[redacted]

Bootstrap basics II 30/10/2018

- Radial quantisation + state-operator correspondence
- OPE ↘ David Tong String theory sec 4 chapter 4.6.
- Bootstrap eqn.

Recap: $\mathcal{O}(x)$ Local operator

Represent conformal generators as differential operators

$$[P_\mu, \mathcal{O}(x)] = -i\partial_\mu \mathcal{O}(x)$$

$$[D, \mathcal{O}(x)] = -i(\Delta + x^\mu \partial_\mu)\mathcal{O}(x)$$

$\vdots$

$\mathcal{O}(0)$ is a primary

$$[K_\mu, \mathcal{O}(0)] = 0, \quad [D, \mathcal{O}(0)] = -i\,\mathcal{O}(0)$$

We also assume unique vacuum $|0\rangle$

$$(P, D, K, M)|0\rangle = 0.$$

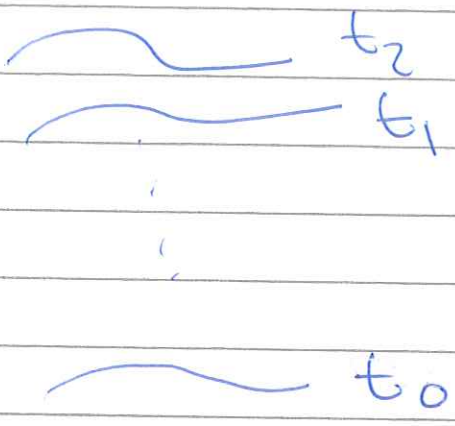
$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\rangle = \langle 0|\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)|0\rangle$$

Radial Quantisation + State-operator correspondence

In QFT, Hilbert space construction depends on a foliation. Usually we foliate by Cauchy surfaces

Operators live @ point in space time
States live on leaves of foliation

$$U = e^{iP^0 \Delta t} = e^{iH\Delta t}$$



moves us between leaves

In CFT it is more convenient to use radial quantisation

$$U = e^{iD\Delta\tau} \quad , \quad \tau = \log r$$

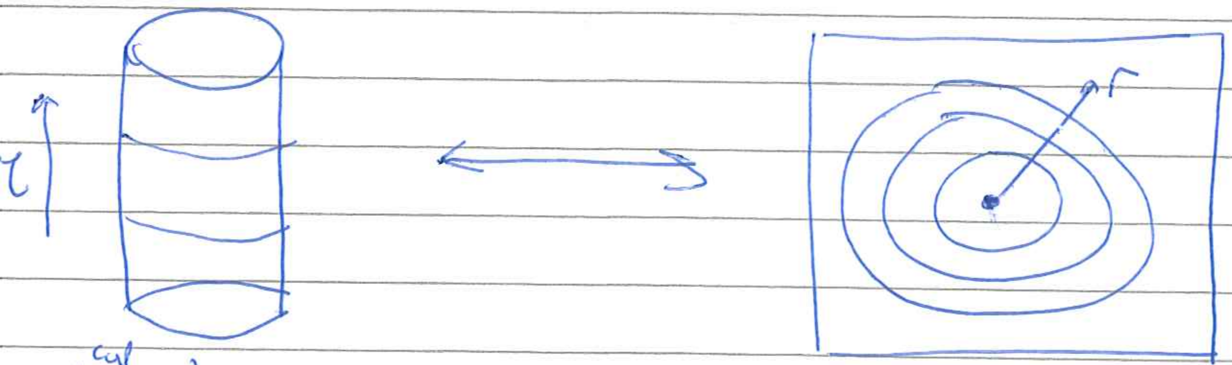
Foliations by S^{d-1}

$\mathbb{R}^{1,d-1}$ is conformally equivalent to $\mathbb{R} \times S^{d-1}$

~~$z = e^{\tau + i}$~~

$$ds^2_{\mathbb{R}^d} = dr^2 + r^2 d\Omega^2_{d-1}$$

$$= r^2 (d\tau^2 + d\Omega^2_{d-1})$$



$H^{cyl} = \frac{\partial}{\partial\tau} \longrightarrow D$ on flat space.

States labelled by Δ, $SO(d)$ spin ℓ

Generate states by inserting operators inside on a given sphere

e.g. $|0\rangle$ vacuum $\Leftrightarrow$ Inserting nothing

Insert $\mathcal{O}(x)$
gives state $|\psi\rangle = \mathcal{O}_\Delta(x)|0\rangle$
is not an eigenstate of D

$$|\psi\rangle = \mathcal{O}(x)|0\rangle = e^{iPx}\,\mathcal{O}(0)\,e^{-iPx}|0\rangle$$
$$= e^{iPx}|\Delta\rangle$$

$$|\Delta\rangle \xrightarrow{P_\mu} |\Delta+1\rangle \xrightarrow{P_\mu} |\Delta+2\rangle \longrightarrow \cdots$$

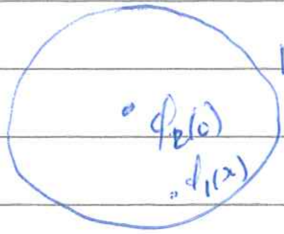
$$0 \xleftarrow{K_\mu} |\Delta\rangle \xleftarrow{K_\mu} |\Delta+1\rangle \longleftarrow \cdots$$

For unitary theories Δ bounded from below.
e.g. in $d=3$ $\Delta \geq \begin{cases} 1+\ell, & \ell \geq 1 \\ 1, & \ell = \frac{1}{2} \end{cases}$
We can generate entire Hilbert space from local operators @ $x=0$ and acting w/ P_μ's.

State operator correspondence
$$|\mathcal{O}\rangle = \lim_{x\to 0} \mathcal{O}(x)|0\rangle$$

OPE (operator product expansion)

Two operators inside sphere



$$|\psi\rangle = \phi_1(x)\,\phi_2(0)\,|0\rangle$$

(radially ordered.

Expand $|\psi\rangle = \sum_n C_n |\Delta_n\rangle$

Basis of Dilatation eigenstates

$$|\Delta_n\rangle \overset{1:1}{\Longleftrightarrow} \text{Local ops.}$$

(primaries or decendents of primaries)

$$|\psi\rangle = \sum_{\mathcal{O}_{\text{primaries}}} \lambda_{\mathcal{O}}\, C_{\mathcal{O}}(x, \partial_y)\, \mathcal{O}(y)\big|_{y=0} |0\rangle$$

fixed by CS

This is the OPE.

It is convergent up to next operator insertion

We can use it to reduce n pt to n-1 pt

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\rangle = \sum_{\mathcal{O}} \lambda_{\mathcal{O}} C_{\mathcal{O}}(x,\partial_y)\,\frac{\delta_{\mathcal{O},\phi_3}}{|y-x_3|^{2\Delta\phi_3}}$$

$$= \lambda_{\phi_3} C_{\phi_3}(x,\partial_y)\,\frac{1}{|y-x_3|^{2\Delta\phi_3}}.$$

$$= \frac{\lambda_{\phi_3}}{(x_{12})(x_{13})(x_{23})}$$

By conformal sym (Last lecture)

$\Rightarrow$ fix $C_{\mathcal{O}}$'s.

$\{\lambda_{\mathcal{O}}, \Delta_{\mathcal{O}}\}$'s specify (local spectrum of) a the CFT completely.

$\Rightarrow$ CFT data.

Boot Strap

Try to fix $\{\lambda_{\mathcal{O}}, \Delta_{\mathcal{O}}\}$ using symmetry + unitarity alone.

4pt f'n $\langle \prod_{i=1}^{4} \phi_i(x_i)\rangle$ of scalars.

$n = \Delta_{\phi_i}$

OPE $\quad \phi_i(x_i)\,\phi_j(x_j) = \sum_{\mathcal{O}} \lambda_{ij\mathcal{O}}\, C_{\mathcal{O}}(x_{ij}, \partial_y)\, \mathcal{O}(y)\big|_{y=\frac{x_i+x_j}{2}}$

OPE 1,2 + 3,4

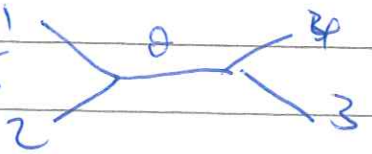
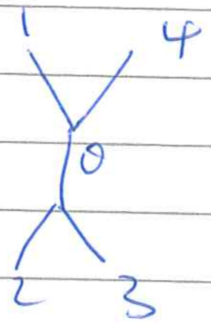
conformal block

$$\langle \phi_1 \ldots \phi_4 \rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}} \lambda_{34\mathcal{O}} \frac{G_{\mathcal{O}}(u,v)}{(x_{12})^{2n}(x_{34})^{2n}}$$

OPE 1,4 + 2,3 $\qquad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$

$$\langle \phi_1 \ldots \phi_4 \rangle = \sum_{\mathcal{O}} \lambda_{14\mathcal{O}} \lambda_{23\mathcal{O}} \frac{G_{\mathcal{O}}(v,u)}{(x_{14})^{2n}(x_{23})^{2n}}.$$

$$\Rightarrow \boxed{\sum_{\mathcal{O}} \lambda_{12\mathcal{O}} \lambda_{34\mathcal{O}} G_{\mathcal{O}}(u,v) = \left(\frac{u}{v}\right)^n \sum_{\mathcal{O}} \lambda_{14\mathcal{O}} \lambda_{23\mathcal{O}} G_{\mathcal{O}}(v,u)}$$

$\sim \sum_{\mathcal{O}} \lambda^2$  $= \sum_{\mathcal{O}} \lambda^2$ 

13, 24 gives no new info.

- Conformal algebra repns

$$\mathfrak{g} = so(d,2)_{\mathbb{C}} \supset \underbrace{so(d)}_{\{M_{\mu\nu}\}} + \underbrace{\cancel{s}u(1)_E}_{\{E\}} = \mathfrak{h}$$

- We want to find unitary irreducible, highest weight repns ρ of $\mathfrak{g}$.

$\mathfrak{g}$ - non compact $\Rightarrow$ Unitary repns ∞ - dim

- To get $\mathfrak{g}$ repns we can induce them from $\mathfrak{h}$ repns $\rho|_{\mathfrak{h}}$, $\rho = \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}} \rho|_{\mathfrak{h}}$

These $\rho|_{\mathfrak{h}}$ repns (should be) well known.

They are labelled by $\{\Delta, j_1, j_2, \ldots, j_n\}$

$n = \text{rank } so(d) = \lfloor \frac{d}{2} \rfloor$ $\qquad j_i = \pm\frac{1}{2}$

Let $|\Lambda\rangle^{h.w.} = |\Delta, j_1, j_2, \ldots, j_n\rangle$

Then $\mathrm{Span}\left\{ \prod_{\mu,\nu=0}^{d-1} \rho|_{\mathfrak{h}}(M_{\mu\nu})^{m_{\mu\nu}} |\Lambda\rangle^{h.w.} \right\}$ is repn space for $\rho|_{\mathfrak{h}}$ and let $|\Lambda\rangle_a$ be a basis for it

- $|\Lambda_a\rangle$'s are conformal primaries iff

$\rho(K_\mu)|\Lambda\rangle_a = 0 \qquad \rho(E)|\Lambda\rangle_a = i\Delta|\Lambda\rangle_a$

The Verma module for $\mathfrak{p}$ is then

$$V^{\Delta}_{j_1,\dots,j_n} = \mathrm{span}\left\{ \prod_{\mu=0}^{d-1} \rho(P_\mu)^{n_\mu} |\Lambda\rangle_a \right\}.$$

Can construct dual module $\bar{V}^{\Delta}_{j_1,\dots,j_n}$
by ${}_a\langle\Lambda| E = i\Delta\, {}_a\langle\Lambda|$, ${}_a\langle\Lambda|\rho(K_\mu) = 0$.

If we demand $\rho(K_\mu)^\dagger = \rho(P_\mu)$ $\quad -\rho(M_{\mu\nu})^\dagger = \rho(M_{\mu\nu})$

$$\rho(E)^\dagger = \rho(E)$$

we have inner product $\langle\cdot|\cdot\rangle : \bar{V}^{\Delta}_{j_1\dots j_n} \times V^{\Delta}_{j_1,\dots,j_n} \to \mathbb{C}$

and $\quad {}_a\langle\Lambda|\Lambda\rangle_b = \delta_{ab}$.

- $V^{\Delta}_{j_1,\dots,j_n}$ is not nessciarily irriducible or unitary.
we demand (unitarity) $\langle\cdot|\cdot\rangle > 0$.

- There may be decendants $\rho(P_\mu)^{n_\mu}|\Lambda\rangle_a = |\Lambda'\rangle_a$
s.t. $\rho(K_\mu)|\Lambda\rangle_a = 0$. $\Rightarrow$ They are also primary.

$\Rightarrow$ Not irriducible. but

$V^{\Delta}_{j_1\dots j_n} / V^{\Delta'}_{j_1'\dots j_n'}$ is irriducible.

Moreover $_a\langle\Lambda'|\Lambda'\rangle_b = 0$.

We can use commutation relations to get bounds on Δ

It turns out that for $d=4$

$$\Delta \geq \begin{cases} 2 + j_1 + j_2 & j_1, j_2 \geq \frac{1}{2} \\ 1 + j_1 & j_1 \geq \frac{1}{2},\ j_2 = 0 \\ 1 + j_2 & j_1 = 0 \quad j_2 \geq \frac{1}{2} \end{cases}$$

When bound saturated $\Rightarrow$ conserved currents (null states)

e.g. Stress tensor multiplet

$$|\Lambda\rangle^{h.w.} = \cancel{|4,} \, |4, 1, 1\rangle$$

$T_{\mu\nu}$

conserved current $\partial_\mu T^{\mu\nu}$

- in 3d

$$\Delta \geq \begin{cases} 1 + j_1 & j_1 \geq 1 \\ 1 & j_1 = \frac{1}{2} \end{cases}$$