All-loop singularities of massless planar amplitudes

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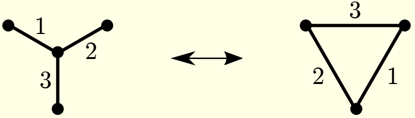
Based on work with Igor Prlina, Stefan Stanojevic, Tristan Dennen, James Stankowicz, and Anastasia Volovich

Five-Minute Version

Singularities of scattering amplitudes in field theory are encoded in the Landau equations.

In massless theories, the set of solutions of the Landau equations for a graph $\mathscr{G}$ is invariant under the graph operations familiar from the theory of electrical circuits, in particular the Y-Δ transformation.



The mathematical problem of classifying planar graphs modulo Y-Δ has a known simple solution.

Therefore we have a handle on all possible (first-type) Landau singularities in any massless, planar field theory.

Interesting implications for planar super-Yang-Mills (SYM) theory...

Introduction and Motivation

A goal of the analytic S-matrix program is to be able to determine scattering amplitudes in quantum field theory based on a few physical principles and a thorough knowledge of their analytic structure.

Inspired by a comment by Maldacena, Simmons-Duffin, Zhiboedov [1509.03612], we have written a series of papers exploring how much can be said, in general, about the locations of singularities of perturbative amplitudes in planar super-Yang-Mills (SYM) theory.

Introduction and Motivation

This talk (mostly) excludes the "trivial" (that means, understood) singularities:

- infrared and collinear singularities, which are completely understood (in any massless gauge theory), to all loop order, based on exponential resummation; and
- poles, which in any massless planar theory can only occur when a sum of cyclically adjacent momenta goes on shell:

$$s_{i+1,j} \equiv (p_{i+1} + \cdots + p_j)^2 = 0$$

or, in momentum twistor language:

$$\langle i\, i{+}1\, j\, j{+}1 \rangle = 0$$

For us the interesting singularities are branch points (really, branch surfaces of codimension one).

Setting the Stage: Branch Surfaces

Branch points can manifest themselves in various different ways. For example, the simplest and best-understood class of amplitudes can be written as linear combinations

$$A = \sum_i R_i P_i$$

where the R_i and P_i are respectively algebraic and polylogarithmic functions of the external momenta describing the scattering.

- The P_i have symbols which encode information about their analytic structure; specifically, the presence of a symbol letter a indicates the presence of a possible logarithmic branch cut between $a = 0$ and $a = \infty$.
- The presence of a symbol letter of the form $a = b + \sqrt{c}$ (for example), indicates the presence of a possible algebraic branch point at $c = 0$, even if $a \notin \{0, \infty\}$ there.
- The R_i can have poles and/or algebraic branch points.

Setting the Stage: Branch Surfaces

Branch points can manifest themselves in various different ways. For example, the simplest and best-understood class of amplitudes can be written as linear combinations

$$A = \sum_i R_i P_i$$

where the R_i and P_i are respectively algebraic and polylogarithmic functions of the momenta of the scattering particles.

I wrote the above formula for motivational purposes only; do not let it prejudice you!

Our results are completely general, and not restricted to the very special class of amplitudes that can be represented as above.

General amplitudes involve more exotic classes of functions that may exhibit their branch surfaces in different ways.

Why Care? One Reason is the Bootstrap

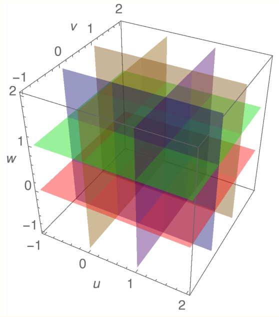
The current state-of-the art for computing multi-loop amplitudes in SYM theory is a bootstrap program that relies on the hypothesis — supported by all available evidence currently available — that all six- (seven-) particle amplitudes can be written in a particular hexagon (heptagon) symbol alphabet containing just nine (forty-two) letters.

[Dixon, Drummond, Duhr, Henn, McLeod, Papathanasiou, von Hippel, MS]

Why Care? One Reason is the Bootstrap

Baked into this ansatz is an even more fundamental hypothesis that six- (and seven-point) amplitudes, to all orders in perturbation theory, can only have branch surfaces on certain particular loci in the space of massless six- (and seven-) particle kinematics.

branch surfaces:



(in coordinates u, v, w defined later)

singularity locus:

$$\mathscr{S}_6 = \{s_{12}=0\} \cup \{s_{23}=0\} \cup \{s_{34}=0\}$$
$$\cup \{s_{45}=0\} \cup \{s_{56}=0\} \cup \{s_{61}=0\}$$
$$\cup \{s_{123}=0\} \cup \{s_{234}=0\} \cup \{s_{345}=0\}$$
$$\cup \{s_{12}s_{45}=s_{123}s_{345}\}$$
$$\cup \{s_{23}s_{56}=s_{234}s_{123}\}$$
$$\cup \{s_{34}s_{61}=s_{345}s_{234}\}$$

where $s_{ij\ldots} \equiv (p_i + p_j + \cdots)^2$

Introduction and Motivation

The key questions that have motivated us since the beginning of our work in this subject include

Can we prove that the hexagon and heptagon symbol alphabets are correct?

Can we determine the symbol alphabets of higher-point amplitudes, with the hope of being able to bootstrap them as well?

Feynman Diagrams

At L-loop order in perturbation theory, the integrand for any n-particle amplitude in any local quantum field theory can be written as a rational function of the n external momenta p_i and L internal momenta ℓ_r.

This rational function can be represented as a sum over Feynman diagrams.

Numerators and Denominators

Consider a term corresponding to any single Feynman diagram.

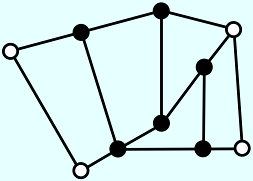
The denominator is the boring part; the propagator structure is entirely determined by the topology of the diagram, and is common to any field theory.

All of the cool stuff, like remarkable cancellations that happen in our favorite quantum field theories, arises due to the delicate structure of interplay between the numerator factors assigned to different Feynman diagrams.

This talk will be about the boring part.

Landau Graphs

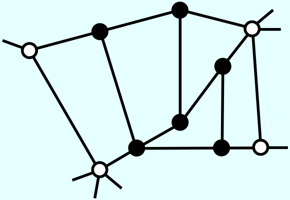
We denote a set of (scalar) propagators of a given topology by a Landau graph.



- To each edge j we assign a four-momentum q_j...
- and a Feynman parameter α_j.
- We impose momentum conservation at each vertex...
- except at certain privileged vertices called terminals.

Landau Graphs

We denote a set of (scalar) propagators of a given topology by a Landau graph.



The terminals are where external edges will be attached to indicate the momentum carried by incoming/outgoing particles.

These are not (yet!) on-shell diagrams; I'm not summing anything and have no specific theory in mind.

Landau graph = depiction of a set of propagators

The Landau Equations

Landau (1959) showed that in quantum field theory, a loop integral with the topology of some given graph $\mathscr{G}$ can have singularities when the external momenta p_i are such that the Landau equations

$$\alpha_j q_j^2 = 0 \text{ for each edge } j\text{, and}$$

$$\sum_{\text{edges } j \in \mathscr{L}} \alpha_j q_j = 0 \text{ for each closed loop } \mathscr{L}$$

admit solutions $\{\alpha_j, q_j\}$. Such singularities are called Landau singularities of the first type.

⚠ Other singularities (second type) also exist in certain theories.

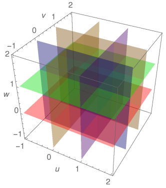
⚠ The Landau equations only know about the scalar propagator skeleton of diagram, so they expose all potential singularities of any generic, local field theory. However, in special theories, some of those potential singularities may be absent due to specially crafted numerators and/or nontrivial cancellation between different graphs.

⚠ Aside: Landau versus Leading Singularities

In our field, the term leading singularity has come to mean a location, in the space of loop integration variables, where an L-loop integrand has a pole of order $4L$; or, more commonly, to a residue of an integrand at such a pole.

By Landau singularity we mean a locus in the space of external kinematics on which an integrated amplitude has a singularity.

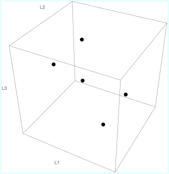
external kinematic space:



Landau singularities = codimension-one branch surfaces

$$\xleftarrow{\int d^{4L} p}$$

loop momentum space:



leading singularities = maximal poles

⚠ Aside: Landau versus Leading Singularities

For example, MHV amplitudes in SYM theory:

are well-known to have no new types of leading singularities beyond tree-level,

but we expect that they have new types of Landau singularities at each loop order

1. tree-level: poles at $\langle i\, i{+}1\, j\, j{+}1 \rangle = 0$
2. one-loop: branch surfaces at $\langle i\, j{-}1\, j\, j{+}1 \rangle = 0$
3. two-loop: branch surfaces at $\langle i(i{-}1\, i{+}1)(j\, j{+}1)(k\, k{+}1) \rangle = 0$ (known from [Caron-Huot 1105.5605])
4. three-loop and higher: not yet explicitly known (beyond the special, degenerate cases $n = 6, 7$)!

The Landau Equations

$$\alpha_j q_j^2 = 0 \text{ for each edge } j, \text{ and}$$

$$\sum_{\text{edges } j \in \mathscr{L}} \alpha_j q_j = 0 \text{ for each closed loop } \mathscr{L}$$

The Landau equations typically admit many branches of solutions.

- We ignore the trivial solution where all $\alpha_j = 0$.
- We also ignore solutions $\{\alpha_j, q_j\}$ arising from soft/collinear regions of loop integration space; these exist for generic P_i.
- Unlike the above cases, we are interested in solutions $\{\alpha_j, q_j\}$ that exist only on codimension-one surfaces in the space P_i; these are potential locations of branch surfaces of amplitudes.

The Landau Equations

⚠ Note that we don't care about the values of the α's.

Solutions that exist for non-negative real values of α (when the external kinematics are real-valued in Minkowski spacetime) are associated with branch singularities on the physical sheet [Coleman-Norton theorem].

We are interested in all possible solutions, even if the corresponding α's are complex; these indicate the presence of singularities that can only be accessed after suitable analytic continuation to some higher sheet.

The Landau Equations and Electrical Circuits

$$\alpha_j q_j^2 = 0 \text{ for each edge } j, \text{ and}$$

$$\sum_{\text{edges } j \in \mathscr{L}} \alpha_j q_j = 0 \text{ for each closed loop } \mathscr{L}$$

The analogy between Feynman diagrams and electrical circuits has long been appreciated; see for example chapter 18 of Bjorken & Drell (1965).

In this analogy the q_j are currents, the α_j are resistances, and the second line above expresses the Kirchhoff rule.

The analogue of the on-shell condition shown in the first line is rather mysterious.

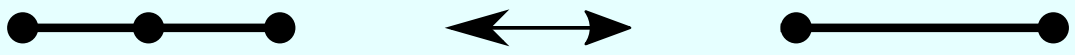
However, a remarkable feature of massless theories is that:

The graph moves familiar from circuit theory preserve the set of first-type Landau singularities in any massless field theory.

The Landau Equations: Series Reduction

$$\alpha_j q_j^2 = 0 \text{ for each edge } j\text{, and}$$

$$\sum_{\text{edges } j \in \mathscr{L}} \alpha_j q_j = 0 \text{ for each closed loop } \mathscr{L}$$

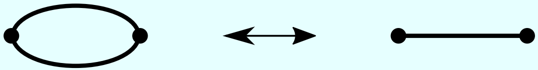


Replace two edges q_1, α_1 and q_2, α_2 in series (hence, $q_1 = q_2$) by a single edge with

$$q \equiv q_1 = q_2\,, \qquad \alpha \equiv \alpha_1 + \alpha_2$$

The Landau Equations: Bubble Reduction

$$\alpha_j q_j^2 = 0 \text{ for each edge } j, \text{ and}$$
$$\sum_{\text{edges } j \in \mathscr{L}} \alpha_j q_j = 0 \text{ for each closed loop } \mathscr{L}$$



The "loop momentum" in the bubble is determined by its Kirchhoff rule.

Consistency of the on-shell conditions follows from momentum conservation at each vertex and the assignment

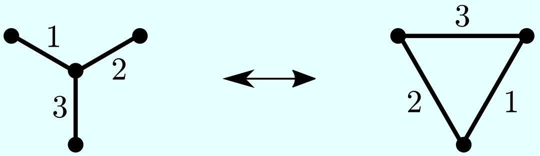
$$\alpha \equiv \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}$$

familiar from electrical circuits.

The Landau Equations: Y-Δ Reduction

$$\alpha_j q_j^2 = 0 \text{ for each edge } j, \text{ and}$$

$$\sum_{\text{edges } j \in \mathscr{L}} \alpha_j q_j = 0 \text{ for each closed loop } \mathscr{L}$$



The "loop momentum" in the triangle is determined by its Kirchhoff rule.

Consistency of the on-shell conditions follows from momentum conservation at each vertex and the assignments

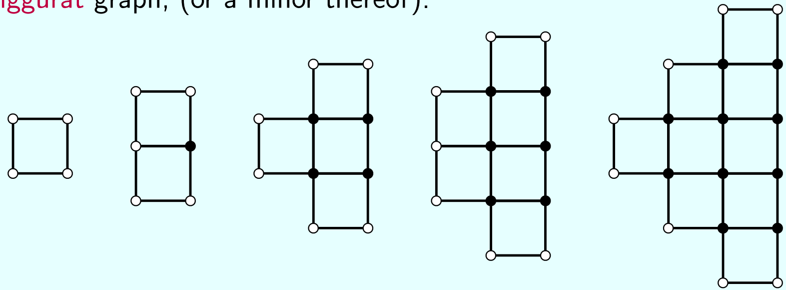
$$\alpha_1' = \frac{\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1}{\alpha_1} \quad \text{etc.}$$

familiar from electrical circuits.

Planar Graph Reduction

The problem of studying the reducibility of m-terminal graphs under the basic circuit operations is well studied in the mathematical literature.

The key result, for our purposes, comes from Isidoro Gitler, who proved in 1991 that any 2-connected m-terminal plane graph, with all terminals lying on a common face (which we take to be the "outer" face), can be Y-Δ reduced to what we call the m-terminal ziggurat graph, (or a minor thereof).

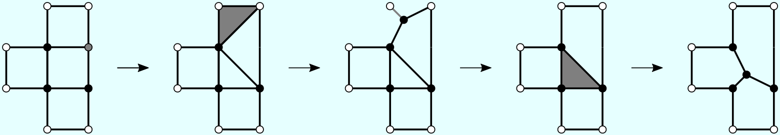


Planar Graph Reduction

The first-type Landau singularities of any n-particle amplitude, in any massless planar field theory, at any finite order in perturbation theory, are a subset of those of the n-particle ziggurat graph (which, we note, has $\lfloor (n-2)^2/4 \rfloor$ loops).

Planar Graph Reduction

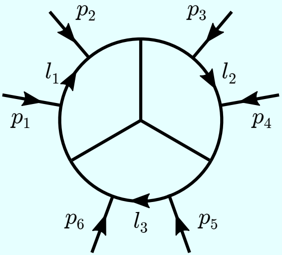
In fact, the loop order $\lfloor (n-2)^2/4 \rfloor$ is unnecessarily high, since ziggurat graphs can in general be further reduced. For example:



It is easy to check, on a case by case basis, that the n-particle ziggurat graph can be reduced to various graphs of lower loop order, but we have not been able to prove a lower bound on the loop order that can be obtained for general n.

Landau Analysis of the Wheel Graph

Let's find the singularities of this graph. First, to find the leading Landau singularities, we put all 12 propagators on shell



$$
\begin{aligned}
(\ell_1 - p_1)^2 = \ell_1^2 = (\ell_1 + p_2)^2 &= 0 \\
(\ell_2 - p_3)^2 = \ell_2^2 = (\ell_2 + p_4)^2 &= 0 \\
(\ell_3 - p_5)^2 = \ell_3^2 = (\ell_3 + p_6)^2 &= 0 \\
(\ell_1 + p_2 - \ell_2 + p_3)^2 &= 0 \\
(\ell_2 + p_4 - \ell_3 + p_5)^2 &= 0 \\
(\ell_3 + p_6 - \ell_1 + p_1)^2 &= 0
\end{aligned}
$$

For generic p_i there are 16 discrete solutions, which are easy to enumerate using the technology of on-shell diagrams.

Landau Analysis of the Wheel Graph

With these solutions in hand, we next turn our attention to the Kirchhoff conditions

$$
\begin{aligned}
0 = & \alpha_1(\ell_1 - p_1) + \alpha_2 \ell_1 + \alpha_3(\ell_1 + p_2) + \\
& \alpha_{10}(\ell_3 + p_6 - \ell_1 + p_1) + \alpha_{11}(\ell_1 + p_2 - \ell_2 + p_3), \\
0 = & \alpha_4(\ell_2 - p_3) + \alpha_5 \ell_2 + \alpha_6(\ell_2 + p_4) + \\
& \alpha_{11}(\ell_1 + p_2 - \ell_2 + p_3) + \alpha_{12}(\ell_2 + p_4 - \ell_3 + p_5), \\
0 = & \alpha_7(\ell_3 - p_5) + \alpha_8 \ell_3 + \alpha_9(\ell_3 + p_6) + \\
& \alpha_{12}(\ell_2 + p_4 - \ell_3 + p_5) + \alpha_{10}(\ell_3 + p_6 - \ell_1 + p_1).
\end{aligned}
$$

Nontrivial solutions to this 12×12 linear system exist only if the associated Kirchhoff determinant K vanishes.

Landau Analysis of the Wheel Graph

By evaluating K on the 16 on-shell solutions, the condition for the existence of a non-trivial solution to the Landau equations can be expressed entirely in terms of the p_i.

Using the familiar variables

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \quad v = \frac{s_{23}s_{56}}{s_{234}s_{123}} \quad w = \frac{s_{34}s_{61}}{s_{345}s_{234}}, \quad s_{ij\cdots} = (p_i + p_j + \cdots)^2$$

we find that $K = 0$ can only be satisfied on the locus

$$\mathscr{S}_6 = \bigcup_{s \in S_6} \{s = 0\}, \qquad S_6 = \{u, v, w, 1-u, 1-v, 1-w, \frac{1}{u}, \frac{1}{v}, \frac{1}{w}\}$$

It is straightforward, if tedious, to check all possible sub-leading Landau singularities; remarkably, they give nothing new.

Landau Analysis of the Wheel Graph

By evaluating K on the 16 on-shell solutions, the condition for the existence of a non-trivial solution to the Landau equations can be expressed entirely in terms of the p_i.

Using the familiar variables

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \quad v = \frac{s_{23}s_{56}}{s_{234}s_{123}} \quad w = \frac{s_{34}s_{61}}{s_{345}s_{234}}, \quad s_{ij\cdots} = (p_i + p_j + \cdots)^2$$

we find that $K = 0$ can only be satisfied on the locus

$$\mathscr{S}_6 = \bigcup_{s \in S_6} \{s = 0\}, \qquad S_6 = \{u, v, w, 1-u, 1-v, 1-w, \frac{1}{u}, \frac{1}{v}, \frac{1}{w}\}$$

Conclusion: any 6-particle amplitude, at any finite loop order, in any massless planar field theory, can have first-type Landau singularities only on the locus $\mathscr{S}_6$.

$\mathscr{S}_6$ and the Hexagon Symbol Alphabet

This claim is consistent with the hexagon symbol alphabet, which contains nine letters

$$A_6 = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

indicating the presence of branch surfaces on the locus

$$\mathscr{A}_6 = \bigcup_{a \in A_6} \{a = 0\} \cup \{\frac{1}{a} = 0\}$$

Here y_u and $1/y_u$ are the two roots of the quadratic equation

$$u(1-v)(1-w)(z^2+1) = \left[u^2 - 2uvw + (1-v-w)^2\right] z$$

with y_v and y_w defined by cyclically exchanging $u \to v \to w \to u$. This relation makes it clear that $y_u \in \{0, \infty\}$ only on the locus $\{u = 0\} \cup \{v = 1\} \cup \{w = 1\}$, etc. Therefore we conclude that $\mathscr{A}_6 = \mathscr{S}_6$.

$\mathscr{S}_6$ and the Hexagon Symbol Alphabet

Six-particle amplitudes in SYM theory exhibit singularities on all branches of the locus $\mathscr{S}_6$ already starting at one loop, even though the nine-letter symbol alphabet is not fully utilized until two loops.

Why, then, did we need to carry out the Landau computation for the wheel graph at three loops— and even that already represented some savings over the original four-loop six-point ziggurat graph?!

It's because there is no single one- or two-loop six-point graph that simultaneously exhibits all branches of the singularity locus $\mathscr{S}_6$.

It's very remarkable that the wheel graph does; and even more that it does so already in its leading singularities.

Second-type Singularities

In addition to the first-type Landau singularities we have classified, there exist second-type singularities that arise in loop integrals as pinch singularities at infinite loop momentum.

We expect that second-type singularities should be absent in dual conformally invariant (DCI) theories, since these have no invariant notion of "infinity" in momentum space.

On the other hand (except for the soft/collinear singularities that we have ignored throughout), all first-type singularities of the ziggurat graphs are manifestly DCI.

Therefore, we can say that the ziggurat graphs capture the "dual conformally invariant part" of the singularity structure of all massless planar theories; this means the singularity loci that do not involve the infinity twistor.

Implications for SYM Theory

In SYM theory, for fixed particle number n, there can be accidental cancellations at low loop order L in certain helicity sectors k.

For example, one-loop MHV amplitudes do not have singularities of three-mass box type, but it is known that two-loop (and higher) MHV amplitudes certainly do [Caron-Huot 1105.5605].

Similarly, one-loop NMHV amplitudes do not have singularities of four-mass box type, but we expect that two-loop (and higher) NMHV and three-loop (and higher) MHV amplitudes do (for $n \geq 8$).

Implications for SYM Theory

In SYM theory, for fixed particle number n, there can be accidental cancellations at low loop order L in certain helicity sectors k.

However, we claim that for any fixed n and k, all such cancellations eventually fail at sufficiently high (but finite!) loop order.

> Claim: Perturbative n-point amplitudes in SYM theory exhibit first-type Landau singularities on such *all* loci that are possible in any massless planar field theory, i.e., on *all* branches of $\mathscr{S}_n$.

Specifically, we claim that for any fixed n and any $0 \leq k \leq n-4$, there is a finite value of $L_{n,k}$ such that the singularity locus of the L-loop n-particle N^{k}MHV amplitude is all of $\mathscr{S}_n$ for all $L \geq L_{n,k}$.

Implications for SYM Theory

This can be seen by blowing up each edge of the ziggurat graph into a bubble; the result is a graph that has the same Landau singularities as the ziggurat but has MHV support, when interpreted as an on-shell diagram.

Equivalently, one can write down an explicit configuration of positive (and mutually positive) lines satisfying the indicated set of cut conditions, indicating that there is support on the (boundary of the) MHV amplituhedron.

Deriving Symbol Alphabets for General Amplitudes?

Can we determine symbol alphabets?

Unfortunately, as the hexagon letters y_u, y_v, y_w indicate, the connection between Landau singularities and symbol alphabets is rather indirect; knowledge of the former tells us about the locus where symbol letters vanish (logarithmic branch surfaces) or have algebraic branch surfaces.

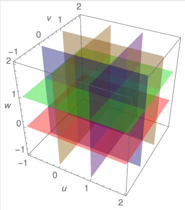
(There are infinitely many algebraic functions of u, v, w that vanish only on $\mathscr{S}_6$.)

In order to determine what the symbol letters of an amplitude actually are, away from this locus, still requires invoking some additional structure.

(Perhaps cluster algebras play a role here [Golden, Goncharov, MS, Vergu, Volovich 1305.1617].)

Conclusion

For each n, the $3(n-5)$-dimensional kinematic configuration space $\mathrm{Gr}(4,\mathbb{C}^n)/(\mathbb{C}^*)^{n-1}$ of n cyclically ordered massless particles modulo dual conformal symmetry has an interesting codimension-one subvariety $\mathscr{S}_n$.



It is a general consequence of locality, in the guise of the Landau equations, that n-particle scattering amplitudes in any massless, planar theory can have first-type Landau singularities only on $\mathscr{S}_n$.

In a random field theory the actual singularity locus might be a proper subset of $\mathscr{S}_n$, but in SYM theory the n-particle has singularities on all of $\mathscr{S}_n$ (at sufficiently high, but finite, loop order).

- Understand the mathematical structure of $\mathscr{S}_n$!
- Harness this knowledge to learn more about amplitudes!