y_{ij} v}{\sqrt{2}} \bar{f}_{L_i} f_{R_j} + \frac{y_{ij}}{\sqrt{2}} h \bar{f}_{L_i} f_{R_j}$$

Fermion Masses

In SM, the Yukawa interactions are the only source of the fermion masses

$$y_{ij}\bar{f}_{L_i}Hf_{R_j} = \frac{y_{ij}v}{\sqrt{2}}\bar{f}_{L_i}f_{R_j} + \frac{y_{ij}}{\sqrt{2}}h\bar{f}_{L_i}f_{R_j}$$

mass

higgs-fermion interactions

both matrices are simultaneously diagonalizable

⇓ ⇓ ⇓

no tree-level Flavor Changing Current induced by the Higgs

Not true anymore if the SM fermions mix with vector-like partners[(*)] or for non-SM Yukawa

$$y_{ij}\left(1+c_{ij}\frac{|H|^2}{f^2}\right)\bar{f}_{L_i}Hf_{R_j} = \frac{y_{ij}v}{\sqrt{2}}\left(1+c_{ij}\frac{v^2}{2f^2}\right)\bar{f}_{L_i}f_{R_j} + \left(1+3c_{ij}\frac{v^2}{2f^2}\right)\frac{y_{ij}}{\sqrt{2}}h\bar{f}_{L_i}f_{R_j}$$

Look for SM forbidden Flavor Violating decays $h \to \mu\tau$ and $h \to e\tau$

(look also at $t \to hc$ ATLAS '14)

- weak indirect constrained by flavor data ($\mu \to e\gamma$): BR<10%
- ATLAS and CMS have the sensitivity to set bounds O(1%)
- ILC/CLIC/FCC-ee can certainly do much better

Blankenburg, Ellis, Isidori '12

Harnik et al '12

Davidson, Verdier '12

CMS-PAS-HIG-2014-005

(*) e.g. Buras, Grojean, Pokorski, Ziegler '11

Fermion Masses

In SM, the Yukawa interactions are the only source of the fermion masses

$$y_{ij}\bar{f}_{L_i} H f_{R_j} = \frac{y_{ij}v}{\sqrt{2}}\bar{f}_{L_i} f_{R_j} + \frac{y_{ij}}{\sqrt{2}} h \bar{f}_{L_i} f_{R_j}$$

mass

higgs-fermion interactions

both matrices are simultaneously diagonalizable

$\Downarrow$ $\Downarrow$ $\Downarrow$

no tree-level Flavor Changing Current induced by the Higgs

Quark mixings

$$\mathcal{L}_{Yuk} = \lambda^L_{ij}(\bar{L}^i_L \phi^c) l^j_R + \lambda^U_{ij}(\bar{Q}^i_{L,\alpha}\phi) u^j_{R,\alpha} + \lambda^D_{ij}(\bar{Q}^i_{L,\alpha}\phi^c) d^j_{R,\alpha} + cc$$

$$\mathcal{L}_L^\dagger \left(\frac{v}{\sqrt{2}}\lambda^L\right)\mathcal{L}_R = \begin{pmatrix} m_e & & \\ & m_\mu & \\ & & m_\tau \end{pmatrix}$$

$$\mathcal{U}_L^\dagger \left(\frac{-v}{\sqrt{2}}\lambda^U\right)\mathcal{U}_R = \begin{pmatrix} m_u & & \\ & m_c & \\ & & m_t \end{pmatrix}$$

$$\mathcal{D}_L^\dagger \left(\frac{v}{\sqrt{2}}\lambda^D\right)\mathcal{D}_R = \begin{pmatrix} m_d & & \\ & m_s & \\ & & m_b \end{pmatrix}$$

$$\begin{aligned} \mathcal{L}_{Yuk\,quad} = & - \left(\bar{e}_L, \bar{\mu}_L, \bar{\tau}_L\right) \begin{pmatrix} m_e & & \\ & m_\mu & \\ & & m_\tau \end{pmatrix} \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} \\ & - \left(\bar{u}_{L,\alpha}, \bar{c}_{L,\alpha}, \bar{t}_{L,\alpha}\right) \begin{pmatrix} m_u & & \\ & m_c & \\ & & m_t \end{pmatrix} \begin{pmatrix} u_{R,\alpha} \\ c_{R,\alpha} \\ t_{R,\alpha} \end{pmatrix} \\ & - \left(\bar{d}_{L,\alpha}, \bar{s}_{L,\alpha}, \bar{b}_{L,\alpha}\right) \mathcal{V}_{KM}^\dagger \begin{pmatrix} m_d & & \\ & m_s & \\ & & m_b \end{pmatrix} \begin{pmatrix} d_{R,\alpha} \\ s_{R,\alpha} \\ b_{R,\alpha} \end{pmatrix} \\ & + \; cc \end{aligned}$$

$$\mathcal{V}_{KM} = \mathcal{D}_L^\dagger \mathcal{U}_L$$

Goldstone Theorem

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Goldstone's theorem [edit]

Goldstone's theorem examines a generic continuous symmetry which is spontaneously broken; i.e., its currents are conserved, but the ground state is not invariant under the action of the corresponding charges. Then, necessarily, new massless (or light, if the symmetry is not exact) scalar particles appear in the spectrum of possible excitations. There is one scalar particle—called a Nambu–Goldstone boson—for each generator of the symmetry that is broken, i.e., that does not preserve the ground state. The Nambu–Goldstone mode is a long-wavelength fluctuation of the corresponding order parameter.

By virtue of their special properties in coupling to the vacuum of the respective symmetry-broken theory, vanishing momentum ("soft") Goldstone bosons involved in field-theoretic amplitudes make such amplitudes vanish ("Adler zeros").

In theories with gauge symmetry, the Goldstone bosons are "eaten" by the gauge bosons. The latter become massive and their new, longitudinal polarization is provided by the Goldstone boson.

QCD example:

For two light quarks, *u* and *d*, the symmetry of the QCD Lagrangian called *chiral symmetry*, and denoted as $U(2)_L \times U(2)_R$, can be decomposed into

$$SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A \,.$$

The quark condensate spontaneously breaks the $SU(2)_L \times SU(2)_R$ down to the diagonal vector subgroup $SU(2)_V$, known as isospin. The resulting effective theory of baryon bound states of QCD (which describes protons and neutrons), then, has mass terms for these, disallowed by the original linear realization of the chiral symmetry, but allowed by the nonlinear (spontaneously broken) realization thus achieved as a result of the strong interactions.[4]

The Nambu-Goldstone bosons corresponding to the three broken generators are the three pions, charged and neutral. More precisely, because of small quark masses which make this chiral symmetry only approximate, the pions are **Pseudo-Goldstone bosons** instead, with a nonzero, but still atypically small mass,[5] $m_\pi \approx \sqrt{v\, m_q} / f_\pi$.

sic!

Goldstone Boson

$U(1)$

$$\phi \to e^{i\alpha}\phi$$

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - \lambda \left(\left|\phi^2\right|^2 - \frac{f^2}{2} \right)^2$$

$$\phi = \frac{1}{\sqrt{2}} \left(f + h(x)\right) e^{i\theta(x)/f}$$

$$h \to h$$

$$\theta \to \theta + \alpha f$$

$U(1)$ non-linearly realized

shift symmetry forbids any mass term for θ

$$\mathcal{L} = \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}\left(\frac{f+h}{f}\right)^2 \partial_\mu \theta \partial^\mu \theta - \lambda \left(f^2 h^2 + f h^3 + \frac{1}{4} h^4 \right)$$

θ remains a massless field

== Goldstone boson ==

To each continuous global symmetry spontaneously broken corresponds a massless field

If the U(1) symmetry is gauged, the Goldstone boson is eaten and it becomes the longitudinal component of the massive gauge boson

Example of Uneaten Goldstone Bosons

$SU(N) \to SU(N-1)$ $\quad (N^2-1) - ((N-1)^2-1) = 2N-1$ Goldstone bosons

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix} \qquad \phi = \exp\left(\frac{i}{f} \left(\begin{array}{ccc|c} -\pi_0 & & & \pi_1 \\ & \ddots & & \vdots \\ & & -\pi_0 & \pi_{N-1} \\ \hline \pi_1^\star & \dots & \pi_{N-1}^\star & (N-1)\pi_0 \end{array} \right) \right) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix}$$

$$\phi = e^{i\pi}\phi_0$$

(N-1) complex, $\vec{\pi}$, and 1 real, π_0, scalars

Let us assume that only SU(N-1) is gauged: then the Goldstone are uneaten.

$SU(N-1)$

$$\phi \to U_{N-1}\phi = U_{N-1}e^{i\pi}U_{N-1}^\dagger U_{N-1}\phi_0 = e^{iU_{N-1}\pi U_{N-1}^\dagger}\phi_0$$

$$\pi \to \left(\begin{array}{c|c} U_{N-1} & \\ \hline & 1 \end{array} \right) \left(\begin{array}{c|c} \pi_0 & \pi \\ \hline \pi^\dagger & \pi_0 \end{array} \right) \left(\begin{array}{c|c} U_{N-1}^\dagger & \\ \hline & 1 \end{array} \right) = \left(\begin{array}{c|c} \pi_0 & U_{N-1}\pi \\ \hline \pi^\dagger U_{N-1}^\dagger & \pi_0 \end{array} \right)$$

linear transformations

$\dfrac{SU(N)}{SU(N-1)}$

$$\phi \to \exp\left(i \left(\begin{array}{c|c} & \vec{\alpha} \\ \hline \vec{\alpha}^\dagger & \end{array} \right) \right) \exp\left(i \left(\begin{array}{c|c} & \vec{\pi} \\ \hline \vec{\pi}^\dagger & \end{array} \right) \right) \phi_0 \approx \exp\left(i \left(\begin{array}{c|c} & \vec{\pi}+\vec{\alpha} \\ \hline \vec{\pi}^\dagger+\vec{\alpha}^\dagger & \end{array} \right) \right) \phi_0$$

non-linear transformations

Appendix I

Lorentz transformations - Dirac equation

Lorentz transformations of scalars

Invariance of integration measure: $d^4x = cdt \cdot dx \cdot dy \cdot dz$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} ct' = \gamma_0\,(ct - \beta_0 x) \\ x' = \gamma_0\,(-\beta_0 ct + x) \\ y' = y \\ z' = z \end{pmatrix}$$

with

$$\beta_0 = \frac{v_0}{c}$$

$$\gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}}$$

$$d^4x' = \underbrace{\left|\frac{\partial x'}{\partial x}\right|}_{\left|\det\begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right| = \gamma^2(1-\beta^2) = 1} d^4x$$

Invariance of scalar kinetic term: $\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

By definition, a scalar remains invariant under Lorentz transformations: $\phi(x) \to \phi'(x') = \phi(x)$

$$\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \to \eta^{\mu\nu}\partial'_\mu\phi'\partial'_\nu\phi' = \eta^{\mu\nu}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\partial_\rho\phi\partial_\sigma\phi$$

$$\begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \gamma^2(1-\beta^2) & & & \\ & \gamma^2(\beta^2-1) & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

More on Lorentz transformations

Covariant form of a Lorentz transformation: $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$

The invariance of the line element: $\Delta^2 = \eta_{\mu\nu} x^{\mu} x^{\nu} \to \Delta'^2 = \eta_{\mu\nu} x'^{\mu} x'^{\nu}$ imposes the following condition

$$\boxed{\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}}$$

We always raise and lower the space time indices with the metric:

$$\Lambda_{\mu\nu} = \eta_{\mu\rho} \Lambda^{\rho}{}_{\nu} \qquad \Lambda_{\mu}{}^{\nu} = \eta_{\mu\rho} \eta^{\nu\sigma} \Lambda^{\rho}{}_{\sigma} \qquad \Lambda^{\mu\nu} = \eta^{\nu\sigma} \Lambda^{\mu}{}_{\sigma}$$

Transformation inverse: $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$ $\qquad x^{\mu} = \Lambda_{\nu}{}^{\mu}\, x'^{\nu}$

Transformation of the space-time derivatives:

$$\partial_{\mu} = \frac{\partial x'^{\nu}}{\partial x^{\mu}} \frac{\partial}{\partial x'^{\nu}} = \Lambda^{\nu}{}_{\mu} \partial'_{\nu}$$

$$\partial'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}} = \Lambda_{\mu}{}^{\nu} \partial_{\nu}$$

Small Lorentz transformations: $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}$

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma} \quad \Leftrightarrow \quad \omega_{\mu\nu} = \omega_{\nu\mu}$$

Lorentz transformations of vectors

Transformation law: $$A^\mu(x) \to A'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu(x) = \Lambda^\mu{}_\nu A^\nu(x)$$

The abelian gauge field strength then transforms as:

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu \to F'_{\mu\nu}(x') = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}$$

The commutator piece of the non-abelian field strength follows the same transformation law

Invariance of vector kinetic term:

$$= \eta^{\sigma\beta}$$

$$F_{\mu\nu}F^{\mu\nu} \to \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \Lambda^{\mu\alpha} \Lambda^{\nu\beta} F_{\rho\sigma} F_{\alpha\beta} = F_{\mu\nu} F^{\mu\nu}$$

$$= \eta^{\rho\alpha}$$

Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

For this equation to be consistent with Einstein equation ($m^2=E^2-p^2$),

the 4x4 γ matrices have to obey the Clifford algebra

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}}$$

Dirac representation of the γ matrices:

$$\gamma^0 = \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Chirality matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\left(\gamma^5\right)^2 = 1_4$$

Lorentz transformations of spinors

Transformation law: $\psi(x) \to \psi'(x') = S(\Lambda)\psi(x)$

We want the Dirac equation to take the same form in the two systems of coordinates x and x'

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \qquad\qquad (i\gamma^\mu \partial'_\mu - m)\psi' = 0$$

This implies the condition: $S\gamma^\mu \Lambda^\nu{}_\mu S^{-1} = \gamma^\nu$

We consider small Lorentz transformations: $\Lambda_\mu{}^\nu = \delta^\mu_\nu + \omega^\mu{}_\nu \qquad S = 1 - \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}$

The covariance of the Dirac equation then implies that the matrices $\sigma^{\mu\nu}$ have to satisfy the relation

$$[\gamma^\nu, \sigma^{\rho\sigma}] = 2i(\eta^{\nu\rho}\gamma^\sigma - \eta^{\nu\sigma}\gamma^\rho)$$

It is easy to check that the following matrices fit the bill: $\sigma^{\rho\sigma} = \frac{i}{2}[\gamma^\rho, \gamma^\sigma]$

Appendix II

Some notes on group theory

SU(N)

ϕ complex N-vector that transforms as $\phi \to \phi' = U\,\phi$

$$UU^\dagger = U^\dagger U = 1 \quad \text{and} \quad \det U = 1$$

Non-abelian action: in general $U_1 U_2 \neq U_2 U_1$.

Infinitesimal transformations: $U = e^{i\alpha^a T^a} \approx 1 + i\alpha^a T^a + \dots$ T^a are the generators of the group

$$T^{a\dagger} = T^a \qquad \mathrm{Tr}(T^a) = 0$$

N-1 independent real diagonal elements (rank N-1)

1/2 (N-1)N independent complex off-diagonal elements

$\Rightarrow$ N^2-1 generators

SU(2): the 3 Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

commutation relations

$$[\sigma^a, \sigma^b] = i\epsilon^{abc}\sigma^c$$

SU(3): the 8 Gell-Man matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Adjoint representation

Consider the N^2-1 generators in the fundamental representation: T^a

They satisfy some non-trivial commutation relations: $[T^a,T^b]=i f^{abc} T^c$

By definition of a commutator, the structure constants satisfy the Jacobi identity:

$$f^{abd}f^{cde} + f^{bcd}f^{ade} + f^{cad}f^{bde} = 0$$

We define N^2-1 matrices of size $(N^2-1)\mathrm{x}(N^2-1)$ by $(\mathcal{T}^a)_{bc}=-i f^{abc}$

The Jacobi identity ensures that these matrices satisfy the same commutation relation

$$[\mathcal{T}^a,\mathcal{T}^b]=i f^{abc} \mathcal{T}^c$$

They form an irreducible representation of SU(N), called the adjoint representation

We show that the product of a fundamental and an anti fundamental is the sum of the trivial representation and the adjoint representation

$$N \otimes \bar{N} = 1 \oplus (N^2-1)$$

SO(N)

ϕ real N-vector that transforms as $\phi \to \phi' = U\,\phi$

$$UU^t = U^tU = 1 \quad \text{and} \quad \det U = 1$$

Non-abelian action: in general $U_1U_2 \neq U_2U_1$.

Infinitesimal transformations: $U = e^{\theta^a T^a} \approx 1 + \theta^a T^a + \ldots$ T^a are the generators of the group

$$T^{a\,t} + T^a = 0 \qquad \mathrm{Tr}(T^a) = 0$$

1/2 (N-1)N independent real off-diagonal elements ➡ 1/2 (N-1)N generators

SO(3)

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad T^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad T^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

commutation relations $[T^a, T^b] = \epsilon^{abc} T^c$

SO(3)≈SU(2)

Symmetries and invariants

SU(N)

the transformations among the components of a complex N-vector that leaves its norm invariant

$$|\phi|^2 = \phi_1^*\phi_1 + \ldots \phi_N^*\phi_N \to |\phi'|^2 = |\phi|^2$$

SU(N,M)

the transformations among the components of a complex (N+M)-vector that leaves its (N,M) norm invariant

$$|\phi|^2 = \phi_1^*\phi_1 + \ldots \phi_N^*\phi_N + \phi_{N+1}^*\phi_{N+1} - \ldots - \phi_{N+M}^*\phi_{N+M} \to |\phi'|^2 = |\phi|^2$$

SO(N)

the transformations among the components of a real N-vector that leaves its norm invariant

$$|\phi|^2 = \phi_1^2 + \ldots \phi_N^2 \to |\phi'|^2 = |\phi|^2$$

SO(N,M)

the transformations among the components of a real (N+M)-vector that leaves its (N,M) norm invariant

$$|\phi|^2 = \phi_1^2 + \ldots \phi_N^2 + \phi_{N+1}^2 - \ldots - \phi_{N+M}^2 \to |\phi'|^2 = |\phi|^2$$

The Lorentz group is thus SO(1,3)

Lorentz transformation

SO(1,3)

The elements of SO(1,3) satisfy $U^t\,\eta\,U = \eta$ where =diag(1,-1,-,1,-1)

The infinitesimal transformations are $U = e^{\theta^a T^a} \approx 1 + \theta^a T^a + \ldots$

The generators satisfy the constraints: ${T^a}^t\eta + \eta T^a = 0$

One particular generator is $T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

We obtain $e^{\theta T} = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

We indeed recover the Lorentz transformation with the identification

$$\gamma = \cosh\theta \quad \text{and} \quad \beta\gamma = \sinh\theta$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \Leftrightarrow \quad \cosh^2\theta - \sinh^2\theta = 1$$