Parton evolution at amplitude level

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- Ultimate goal: systematically to go beyond leading colour in an event generator

Plan of this talk

1. The algorithm we have currently implemented
2. Beyond the soft approximation
3. Factorization
4. A remarkable(?) result concerning the loop corrections

Nagy & Soper: DEDUCTOR
JHEP 0709 (2007) 114
JHEP 0803 (2008) 030
JHEP 0807 (2008) 025
JHEP 1206 (2012) 044
JHEP 1406 (2014) 097
JHEP 1507 (2015) 119
Phys.Rev. D99 (2019) no.5, 054009

Parton showers with more exact color evolution

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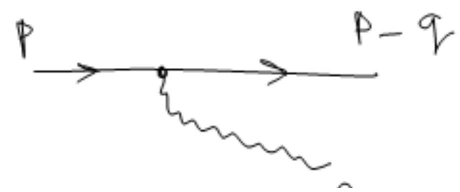
(Dated: 7 March 2019)

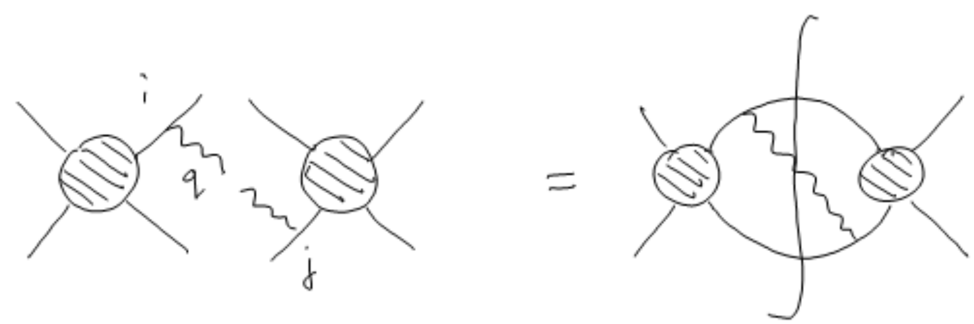
Parton shower event generators typically approximate evolution of QCD color so that only contributions that are leading in the limit of an infinite number of colors are retained. Our parton shower generator, DEDUCTOR, has used an "LC+" approximation that is better, but still quite limited. In this paper, we introduce a new scheme for color in which the approximations can be systematically improved. That is, one can choose the theoretical accuracy level, but the accuracy level that is practical is limited by the computer resources available.

We expect that the algorithm presented here will not be the last word in algorithms for this purpose. Surely it is possible to do better. Indeed, Ángeles Martínez, De Angelis, Forshaw, Plätzer, and Seymour [16] have provided a formalism for the description of soft gluon emissions that is similar in some ways to the general formalism [4, 7] on which this paper is based. If the approach of Ref. [16] can be extended to include the collinear singularities of QCD, then it will be of great interest to see if there can be a practical implementation of the resulting formalism. Perhaps such an implementation will be able to outperform what this paper provides.

Soft gluons



$$= 2 g p_\mu \delta_{\lambda\lambda'} T^a_{ij}$$

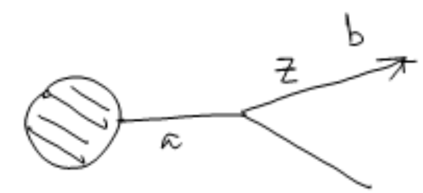


$$d\sigma_{n+1} = d\sigma_n \frac{\alpha_s}{2\pi}\frac{dE}{E}\frac{d\Omega}{2\pi}\sum_{ij} C_{ij} E^2 \, \frac{p_i \cdot p_j}{p_i \cdot q \; p_j \cdot q}$$

Colour factor is nasty

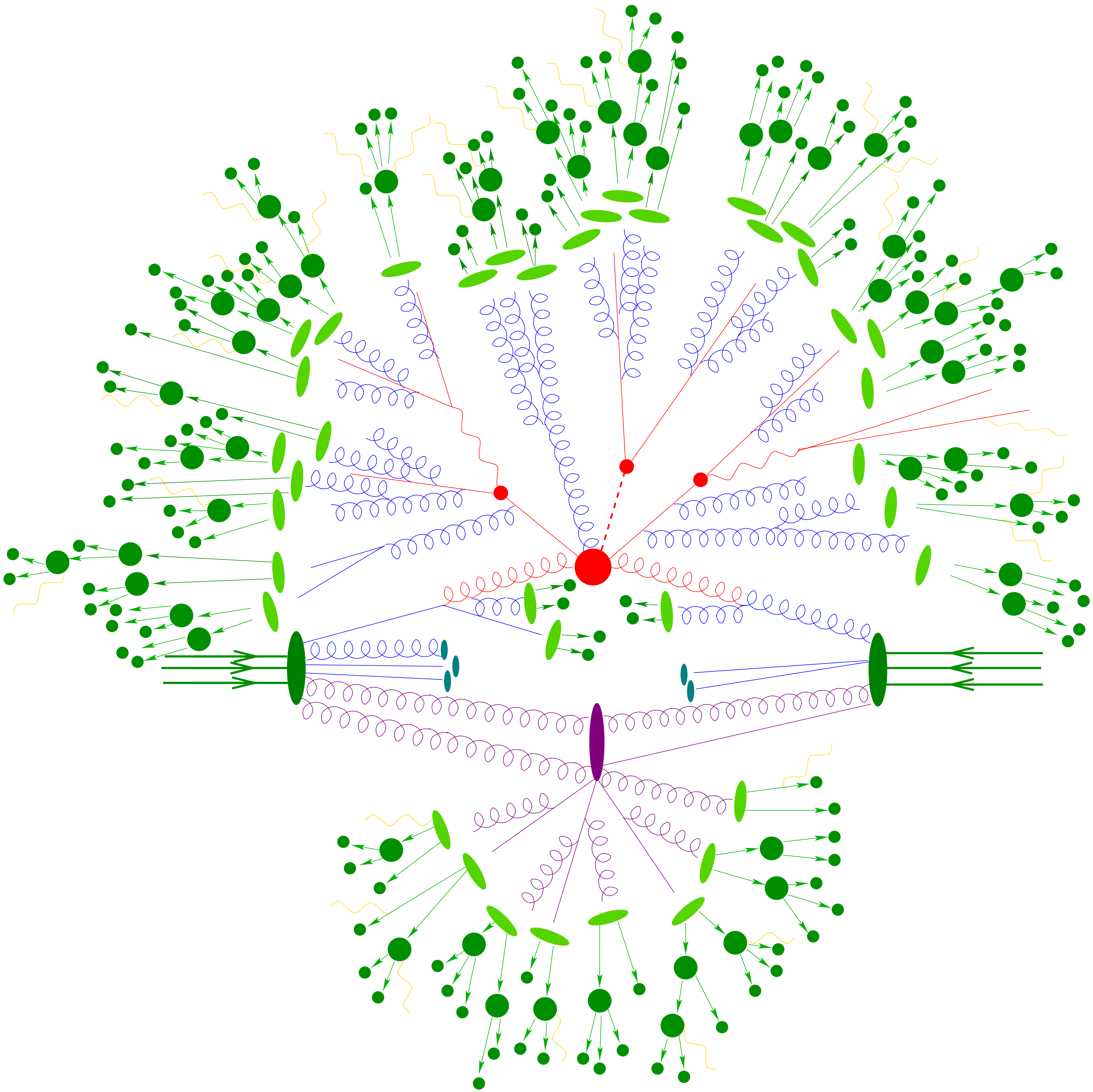
Collinear emissions

Colour structure is easier. It is as if emission is off the parton to which it is collinear.



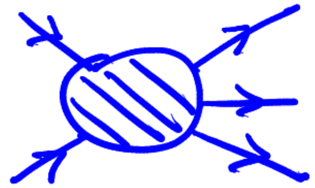
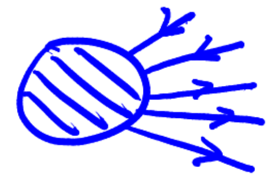
$$d\sigma_{n+1} = d\sigma_n \, \frac{\alpha_s}{2\pi}\frac{dq^2}{q^2} dz \; P_{ba}(z)$$

Simulation codes exploit the fact that in the "**large $\mathbf{N_c}$**" approximation both wide-angle soft and collinear emissions can be included via a classical branching algorithm, i.e. quantum interference included by clever re-arrangement of the interference terms = the HERWIG project.



https://sciencenode.org/feature/sherpa-and-open-science-grid-predicting-emergence-jets.php

Soft gluon evolution

$|H\rangle =$  $=$ 

$|H\rangle\langle H| = H =$ 

$$D_i = \sum_j T_j \, E_i \frac{p_j}{p_j \cdot q_i}$$

$$V_{a,b} = \exp\left[\int_a^b \frac{dE}{E} \Gamma\right]$$

$$\Gamma = \frac{\alpha_s}{\pi} \sum_{i<j} \left(-T_i \cdot T_j\right) \left\{ \int \frac{d\Omega_k}{4\pi} w_{ij}(\hat{k}) - i\pi \tilde{\delta}_{ij} \right\}$$

$$w_{ij}(\hat{k}) = E_k^2 \frac{p_i \cdot p_j}{p_i \cdot k \; p_j \cdot k}$$

e.g. $A_1 =$

$$d\sigma_n = \mathrm{Tr}\, A_n(\mu)\, d\Pi_n$$

$\mu = 0$ always
& $\mu = Q_0$ if inclusive
for $E < Q_0$

$$\Sigma = \sum_n \int d\sigma_n \; u_n(q_1, \dots, q_n)$$

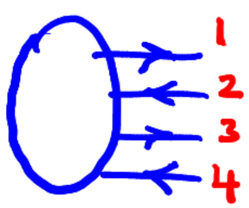
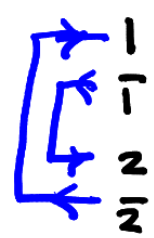
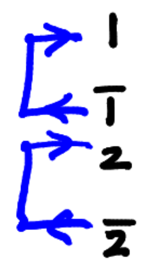
measurement function

$$A_n(E) = V_{E,E_n} D_n A_{n-1}(E_n) D_n^\dagger V_{E,E_n}^\dagger \, \Theta(E \leq E_n)$$

Ángeles Martínez, de Angelis, JF, Plätzer, Seymour
JHEP 05(2018) 044
arXiv:1802.08531

Colour evolution

Work in colour flow basis

 $= \alpha$  $+ \; \beta$ 

i	c_i	$\bar{c}_i$
1	1	0
2	0	$\bar{1}$
3	2	0
4	0	$\bar{2}$

$$|M\rangle = \alpha \left| \begin{matrix} 1 & 2 \\ \bar{2} & \bar{1} \end{matrix} \right\rangle + \beta \left| \begin{matrix} 1 & 2 \\ \bar{1} & \bar{2} \end{matrix} \right\rangle$$

$$= \alpha \, |2\,1\rangle + \beta \, |1\,2\rangle$$

length of σ, τ

$$\langle \sigma | \tau \rangle = N_c^{\,n - \#(\sigma,\tau)}$$

\# transpositions by which σ & τ differ

e.g. $\langle 1 2 | 1 2 \rangle = N_c^2$

$\langle 2 1 | 1 2 \rangle = N_c$

$$\sum_{\sigma} |\sigma\rangle[\sigma| = \mathbb{1}$$

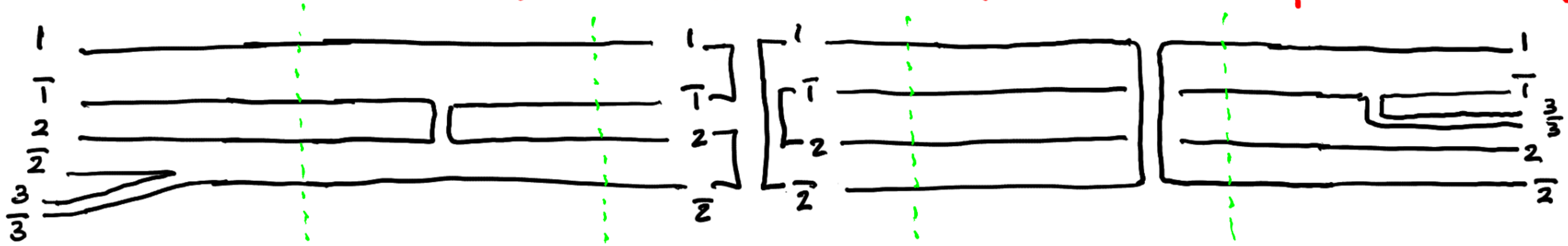
not orthonormal

$$[\sigma|\tau\rangle = \langle\sigma|\tau] = \delta_{\sigma,\tau}$$

"scalar product matrix"

$$\mathrm{Tr}\, A = \sum_{\tau,\sigma} [\tau|A|\sigma]\langle\sigma|\tau\rangle$$

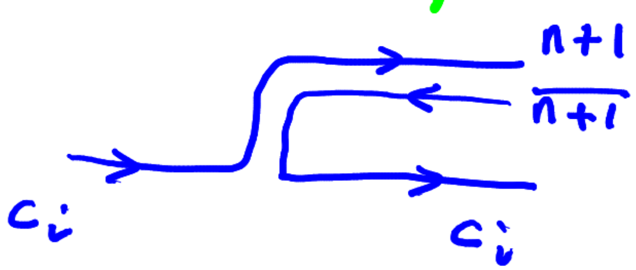
$$|\sigma_3\rangle[\sigma_3|D|\sigma_2\rangle[\sigma_2|V|\sigma_1\rangle[\sigma_1|H|\tau_1]\langle\tau_1|V^\dagger|\tau_2]\langle\tau_2|D^\dagger|\tau_3]\langle\tau_3|$$



$\sigma_3 = (3\,1\,2)$ $\quad$ $\sigma_2 = (2\,1)$ $\quad$ $\sigma_1 = (1\,2)$ $\quad$ $\tau_1 = (2\,1)$ $\quad$ $\tau_2 = (2\,1)$ $\quad$ $\tau_3 = (2\,3\,1)$

Real emissions:

$$T_i = \lambda_i t_{c_i} - \bar{\lambda}_i \bar{t}_{\bar{c}_i} - \frac{1}{N_c}(\lambda_i - \bar{\lambda}_i)\, s$$



$$t_\alpha \left| \begin{matrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{matrix} \right\rangle = \left| \begin{matrix} 1 & \dots & \alpha & \dots & n & n+1 \\ \sigma(1) & \dots & n+1 & \dots & \sigma(n) & \sigma(\alpha) \end{matrix} \right\rangle$$

$$s \left| \begin{matrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{matrix} \right\rangle = \left| \begin{matrix} 1 & 2 & \dots & n & n+1 \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) & n+1 \end{matrix} \right\rangle$$

$\lambda_i = 1/\sqrt{2}$ out q / in $\bar{q}$
$\bar{\lambda}_i = 0$

$\lambda_i = 0$ in q
$\bar{\lambda}_i = 1/\sqrt{2}$ out $\bar{q}$

$\lambda_i = \bar{\lambda}_i = 1/\sqrt{2}$ g

$$T\,|\sigma_n\rangle \cdots\cdots\cdots \langle\tau_n|\,T = |\sigma_{n+1}\rangle \cdots \langle\tau_{n+1}|$$

if σ_n & τ_n differ by n transpositions

then σ_{n+1} & τ_{n+1} differ by n or $n+2$ transpositions

or $n+1$ transpositions

$t \ldots s$ $\quad 1/N_c$

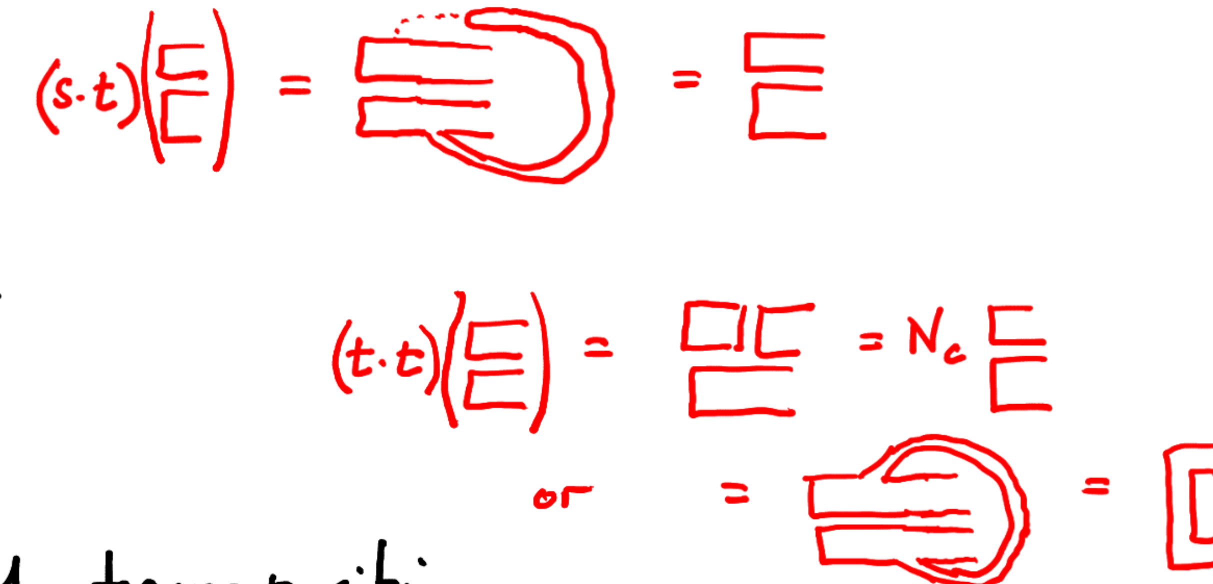
$t \ldots t$

$s \ldots s$ $\quad 1/N_c^2$

$t \ldots t$

Note: cannot reduce # transpositions via real emissions

Virtual corrections



$$s\cdot t = t\cdot s = \mathbb{1}$$

$$s\cdot s = N_c\, \mathbb{1}$$

$$t\cdot t = N_c\, \mathbb{1} \quad \text{or} \quad 1 \text{ transposition}$$

$$\Gamma_\tau \big(\underset{T_i\cdot T_j}{\Gamma} \big| \sigma \big\rangle = N_c\, \delta_{\tau\sigma}\, \Gamma_\sigma + \underset{\#(\sigma,\tau)=1}{\Sigma_{\sigma\tau}} + \frac{1}{N_c}\, \delta_{\tau\sigma}\, \rho$$

Plätzer: Eur. Phys. J. C (2014) 74, arXiv: 1312.2448

$$[\tau|e^{\Gamma}|\sigma\rangle = \sum_{l=0}^{\infty \to d} \frac{(-1)^l}{N_c^l} \sum_{\sigma_0,\sigma_1,\ldots\sigma_l} \delta_{\tau\sigma_0}\delta_{\sigma_l\sigma} \left(\prod_{\alpha=0}^{l-1}\Sigma_{\sigma_\alpha\sigma_{\alpha+1}}\right) R(\{\sigma_0,\sigma_1,\ldots,\sigma_l\})$$

e.g. $d=1$

$$[\tau|e^{\Gamma}|\sigma\rangle = \delta_{\tau\sigma}\, e^{-N_c\Gamma'_\sigma} - \frac{1}{N_c}\Sigma_{\tau\sigma} \underbrace{\frac{e^{-N_c\Gamma'_\tau} - e^{-N_c\Gamma'_\sigma}}{\Gamma'_\tau - \Gamma'_\sigma}}_{= -N_c e^{-N_c\Gamma'_\sigma} \text{ if } \sigma=\tau}$$

$$\left(\Gamma' = \Gamma - \frac{\rho}{N_c^2}\right)$$

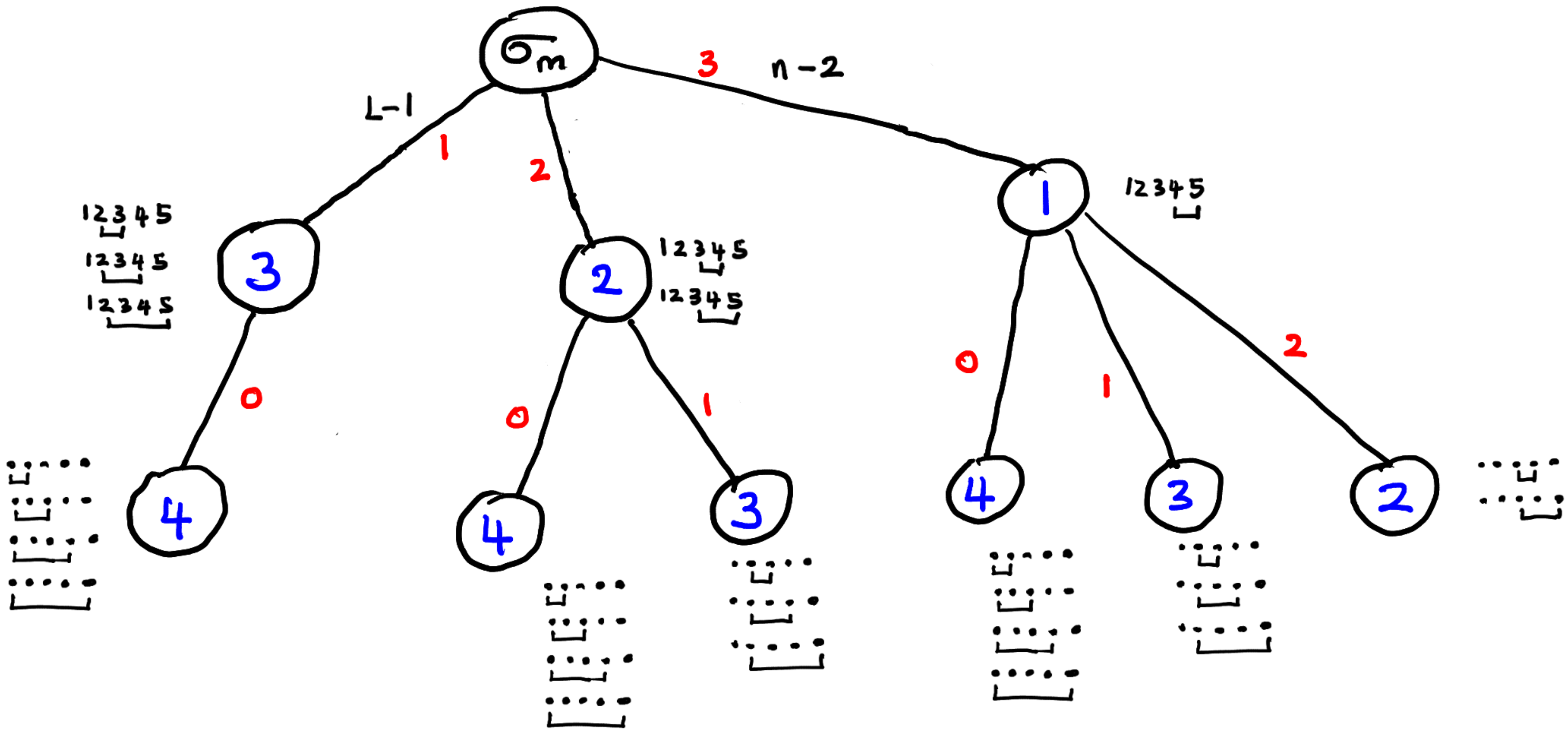
"N^dLC"

= much more than N^dLC for observables

Key component is how to select σ_{m+1} from σ_m

s.t. $\#(\sigma_{m+1}, \sigma_m) = L$

e.g. $n = 5$, $L = 2$ $\qquad \sigma_m = (1\ 2\ 3\ 4\ 5)$



Collinear emissions

$$\ln \mathbf{V}_{ab} = \ln \mathbf{W}_{ab} + \ln \mathbf{K}_{ab},$$

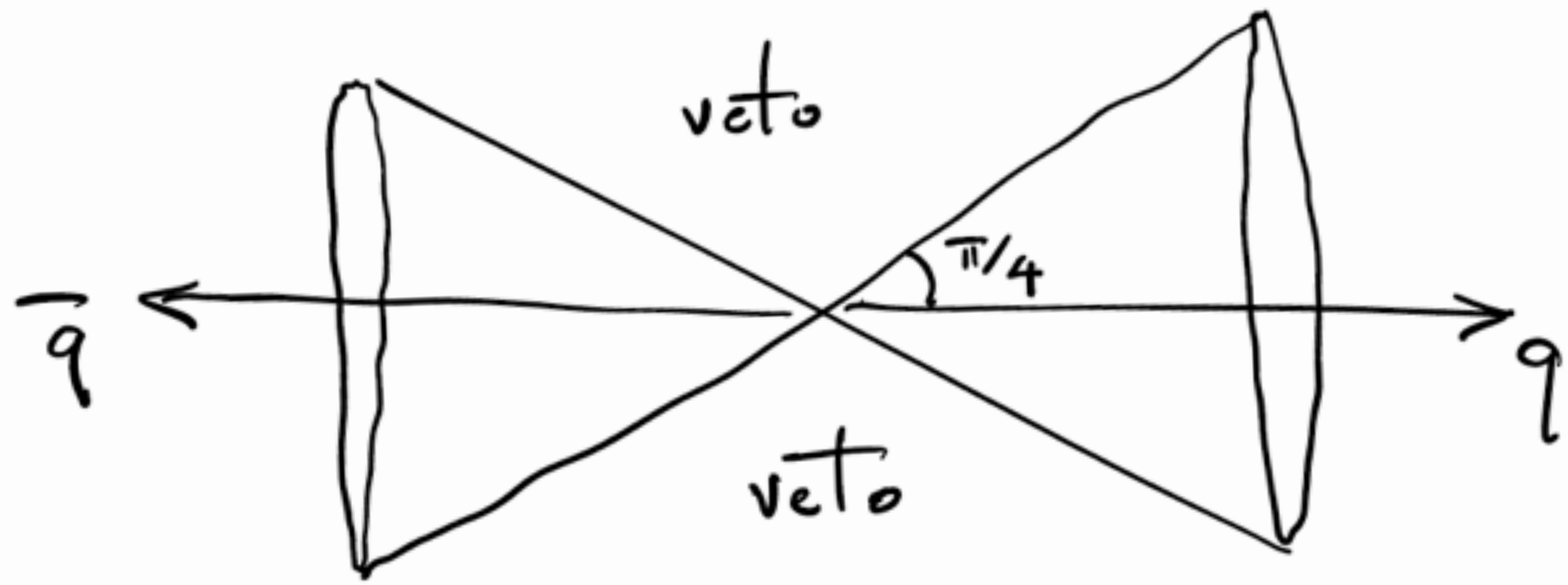
$$\ln \mathbf{W}_{ab}\Big|_{\text{energy}} = \frac{\alpha_s}{\pi} \sum_{i<j} \mathbb{T}_i^g \cdot \mathbb{T}_j^g \int_a^b \frac{\mathrm{d}E}{E} \int \frac{\mathrm{d}\Omega}{4\pi} \left(\frac{n_i \cdot n_j - n_i \cdot n - n_j \cdot n}{n_i \cdot n \; n \cdot n_j} \right)$$

$$= \frac{\alpha_s}{\pi} \sum_{i<j} \mathbb{T}_i^g \cdot \mathbb{T}_j^g \int_a^b \frac{\mathrm{d}E}{E} \ln \frac{n_i \cdot n_j}{2}$$

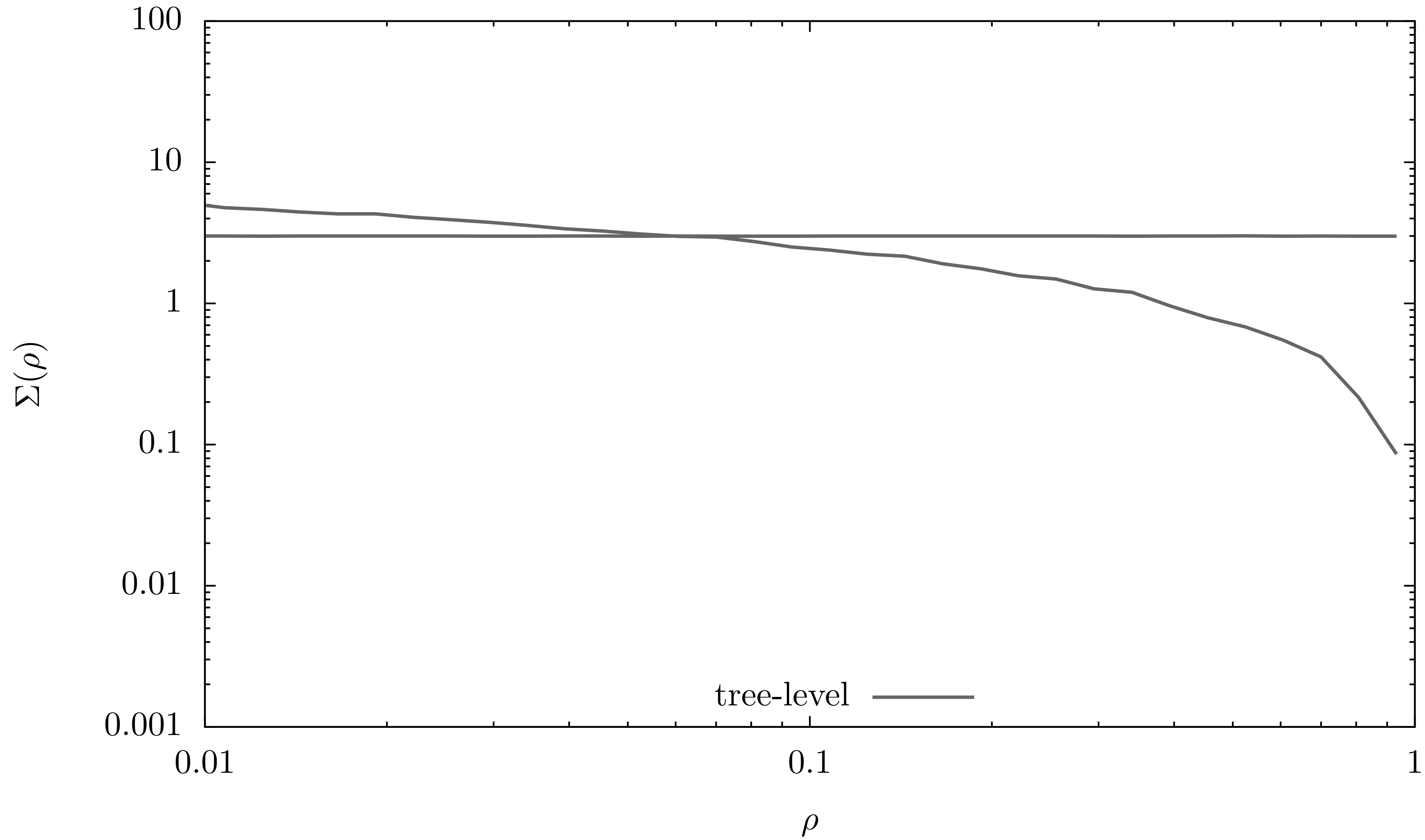
5th form factor of
Dokshitzer & Marchesini
Phys.Lett. B631 (2005) 118-125

$$\ln \mathbf{K}_{ab}\Big|_{\text{energy}} = \frac{\alpha_s}{\pi} \sum_{i} (\mathbb{T}_i^g)^2 \int_a^b \frac{\mathrm{d}E}{E} \int \frac{\mathrm{d}\Omega}{4\pi} \frac{2}{n_i \cdot n}$$

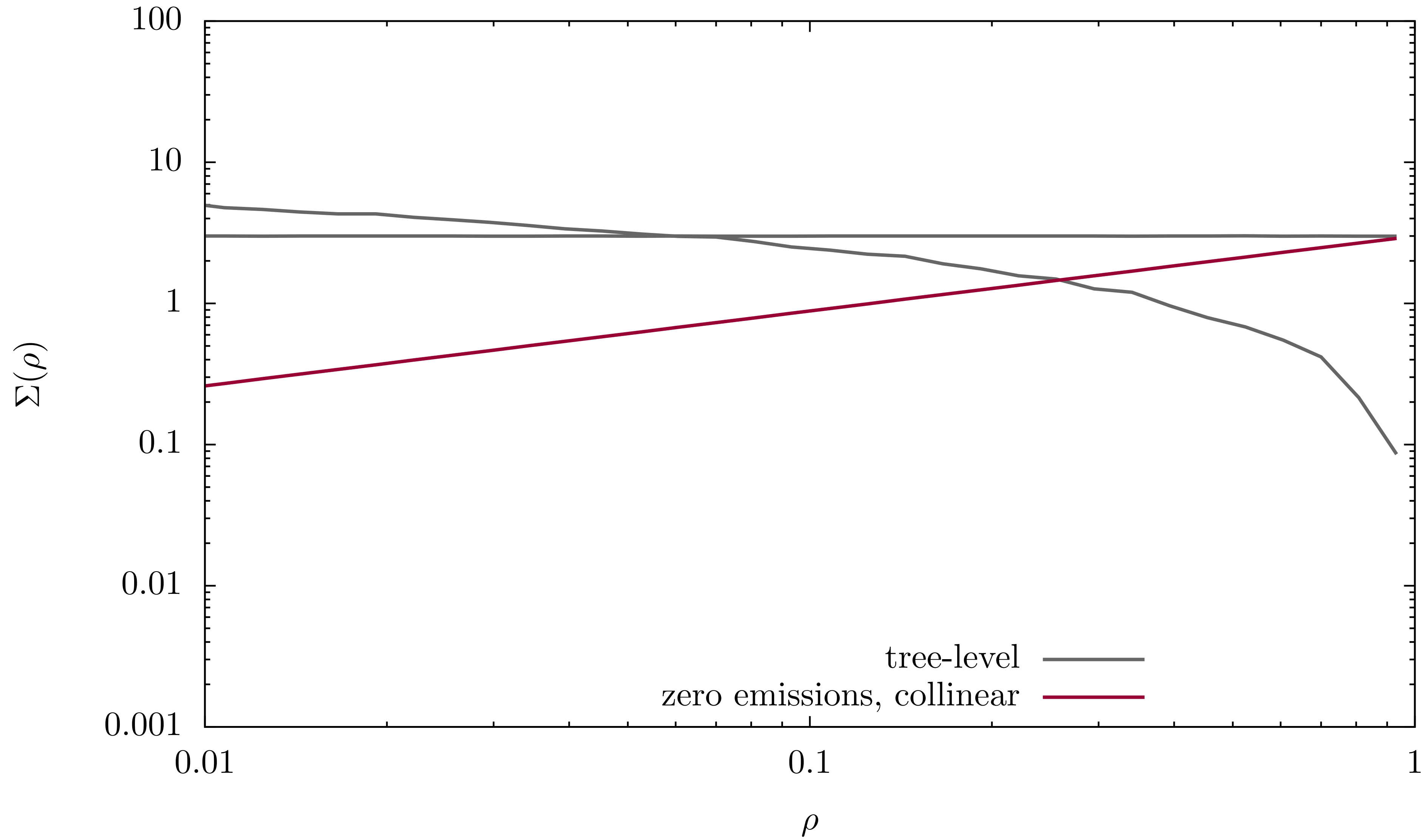
Collinear divergence isolated in abelian Sudakov



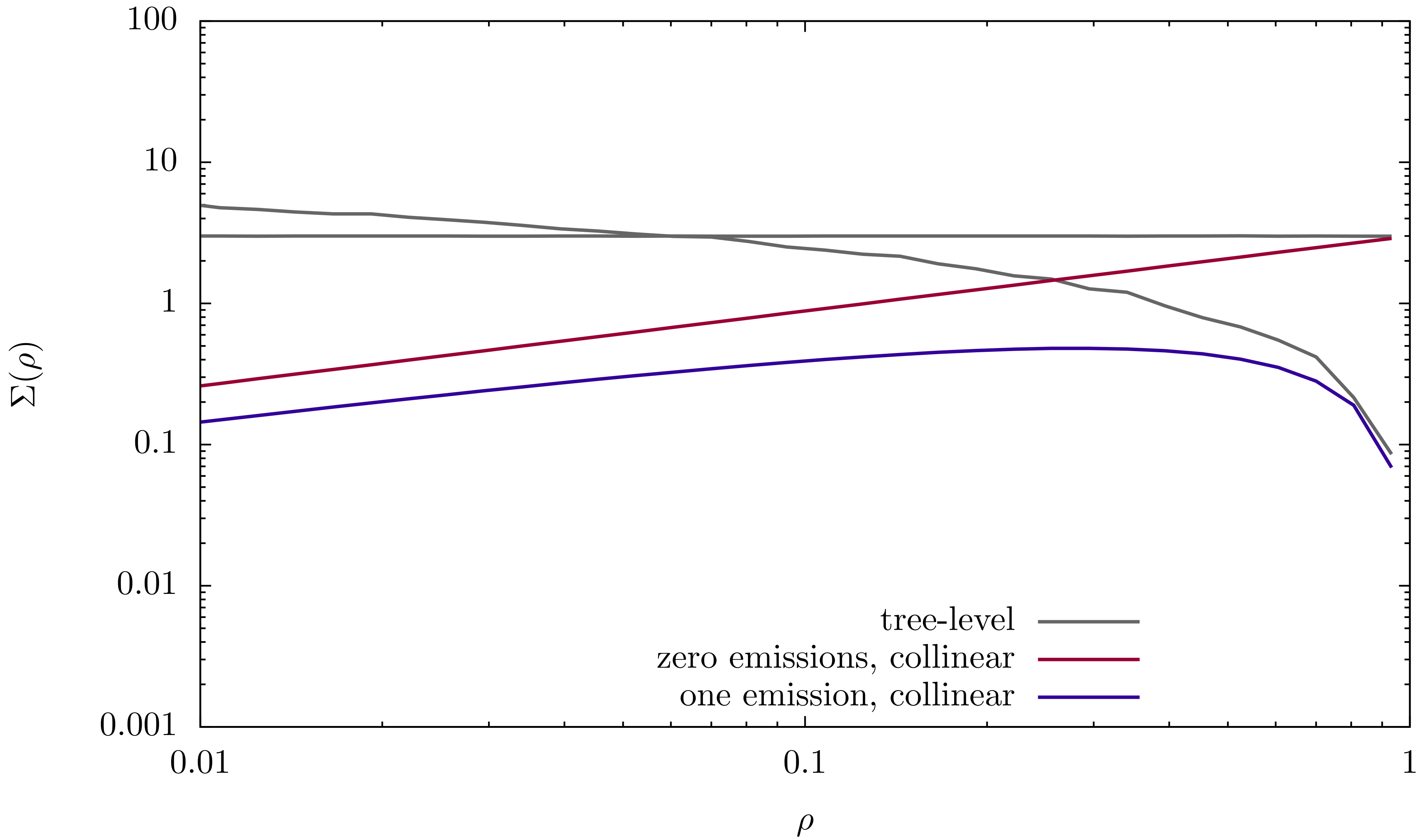
$$\rho = Q_0/Q$$



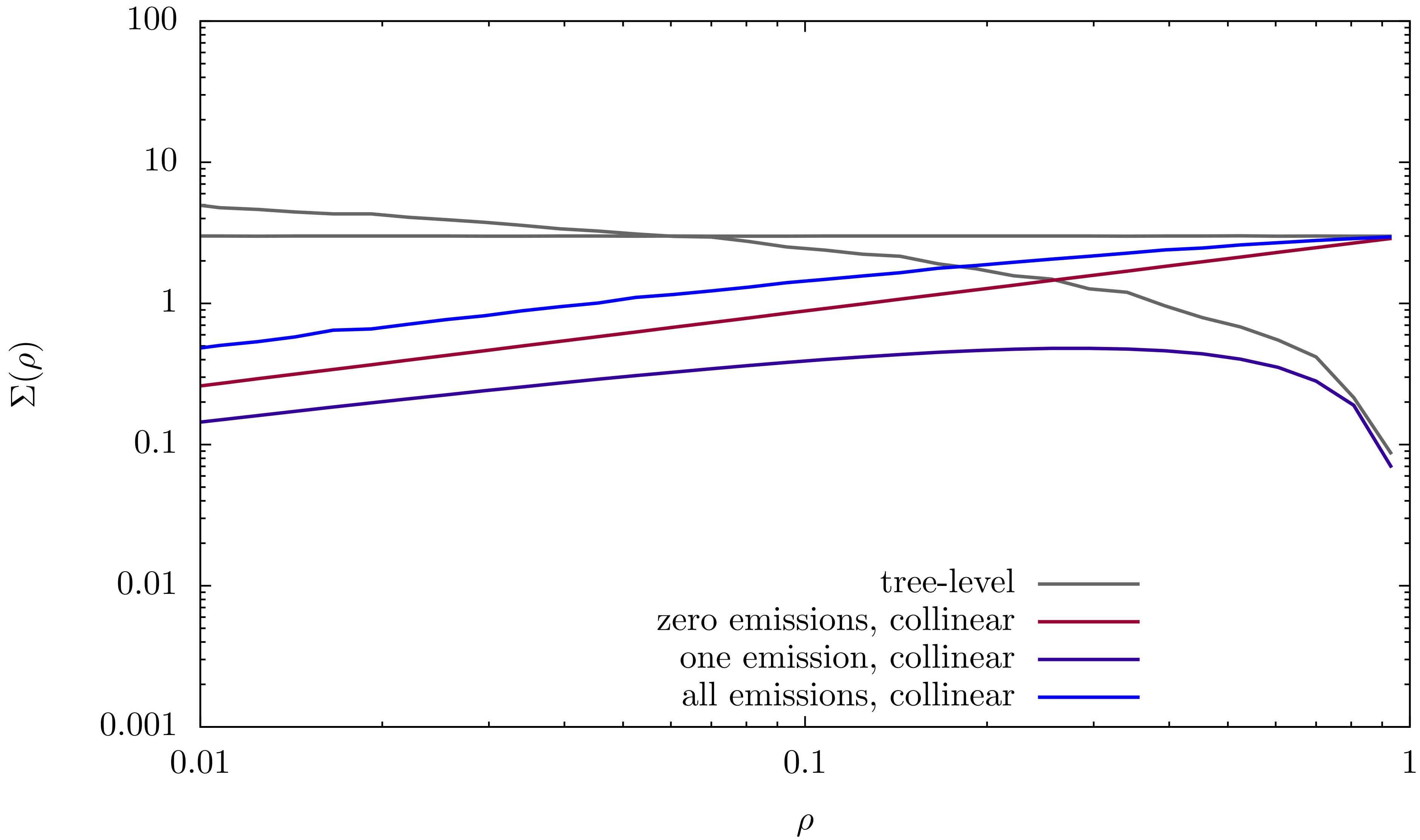
Zero and one emission



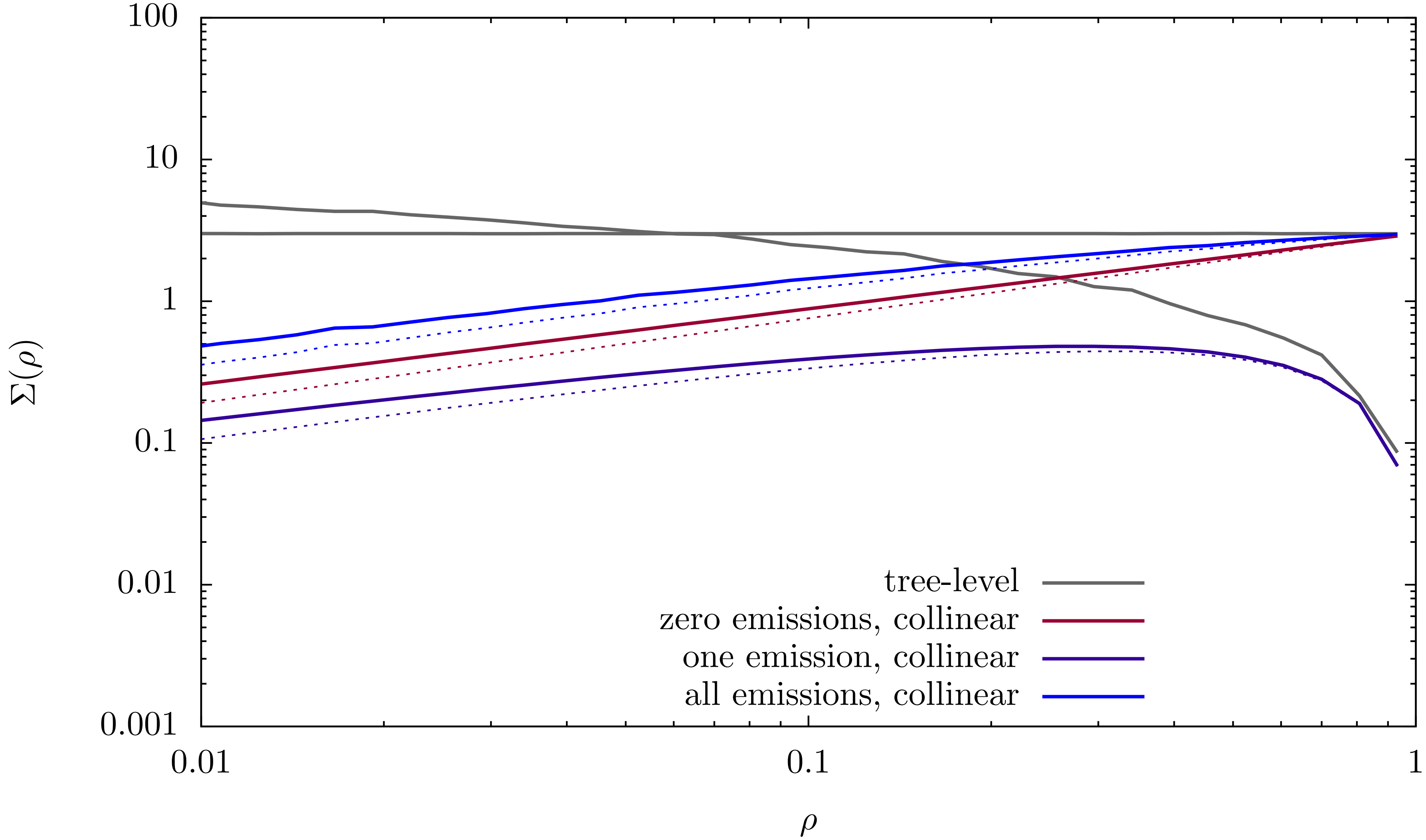
zero emission plus collinear Sudakov



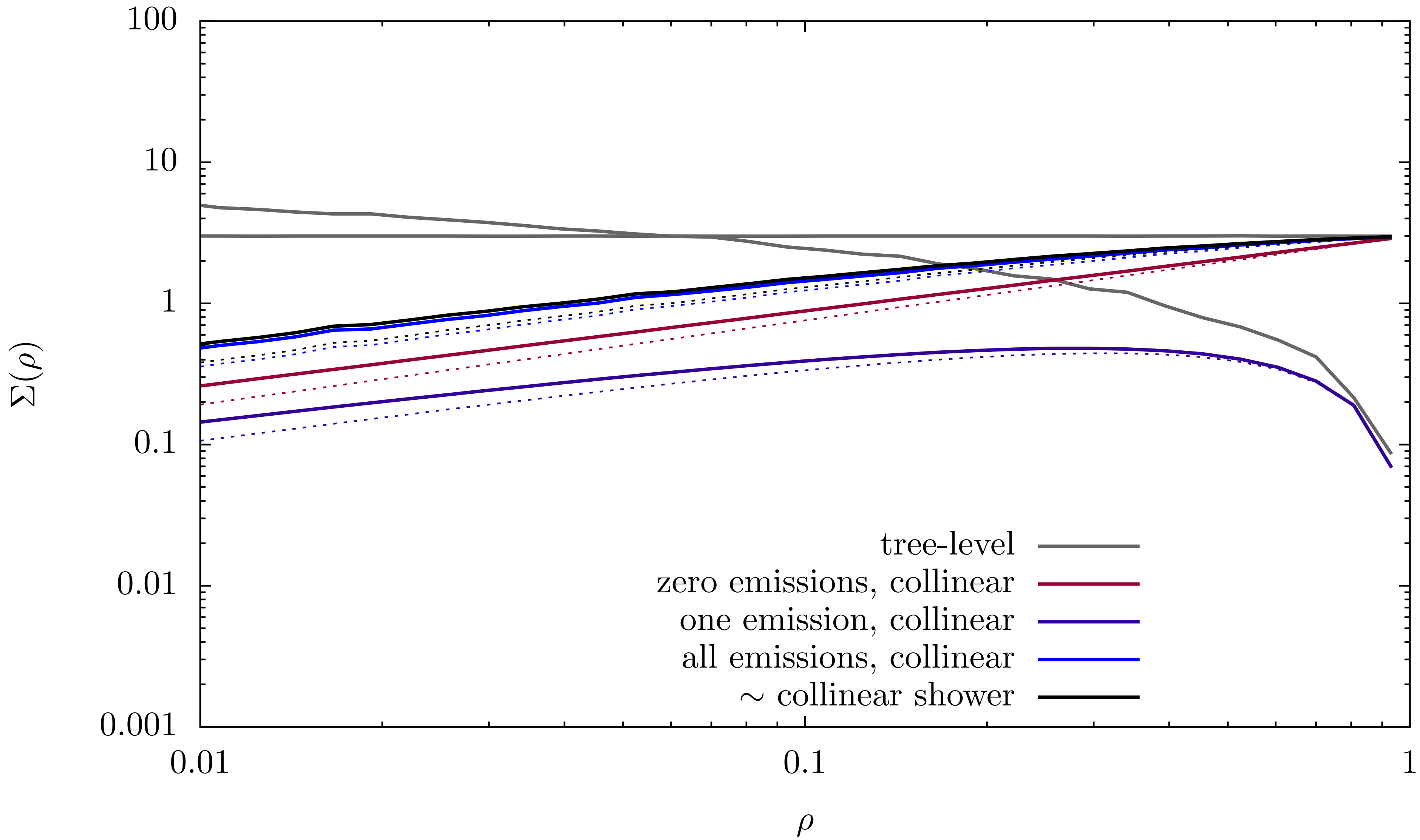
one emission plus collinear Sudakov



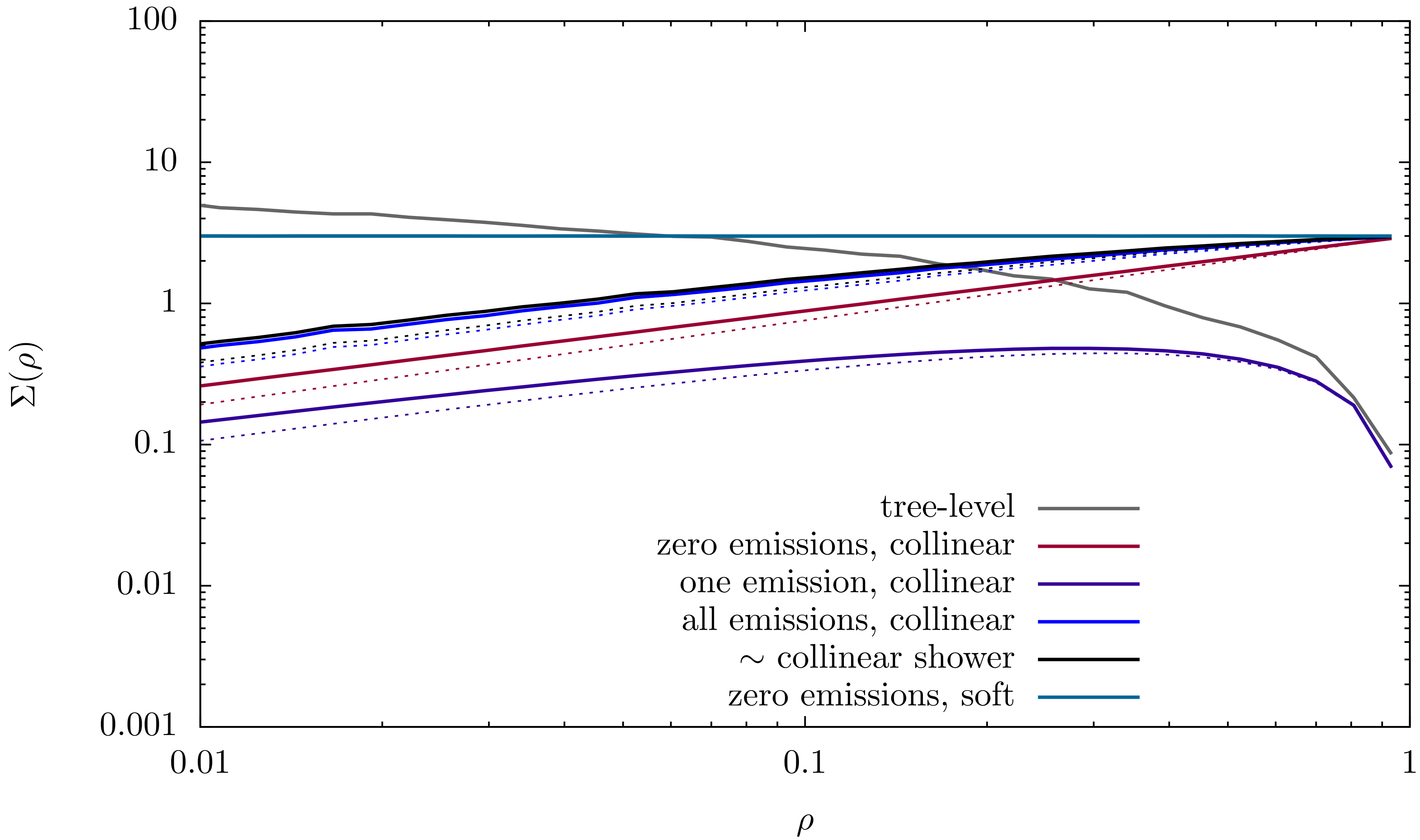
all emissions plus collinear Sudakov



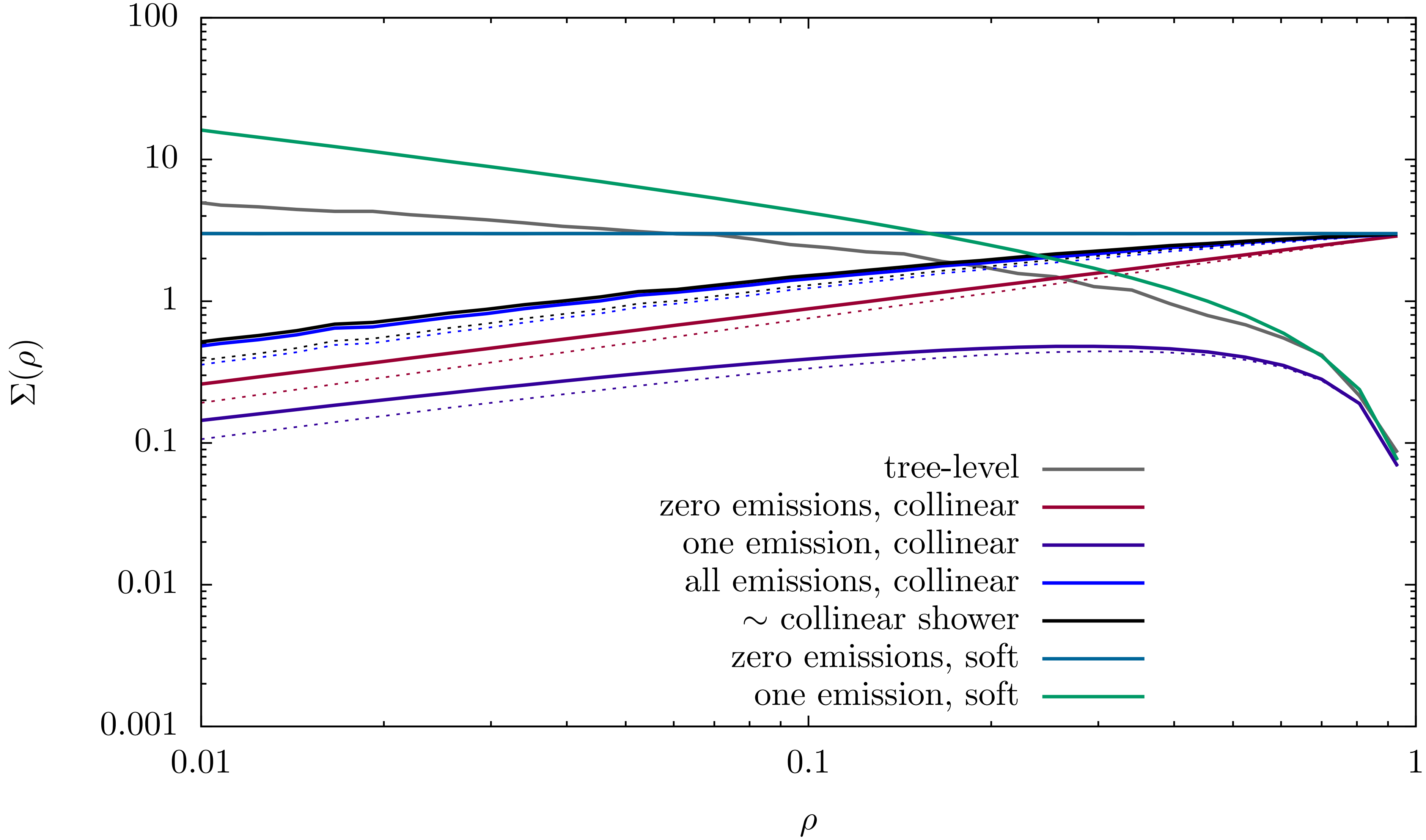
Strictly leading colour



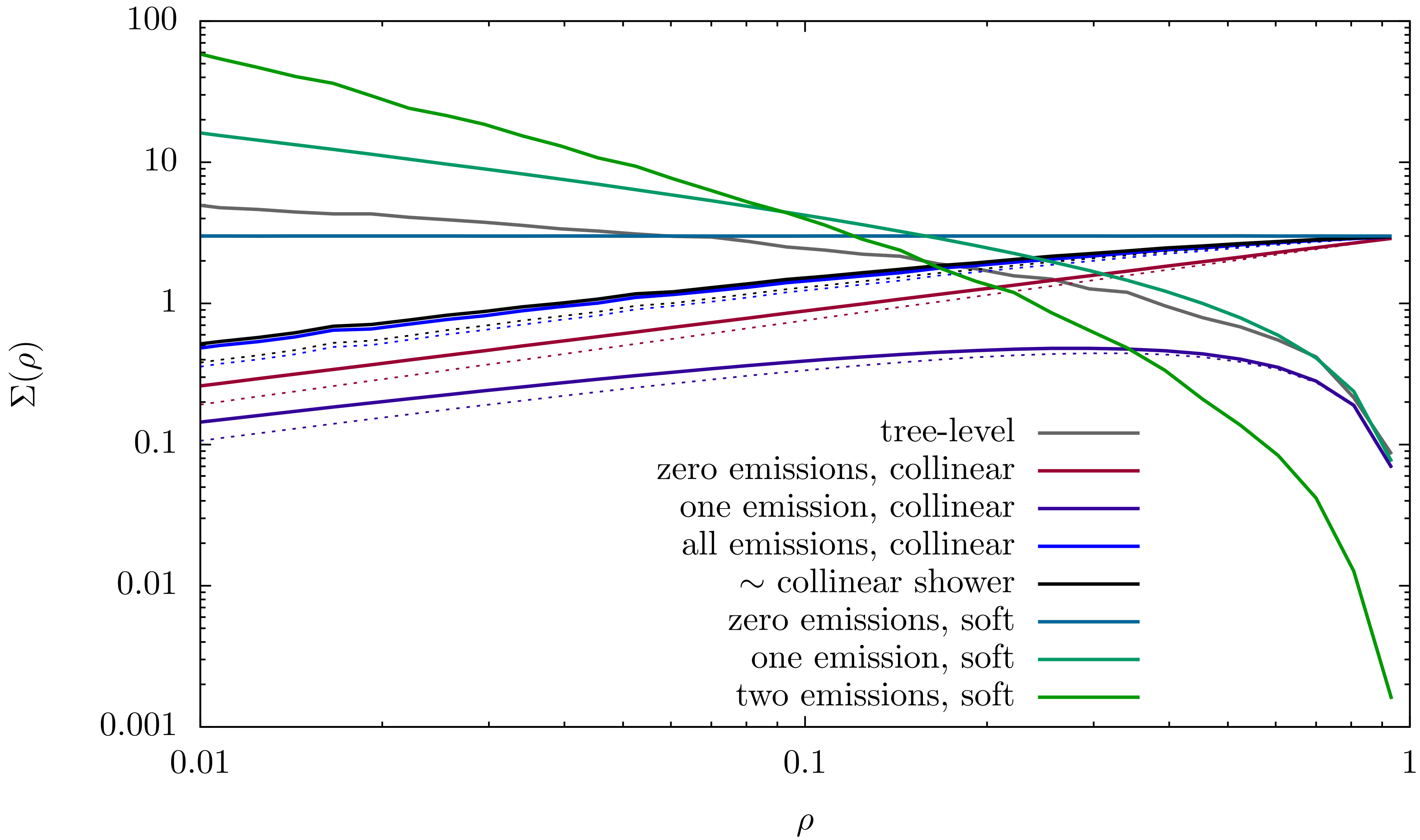
leading N in reals (~ Herwig et al)



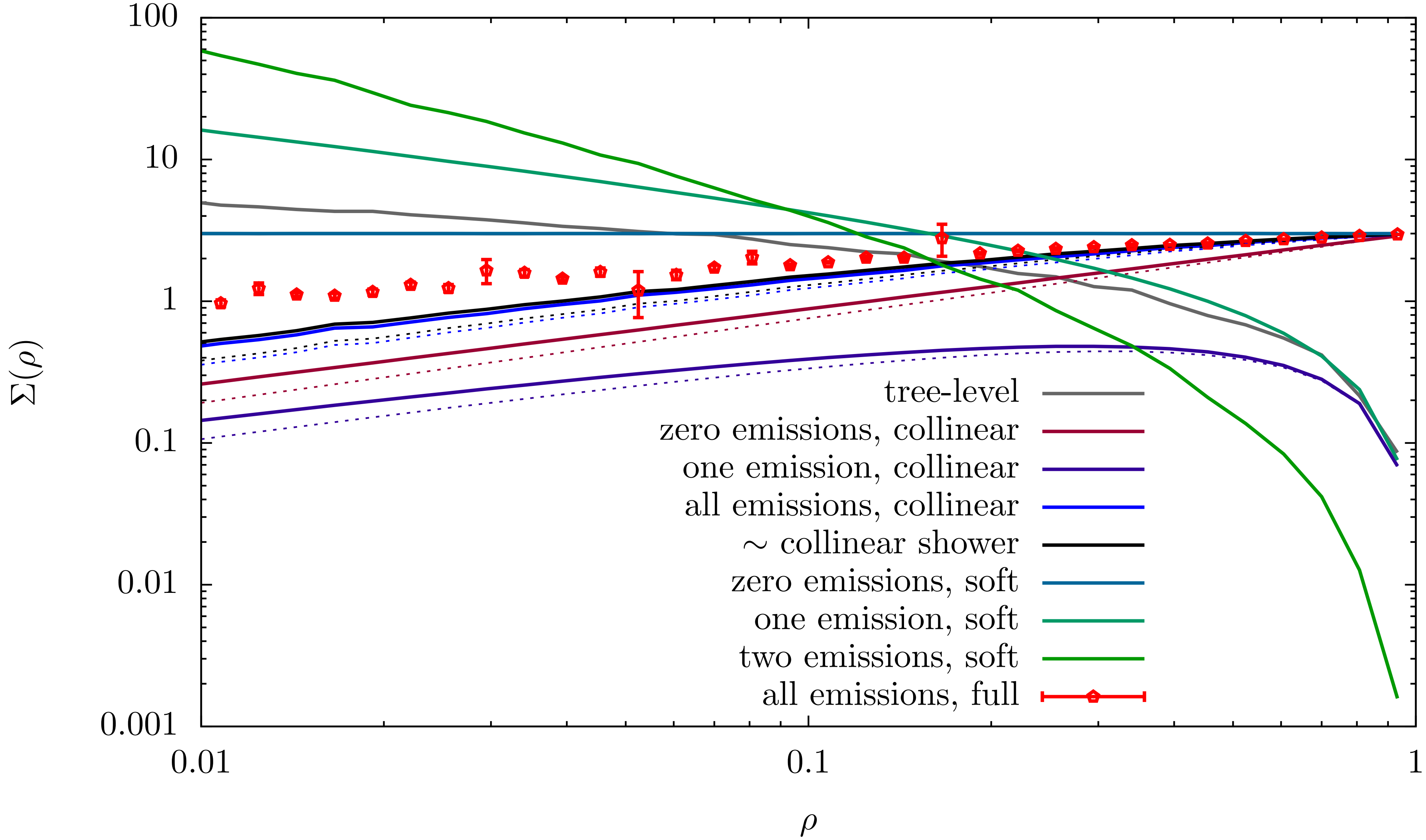
zero emissions + soft Sudakov only



one emission + soft Sudakov only



two emissions + soft Sudakov only



all emissions + soft Sudakov + collinear Sudakov

Going beyond the soft approximation

- Hard-collinear emissions / loops
- Recoils
- Spin

$$\mathrm{Tr}\left[\mathbf{V}_{\mu,q_{1\perp}}\,\mathbf{D}_1\,\mathbf{V}_{q_{1\perp},Q}\,|\mathcal{M}\rangle\langle\mathcal{M}|\,\mathbf{V}^\dagger_{q_{1\perp},Q}\,\mathbf{D}^\dagger_1\,\mathbf{V}^\dagger_{\mu,q_{1\perp}}\right] = \mathrm{d}\sigma_1$$

$$\boxed{\begin{aligned}\Sigma(\mu) &= \int \sum_n \mathrm{d}\sigma_n\, u_n(q_1, \ldots, q_n), \\ &= \int \sum_n \left(\prod_{i=1}^{n} \mathrm{d}\Pi_i\right) \mathrm{Tr}\, \mathbf{A}_n(\mu; \{p\}_n)\, u_n(q_1, \ldots, q_n),\end{aligned}}$$

JRF, Jack Holguin, Simon Plätzer: arXiv:1905.08686

$$
\begin{aligned}
d\sigma_0 &= \mathrm{Tr}\left(\mathbf{V}_{\mu,Q}\mathbf{H}(Q;\{p\})\mathbf{V}^\dagger_{\mu,Q}\right) = \mathrm{Tr}\,\mathbf{A}_0(\mu;\{p\}),\\
d\sigma_1 &= \int \prod_{i=1}^{n_\mathrm{H}+1} d^4p_i\,\mathrm{Tr}\left(\mathbf{V}_{\mu,q_{1\perp}}\mathbf{D}_1\mathbf{V}_{q_{1\perp},Q}\mathbf{H}(Q;\{p\})\mathbf{V}^\dagger_{q_{1\perp},Q}\mathbf{D}^\dagger_1\mathbf{V}^\dagger_{\mu,q_{1\perp}}\right) d\Pi_1\\
&= \mathrm{Tr}\,\mathbf{A}_1(\mu;\{\tilde{p}\}\cup q_1)\,d\Pi_1,\\
d\sigma_n &= \mathrm{Tr}\,\mathbf{A}_n(\mu;\{p\}_n)\prod_{i=1}^{n} d\Pi_i,
\end{aligned}
$$

$$
\boxed{...\mathbf{D}_i\mathcal{O}\mathbf{D}_i^\dagger... = ...\mathbf{S}_i\mathcal{O}\mathbf{S}_i^\dagger... + ...\mathbf{C}_i\mathcal{O}\mathbf{C}_i^\dagger....}
$$

$$
\boxed{
\begin{aligned}
\mathbf{S}_i &= \sum_j \left(\frac{q_{i\perp}^{(j\vec{m})}}{2\tilde{p}_j\cdot q_i}\mathbb{T}_j^g\otimes\left(\tilde{p}_j\cdot\epsilon_+^*(q_i)\mathbb{S}^{1_i}+\tilde{p}_j\cdot\epsilon_-^*(q_i)\mathbb{S}^{-1_i}\right)\right)\mathfrak{R}_{ij}^{\mathrm{soft}}(\{p\},\{\tilde{p}\},q_i),\\
\mathbf{C}_i &= \sum_j \frac{q_{i\perp}^{(j\vec{n})}}{2\sqrt{z_i}}\Delta_{ij}\,\overline{\mathbf{P}}_{ij}\,\mathfrak{R}_{ij}^{\mathrm{coll}}(\{p\},\{\tilde{p}\},q_i),
\end{aligned}
}
$$

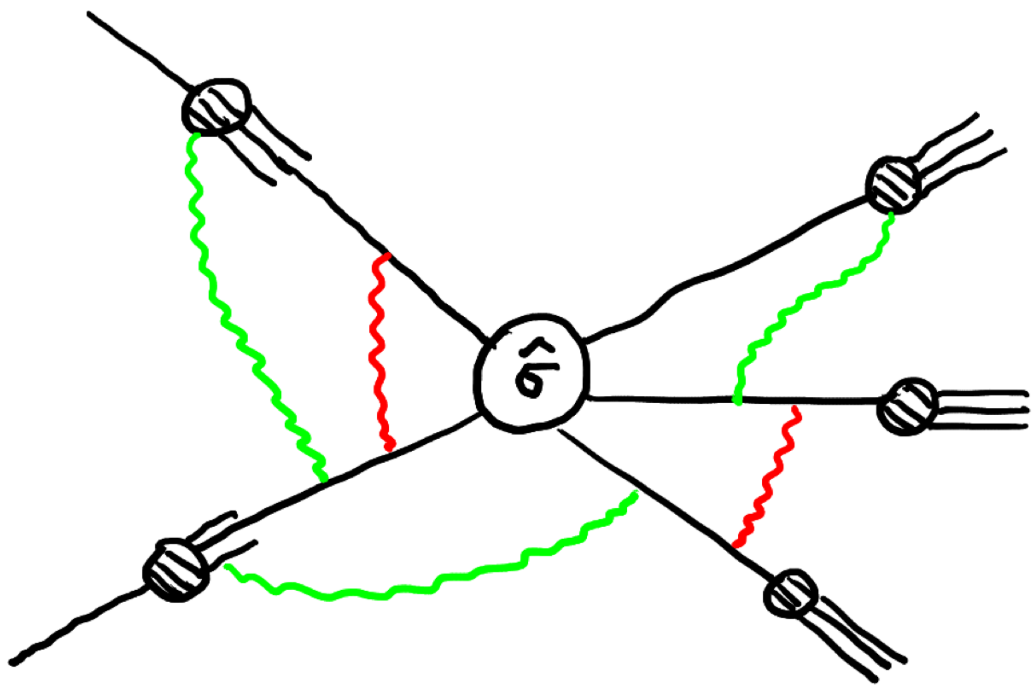
where

$$
\mathbf{A}_n(q_\perp;\{\tilde{p}\}_{n-1}\cup q_n) = \int \prod_{i=1}^{n_\mathrm{H}+n} d^4p_i\mathbf{V}_{q_\perp,q_{n\perp}}\mathbf{D}_n\mathbf{A}_{n-1}(q_{n\perp};\{p\}_{n-1})\mathbf{D}_n^\dagger\mathbf{V}^\dagger_{q_\perp,q_{n\perp}}\Theta(q_\perp\leq q_{n\perp}). \tag{2.2}
$$

$$
\begin{aligned}
\mathbf{P}_{ij} = \delta_{s_j,\frac{1}{2}}\delta_j^{\text{final}} \Bigg(& \sqrt{\frac{\mathcal{P}_{qq}}{2\mathcal{C}_{\mathrm{F}}(1+z_i^2)}}\frac{1}{\langle q_i\tilde{p}_j\rangle}(\mathbb{T}_j^g\otimes\mathbb{S}^{+1_i}) + \sqrt{\frac{z_i^2\mathcal{P}_{qq}}{2\mathcal{C}_{\mathrm{F}}(1+z_i^2)}}\frac{1}{[\tilde{p}_j q_i]}(\mathbb{T}_j^g\otimes\mathbb{S}^{-1_i}) \\
& + \sqrt{\frac{\mathcal{P}_{gq}}{2\mathcal{C}_{\mathrm{F}}(2-2z_i+z_i^2)}}\frac{1}{\langle\tilde{p}_j q_i\rangle}\mathbb{W}^{ij}(\mathbb{T}_j^g\otimes\mathbb{S}^{+1_i}) + \sqrt{\frac{(1-z_i)^2\mathcal{P}_{gq}}{2\mathcal{C}_{\mathrm{F}}(2-2z_i+z_i^2)}}\frac{1}{[q_i\tilde{p}_j]}\mathbb{W}^{ij}(\mathbb{T}_j^g\otimes\mathbb{S}^{-1_i})\Bigg) \\
+ \delta_{s_j,-\frac{1}{2}}\delta_j^{\text{final}} \Bigg(& \sqrt{\frac{\mathcal{P}_{qq}}{2\mathcal{C}_{\mathrm{F}}(1+z_i^2)}}\frac{1}{[\tilde{p}_j q_i]}(\mathbb{T}_j^g\otimes\mathbb{S}^{-1_i}) + \sqrt{\frac{z_i^2\mathcal{P}_{qq}}{2\mathcal{C}_{\mathrm{F}}(1+z_i^2)}}\frac{1}{\langle q_i\tilde{p}_j\rangle}(\mathbb{T}_j^g\otimes\mathbb{S}^{+1_i}) \\
& + \sqrt{\frac{\mathcal{P}_{gq}}{2\mathcal{C}_{\mathrm{F}}(2-2z_i+z_i^2)}}\frac{1}{[q_i\tilde{p}_j]}\mathbb{W}^{ij}(\mathbb{T}_j^g\otimes\mathbb{S}^{-1_i}) + \sqrt{\frac{(1-z_i)^2\mathcal{P}_{gq}}{2\mathcal{C}_{\mathrm{F}}(2-2z_i+z_i^2)}}\frac{1}{\langle\tilde{p}_j q_i\rangle}\mathbb{W}^{ij}(\mathbb{T}_j^g\otimes\mathbb{S}^{+1_i})\Bigg) \\
+ \delta_{s_j,1}\delta_j^{\text{final}} \Bigg(& \sqrt{\frac{(1-z_i)^2\mathcal{P}_{qg}}{2T_{\mathrm{R}}(1-2z_i(1-z_i))}}\frac{1}{[\tilde{p}_j q_i]}(\mathbb{W}^{ij}-1)(\mathbb{T}_j^g\otimes\mathbb{P}_j^1\mathbb{P}_j^2\mathbb{S}^{+\frac{1}{2}_i}) \\
& + \sqrt{\frac{z_i^2\mathcal{P}_{qg}}{2T_{\mathrm{R}}(1-2z_i(1-z_i))}}\frac{1}{[\tilde{p}_j q_i]}(\mathbb{W}^{ij}-1)(\mathbb{T}_j^g\otimes\mathbb{P}_j^2\mathbb{S}^{-\frac{1}{2}_i}) \\
& + \sqrt{\frac{\mathcal{P}_{gg}}{2\mathcal{C}_{\mathrm{A}}(1-z_i+z_i^2)^2}}\frac{1}{\langle q_i\tilde{p}_j\rangle}(\mathbb{T}_j^g\otimes\mathbb{S}^{+1_i}) \\
& + \sqrt{\frac{z_i^4\mathcal{P}_{gg}}{2\mathcal{C}_{\mathrm{A}}(1-z_i+z_i^2)^2}}\frac{1}{[q_i\tilde{p}_j]}(\mathbb{T}_j^g\otimes\mathbb{S}^{-1_i}) + \sqrt{\frac{\mathcal{P}_{gg}(1-z_i)^4}{2\mathcal{C}_{\mathrm{A}}(1-z_i+z_i^2)^2}}\frac{1}{[\tilde{p}_j q_i]}(\mathbb{T}_j^g\otimes\mathbb{P}_j^1\mathbb{S}^{+1_i})\Bigg) \\
+ \delta_{s_j,-1}\delta_j^{\text{final}} \Bigg(& \sqrt{\frac{(1-z_i)^2\mathcal{P}_{qg}}{2T_{\mathrm{R}}(1-2z_i(1-z_i))}}\frac{1}{\langle q_i\tilde{p}_j\rangle}(\mathbb{W}^{ij}-1)(\mathbb{T}_j^g\otimes\mathbb{P}_j^1\mathbb{P}_j^2\mathbb{S}^{-\frac{1}{2}_i}) \\
& + \sqrt{\frac{z_i^2\mathcal{P}_{qg}}{2T_{\mathrm{R}}(1-2z_i(1-z_i))}}\frac{1}{\langle q_i\tilde{p}_j\rangle}(\mathbb{W}^{ij}-1)(\mathbb{T}_j^g\otimes\mathbb{P}_j^2\mathbb{S}^{+\frac{1}{2}_i}) \\
& + \sqrt{\frac{\mathcal{P}_{gg}}{2\mathcal{C}_{\mathrm{A}}(1-z_i+z_i^2)^2}}\frac{1}{[\tilde{p}_j q_i]}(\mathbb{T}_j^g\otimes\mathbb{S}^{-1_i}) \\
& + \sqrt{\frac{z_i^4\mathcal{P}_{gg}}{2\mathcal{C}_{\mathrm{A}}(1-z_i+z_i^2)^2}}\frac{1}{\langle\tilde{p}_j q_i\rangle}(\mathbb{T}_j^g\otimes\mathbb{S}^{+1_i}) + \sqrt{\frac{\mathcal{P}_{gg}(1-z_i)^4}{2\mathcal{C}_{\mathrm{A}}(1-z_i+z_i^2)^2}}\frac{1}{\langle q_i\tilde{p}_j\rangle}(\mathbb{T}_j^g\otimes\mathbb{P}_j^1\mathbb{S}^{-1_i})\Bigg) \\
+ \delta_{s_j,\frac{1}{2}}\delta_j^{\text{initial}}\sqrt{\frac{1}{z_i}} \Bigg(& \sqrt{\frac{\mathcal{P}_{qq}}{\mathcal{C}_{\mathrm{F}}(1+z_i^2)}}\frac{1}{\langle q_i p_j\rangle}(\mathbb{T}_j^g\otimes\mathbb{S}^{+1_i}) + \sqrt{\frac{z_i^2\mathcal{P}_{qq}}{\mathcal{C}_{\mathrm{F}}(1+z_i^2)}}\frac{1}{[p_j q_i]}(\mathbb{T}_j^g\otimes\mathbb{S}^{-1_i}) \\
& + \sqrt{\frac{(1-z_i)^2\mathcal{P}_{qg}}{n_f\mathcal{C}_F(1-2z_i(1-z_i))}}\frac{1}{[p_j q_i]}\mathbb{W}^{ij}(\mathbb{T}_j^g\otimes\mathbb{S}^{+1_i}) \\
& + \sqrt{\frac{z_i^2\mathcal{P}_{qg}}{n_f\mathcal{C}_F(1-2z_i(1-z_i))}}\frac{1}{\langle q_i p_j\rangle}\mathbb{W}^{ij}(\mathbb{T}_j^g\otimes\mathbb{S}^{-1_i})\Bigg) \\
+ \delta_{s_j,-\frac{1}{2}}\delta_j^{\text{initial}}\sqrt{\frac{1}{z_i}} \Bigg(& \sqrt{\frac{\mathcal{P}_{qq}}{\mathcal{C}_{\mathrm{F}}(1+z_i^2)}}\frac{1}{[p_j q_i]}(\mathbb{T}_j^g\otimes\mathbb{S}^{-1_i}) + \sqrt{\frac{z_i^2\mathcal{P}_{qq}}{\mathcal{C}_{\mathrm{F}}(1+z_i^2)}}\frac{1}{\langle q_i p_j\rangle}(\mathbb{T}_j^g\otimes\mathbb{S}^{+1_i}) \\
& + \sqrt{\frac{(1-z_i)^2\mathcal{P}_{qg}}{n_f\mathcal{C}_F(1-2z_i(1-z_i))}}\frac{1}{\langle q_i p_j\rangle}\mathbb{W}^{ij}(\mathbb{T}_j^g\otimes\mathbb{S}^{-1_i}) \\
& + \sqrt{\frac{z_i^2\mathcal{P}_{qg}}{n_f\mathcal{C}_F(1-2z_i(1-z_i))}}\frac{1}{[p_j q_i]}\mathbb{W}^{ij}(\mathbb{T}_j^g\otimes\mathbb{S}^{+1_i})\Bigg) \\
+ \delta_{s_j,1}\delta_j^{\text{initial}}\sqrt{\frac{1}{z_i}} \Bigg(& \sqrt{\frac{2n_f\mathcal{P}_{gq}}{T_{\mathrm{R}}(2-2z_i+z_i^2)}}\frac{1}{\langle p_j q_i\rangle}(\mathbb{T}_j^q\otimes\mathbb{P}_j^2\mathbb{S}^{+\frac{1}{2}_i}) \\
& + \sqrt{\frac{2n_f(1-z_i)^2\mathcal{P}_{gq}}{T_{\mathrm{R}}(2-2z_i+z_i^2)}}\frac{1}{[q_i p_j]}(\mathbb{T}_j^q\otimes\mathbb{P}_j^1\mathbb{P}_j^2\mathbb{S}^{-\frac{1}{2}_i}) + \sqrt{\frac{\mathcal{P}_{gg}}{\mathcal{C}_{\mathrm{A}}(1-z_i+z_i^2)^2}}\frac{1}{\langle q_i p_j\rangle}(\mathbb{T}_j^g\otimes\mathbb{S}^{+1_i}) \\
& + \sqrt{\frac{z_i^4\mathcal{P}_{gg}}{\mathcal{C}_{\mathrm{A}}(1-z_i+z_i^2)^2}}\frac{1}{[q_i p_j]}(\mathbb{T}_j^g\otimes\mathbb{S}^{-1_i}) + \sqrt{\frac{\mathcal{P}_{gg}(1-z_i)^4}{\mathcal{C}_{\mathrm{A}}(1-z_i+z_i^2)^2}}\frac{1}{\langle q_i p_j\rangle}(\mathbb{T}_j^g\otimes\mathbb{P}_j^1\mathbb{S}^{-1_i})\Bigg) \\
+ \delta_{s_j,-1}\delta_j^{\text{initial}}\sqrt{\frac{1}{z_i}} \Bigg(& \sqrt{\frac{2n_f\mathcal{P}_{gq}}{T_{\mathrm{R}}(2-2z_i+z_i^2)}}\frac{1}{[q_i p_j]}(\mathbb{T}_j^q\otimes\mathbb{P}_j^2\mathbb{S}^{-\frac{1}{2}_i}) \\
& + \sqrt{\frac{2n_f(1-z_i)^2\mathcal{P}_{gq}}{T_{\mathrm{R}}(2-2z_i+z_i^2)}}\frac{1}{\langle p_j q_i\rangle}(\mathbb{T}_j^q\otimes\mathbb{P}_j^1\mathbb{P}_j^2\mathbb{S}^{+\frac{1}{2}_i}) + \sqrt{\frac{\mathcal{P}_{gg}}{\mathcal{C}_{\mathrm{A}}(1-z_i+z_i^2)^2}}\frac{1}{[p_j q_i]}(\mathbb{T}_j^g\otimes\mathbb{S}^{-1_i}) \\
& + \sqrt{\frac{z_i^4\mathcal{P}_{gg}}{\mathcal{C}_{\mathrm{A}}(1-z_i+z_i^2)^2}}\frac{1}{\langle p_j q_i\rangle}(\mathbb{T}_j^g\otimes\mathbb{S}^{+1_i}) + \sqrt{\frac{\mathcal{P}_{gg}(1-z_i)^4}{\mathcal{C}_{\mathrm{A}}(1-z_i+z_i^2)^2}}\frac{1}{[p_j q_i]}(\mathbb{T}_j^g\otimes\mathbb{P}_j^1\mathbb{S}^{+1_i})\Bigg).
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
\overline{\mathcal{P}}_{qq} &= \mathcal{P}_{qq} - 2\mathcal{C}_{\mathrm{F}}\frac{1}{1-z} = -\mathcal{C}_{\mathrm{F}}(1+z), \\
\overline{\mathcal{P}}_{gg}^{\text{initial}} &= \mathcal{P}_{gg} - 2\mathcal{C}_{\mathrm{A}}\frac{1}{1-z} = 2\mathcal{C}_{\mathrm{A}}\left(\frac{1}{z} + z(1-z) - 2\right) \\
\overline{\mathcal{P}}_{gg}^{\text{final}} &= \mathcal{P}_{gg} - 2\mathcal{C}_{\mathrm{A}}\frac{1}{1-z} - 2\mathcal{C}_{\mathrm{A}}\frac{1}{z} = 2\mathcal{C}_{\mathrm{A}}\left(z(1-z) - 2\right) \\
\overline{\mathcal{P}}_{gq}^{\text{final}} &= \mathcal{P}_{gq} - 2\mathcal{C}_{\mathrm{F}}\frac{1}{z} = \mathcal{C}_{\mathrm{F}}\left(\frac{1+(1-z)^2}{z} - \frac{2}{z}\right), \qquad \overline{\mathcal{P}}_{gq}^{\text{initial}} = \mathcal{P}_{gq}, \\
\overline{\mathcal{P}}_{qg} &= \mathcal{P}_{qg}.
\end{aligned}
$$

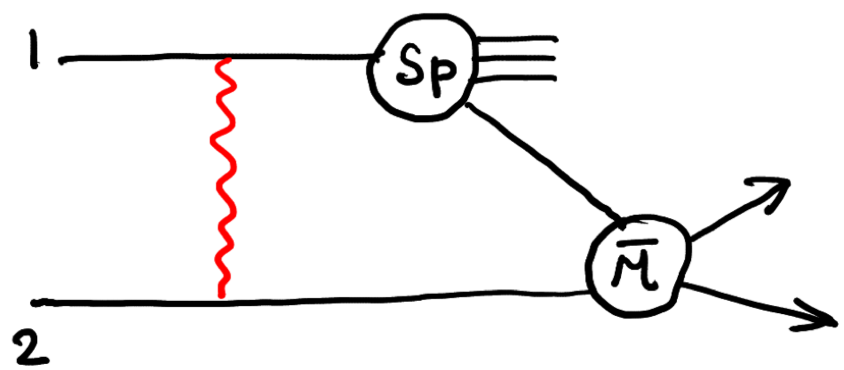
Factorization



Coherence allows us to "unhook" eikonal gluons & recover collinear factorization.

But it fails for Coulomb gluons.

One loop



$$|\mathcal{M}^{(1)}\rangle = \mathrm{Sp}\,|\bar{\mathcal{M}}\rangle$$

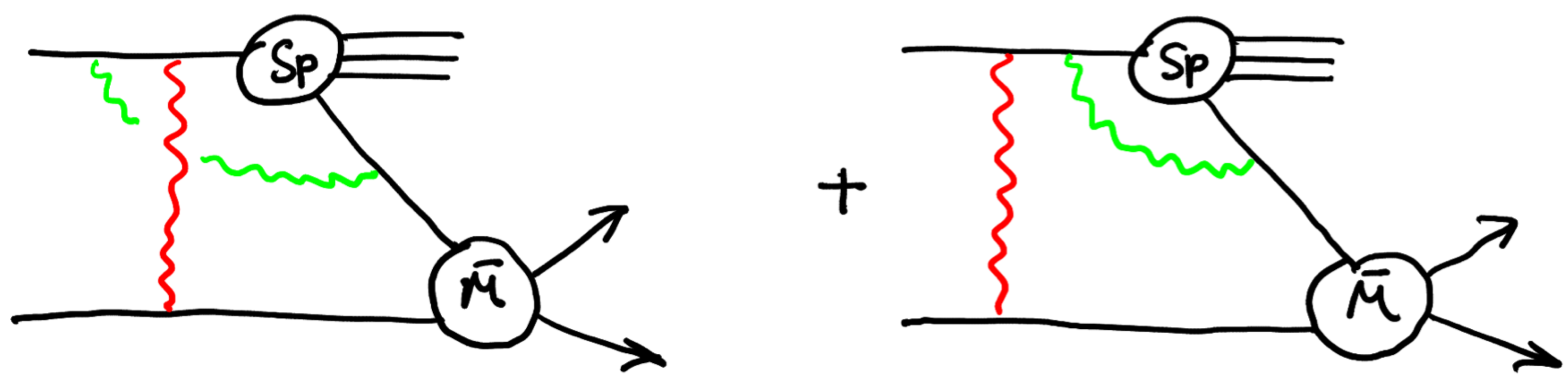
$$\mathrm{Sp} \sim i\pi\, \underline{T}_1 \cdot \underline{T}_2 = \Delta_c$$

But $\sigma \sim \langle \mathcal{M}^{(1)} | \mathcal{M}^{(0)} \rangle + \langle \mathcal{M}^{(0)} | \mathcal{M}^{(1)} \rangle$

So no breakdown of collinear factorization

$\because$ Δ_c is anti-hermitian

Two loops



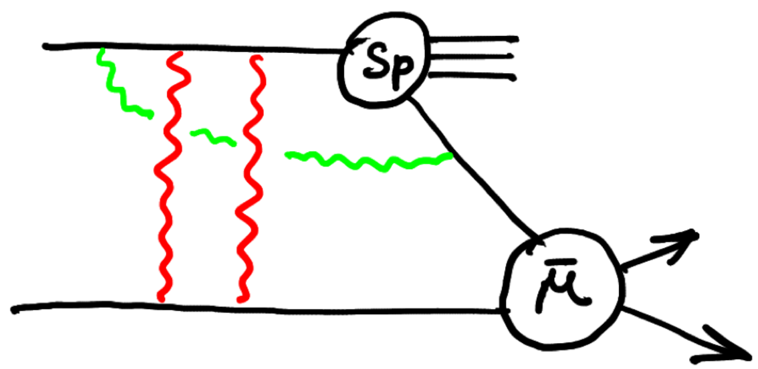
$$\mathrm{Sp} \sim [\,I,\ \Delta_c\,]$$

But $\sigma \sim \mathrm{Tr}\left(H\,[I, \Delta_c]\right) = 0$ in QCD

$\because$ symmetric anti-symmetric

$$H = |\overline{\mathcal{M}}\rangle\langle\overline{\mathcal{M}}|$$

Three loops



$+ \ \cdots\cdots$

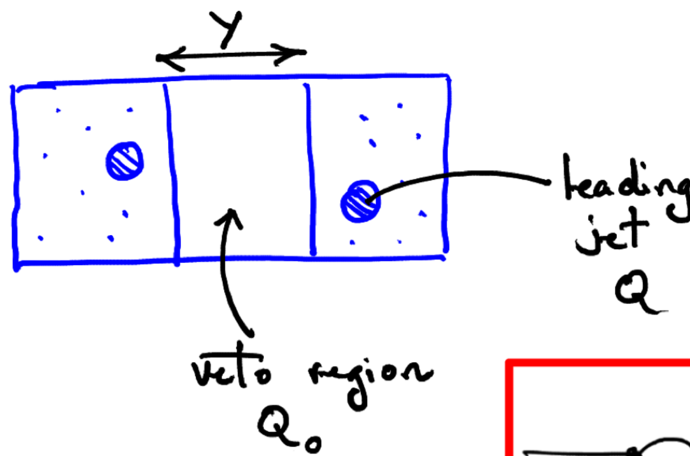
$$\mathrm{Sp} \sim [\Delta_c, [\Delta_c, I]]$$

No escape!

Only cancels if we integrate over the full green (“eikonal”) gluon phase-space, i.e. no breakdown of the factorization theorems as originally proven.

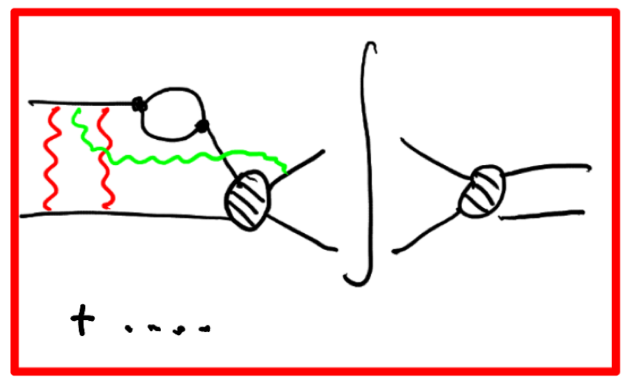
JRF, Kyrieleis, Seymour, arXiv:08081269
JRF, Seymour, Siodmok, arXiv: 1206.6363
Catani, de Florian, Rodrigo: arXiv: 1112.4405

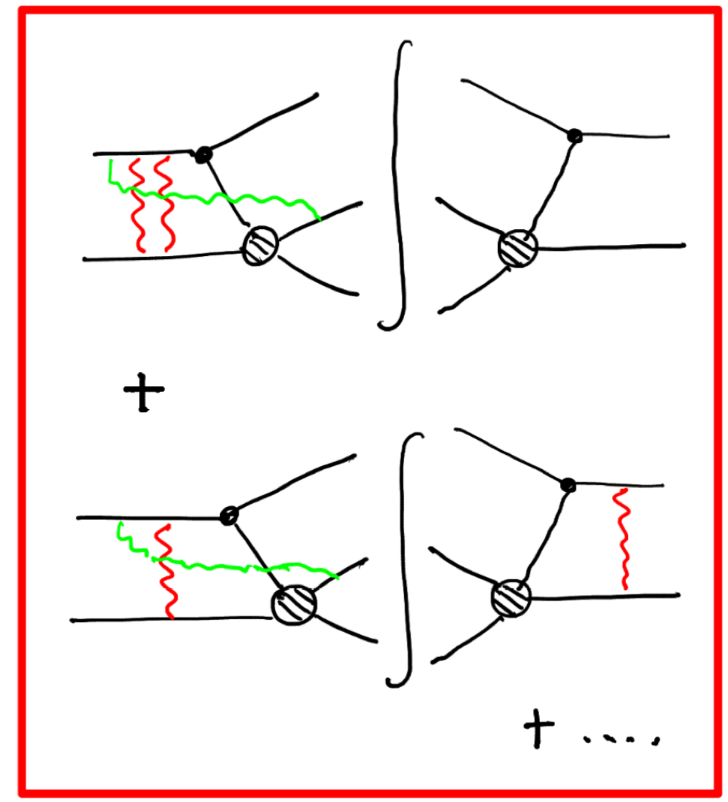
The impact on "gaps between jets"



$$\sigma = \sigma_0 + \sigma_1 + \sigma_2 + \ldots\ldots$$

$$\sigma_1 = \sigma_1^V - \sigma_1^R$$





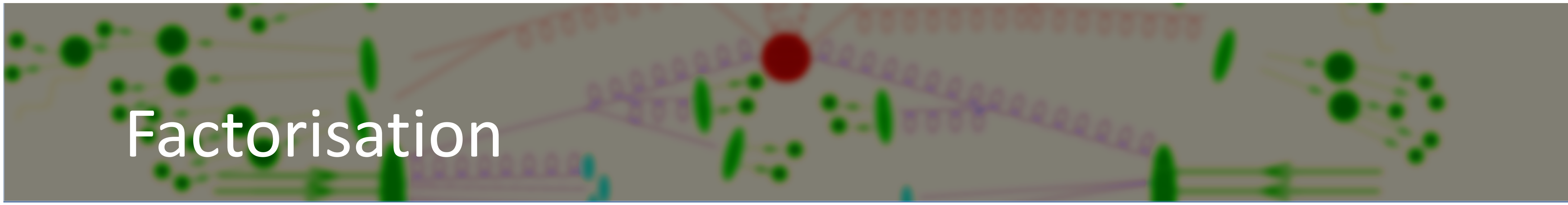
$$\sigma_1 \sim \frac{\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_\perp}{k_\perp}\int^{\ln Q/k_T} dy \times \pi^2 Y \left[\frac{\alpha_s}{\pi}\int_{Q_0}^{k_T}\frac{dk_\perp'}{k_\perp'}\right]^3$$

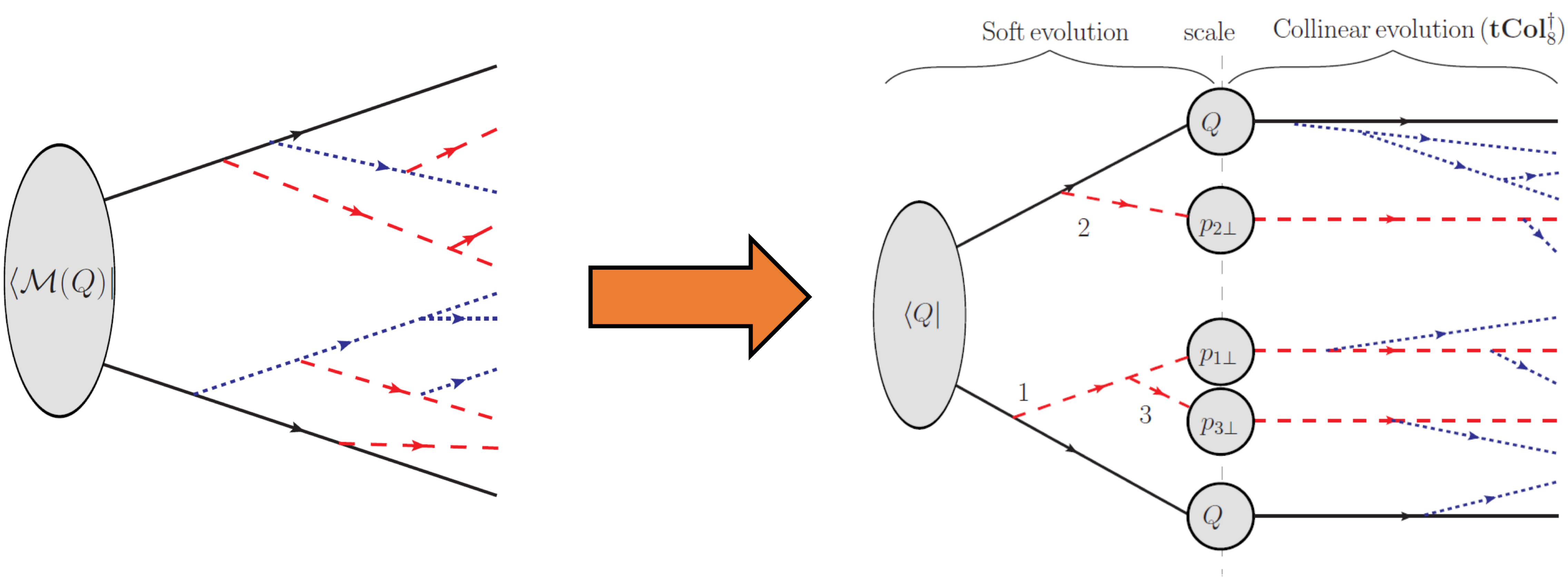
$$\times \left\{ t_1^2 \left[\left[t_t^2, \Delta_c\right], \Delta_c\right] - t_1^\dagger \left[\left[t_t^2, \Delta_c\right], \Delta_c\right] t_1 \right\}$$

$$\sigma_1 \sim \alpha_s^4 \log^5\left(\frac{Q}{Q_0}\right)\pi^2 Y$$

JRF, Kyrieleis, Seymour hep-ph/0604094

Ignoring Coulomb/Glauber exchanges



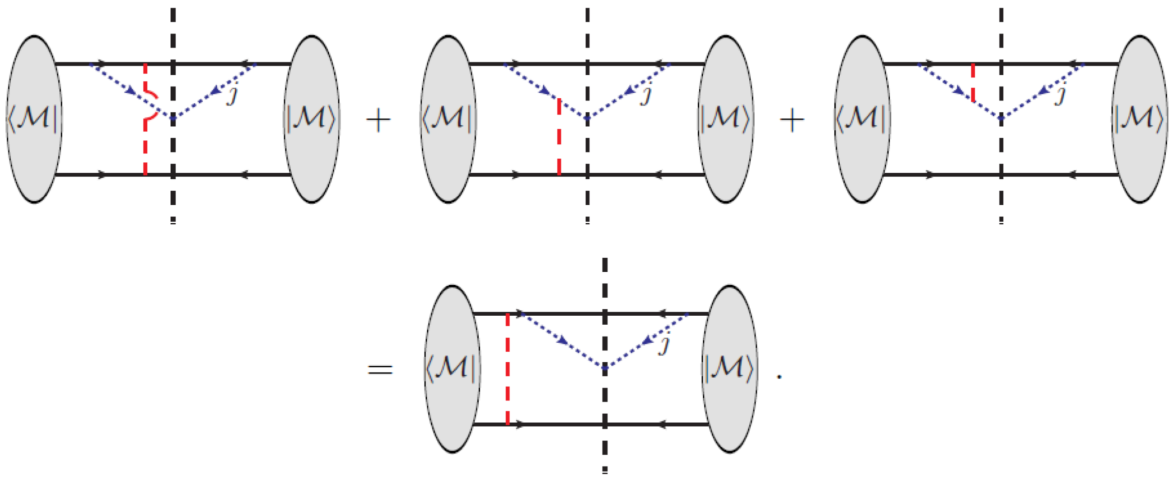


Factorisation

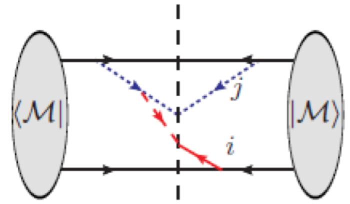
$$\left[\mathbf{D}_i - \overline{\mathbf{C}}_i, \overline{\mathbf{C}}_j\right] \simeq 0, \qquad \left[\mathbf{V}_{a,b}(\mathbf{V}^{\mathrm{col}}_{a,b})^{-1}, \overline{\mathbf{C}}_j\right] \simeq 0,$$

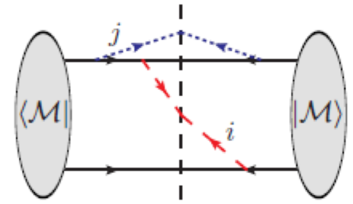
$$\left[\mathbf{V}_{a,b}(\mathbf{V}^{\mathrm{col}}_{a,b})^{-1}, \mathbf{V}^{\mathrm{col}}_{c,d}\right] \simeq 0, \qquad \left[\mathbf{D}_i - \overline{\mathbf{C}}_i, \mathbf{V}^{\mathrm{col}}_{a,b}\right] \simeq 0.$$

The equality only holds when considering only the real part of these diagrams. The soft loop also generates imaginary parts; Coulomb/Glauber exchanges.

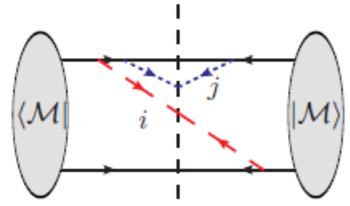


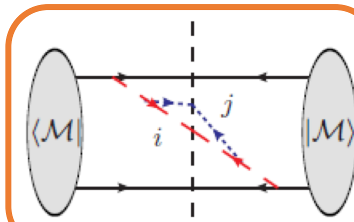
Factorisation

[

+ 

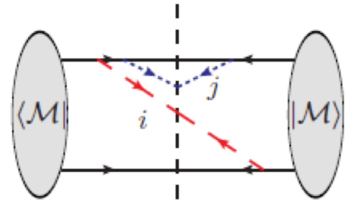
] $\Theta(q_{j\perp} - q_{i\perp})\Theta(q_{i\perp} - \mu)\Theta(Q - q_{j\perp})$

+ [

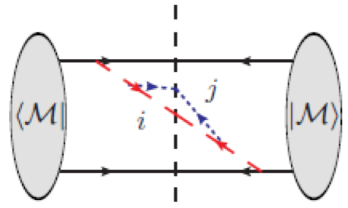
+ 

] $\Theta(q_{i\perp} - q_{j\perp})\Theta(q_{j\perp} - \mu)\Theta(Q - q_{i\perp})$

(3.32)

= 

$\Theta(q_{i\perp} - \mu)\Theta(Q - q_{i\perp})\Theta(q_{j\perp} - \mu)\Theta(Q - q_{j\perp})$

+ 

$\Theta(q_{i\perp} - q_{j\perp})\Theta(q_{j\perp} - \mu)\Theta(Q - q_{i\perp})$.

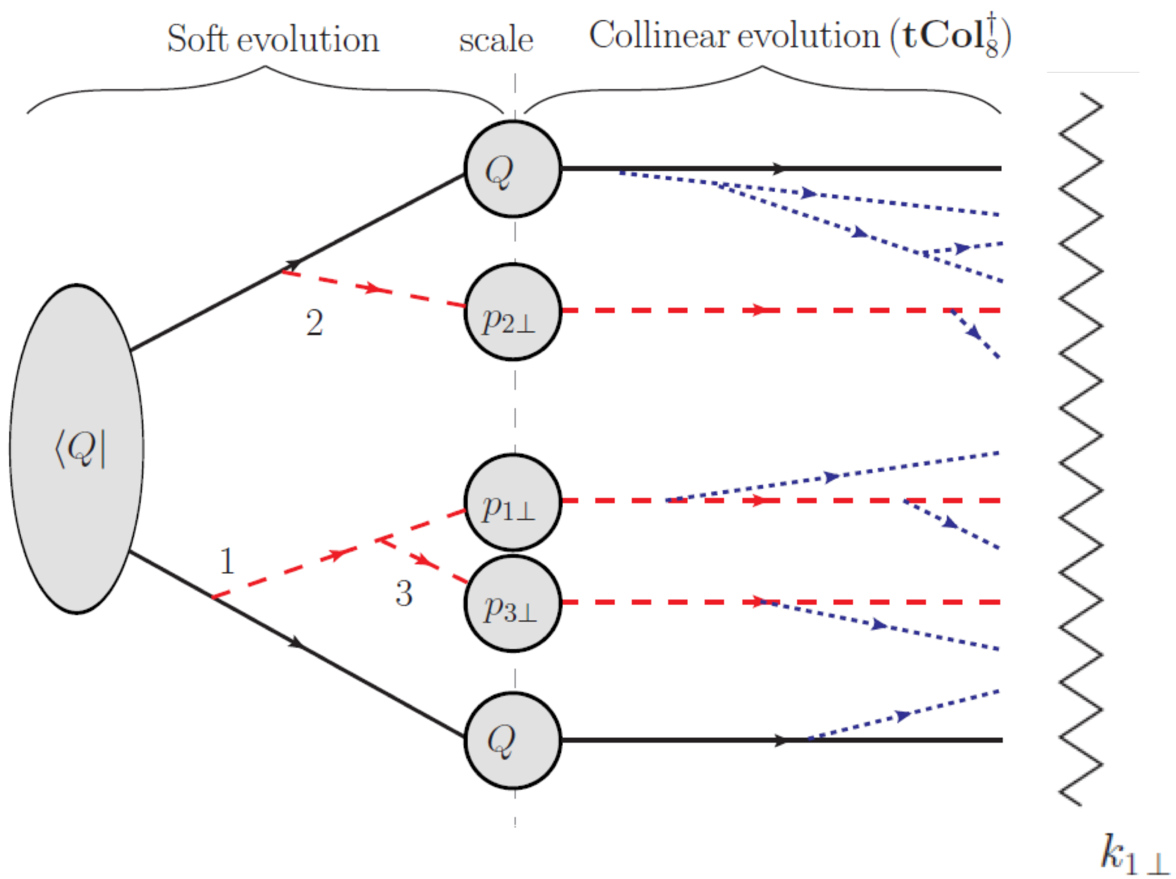
Factorisation

$$
\begin{aligned}
\mathbf{V}_{a,b} = \mathrm{P}\ \exp \Bigg[& -\frac{\alpha_s}{\pi} \sum_{i<j} \int_a^b \frac{\mathrm{d}k_\perp^{(ij)}}{k_\perp^{(ij)}} (-\mathbb{T}_i^g \cdot \mathbb{T}_j^g) \left\{ \int \frac{\mathrm{d}y\,\mathrm{d}\phi}{4\pi} (k_\perp^{(ij)})^2 \frac{\tilde{p}_i \cdot \tilde{p}_j}{(\tilde{p}_i \cdot k)(\tilde{p}_j \cdot k)} \theta_{ij}(k) - i\pi\, \tilde{\delta}_{ij} \right\} \\
& \times \mathcal{R}_{ij}^{\mathrm{soft}}(k, \{\tilde{p}\}) - \frac{\alpha_s}{\pi} \sum_i \int_a^b \frac{\mathrm{d}k_\perp^{(i\vec{n})}}{k_\perp^{(i\vec{n})}} \sum_{\upsilon \in \{q,g\}} \mathbb{T}_i^{\bar{\upsilon}\,2} \int \frac{\mathrm{d}z\,\mathrm{d}\phi}{8\pi} \overline{\mathcal{P}}^{\circ}_{\upsilon\upsilon_i}(z)\, \theta_i(k)\, \mathcal{R}_i^{\mathrm{coll}}(k, \{\tilde{p}\}) \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{V}_{a,b} = & \hat{\mathbf{V}}_{a,b} - \frac{\alpha_s}{\pi} \sum_{i_1<j_1} \int_a^b \frac{\mathrm{d}k_{1\perp}^{(i_1j_1)}}{k_{1\perp}^{(i_1j_1)}} \hat{\mathbf{V}}_{a,k_{1\perp}} (\mathbb{T}_{i_1}^g \cdot \mathbb{T}_{j_1}^g)\, i\pi\, \tilde{\delta}_{i_1j_1} \hat{\mathbf{V}}_{k_{1\perp},b} \\
& + \left(\frac{\alpha_s}{\pi}\right)^2 \sum_{i_2<j_2} \int_a^b \frac{\mathrm{d}k_{1\perp}^{(i_1j_1)}}{k_{1\perp}^{(i_1j_1)}} \sum_{i_1<j_1} \int_a^{k_{1\perp}^{(i_1j_1)}} \frac{\mathrm{d}k_{2\perp}^{(i_2j_2)}}{k_{2\perp}^{(i_2j_2)}} \hat{\mathbf{V}}_{a,k_{2\perp}} (\mathbb{T}_{i_2}^g \cdot \mathbb{T}_{j_2}^g)\, i\pi\, \tilde{\delta}_{i_2j_2} \\
& \times \hat{\mathbf{V}}_{k_{2\perp},k_{1\perp}} (\mathbb{T}_{i_1}^g \cdot \mathbb{T}_{j_1}^g)\, i\pi\tilde{\delta}_{i_1j_1} \hat{\mathbf{V}}_{k_{1\perp},b} - \ldots,
\end{aligned}
$$

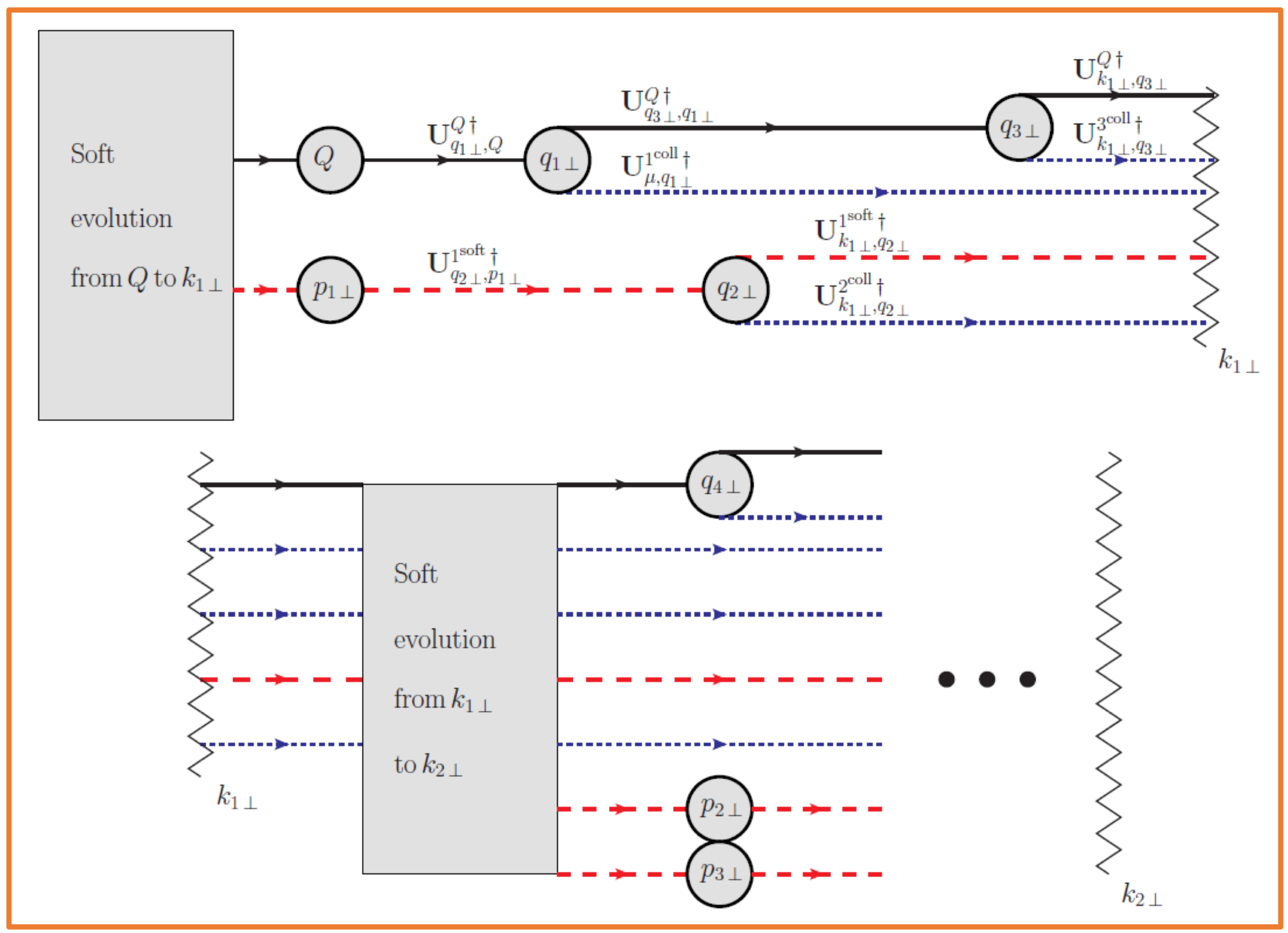
To interleave Coulomb terms we use a path ordered expansion around the "$i\pi$" terms. Following this we can carefully interleave them into a factorised evolution.

Factorisation



In a practical calculation, this means we can include Coulomb terms by using the factorised algorithm and terminating the evolution at the coulomb scale. After this you then perform a second evolution, using the output of the first as the hard process (initial condition). This second evolution runs from the first Coulomb scale and terminates on a second. Etc.
Finally we must integrate the Coulomb scales over the full ranged allowed by the ordering.

Factorisation

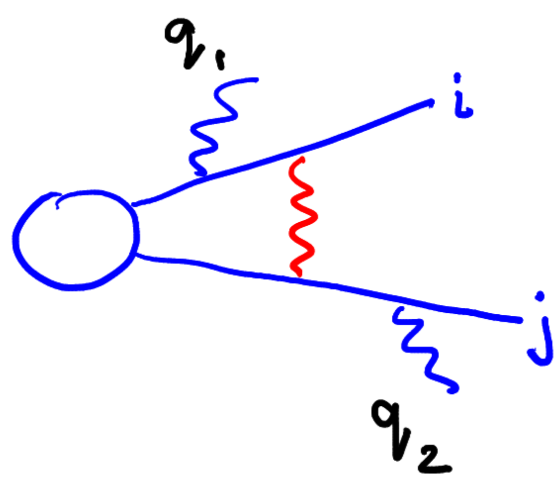


$$\mathbf{V}^{\rm tcol}_{a,b} = \prod_j \mathbf{U}^j_{a,b}$$

$$\mathbf{U}^j_{a,b} = \exp\left[-\frac{\alpha_s}{\pi} \int_a^b \frac{\mathrm{d}k_\perp^{(j\vec{n})}}{k_\perp^{(j\vec{n})}} \Theta(q_{i\perp}^{(j\vec{n})} \le p_{j\perp}) \sum_\upsilon \mathbb{T}_j^{\bar{\upsilon}\,2} \int \frac{\mathrm{d}z\,\mathrm{d}\phi}{8\pi} \mathcal{P}^{\circ}_{\upsilon\upsilon_j} \right]$$

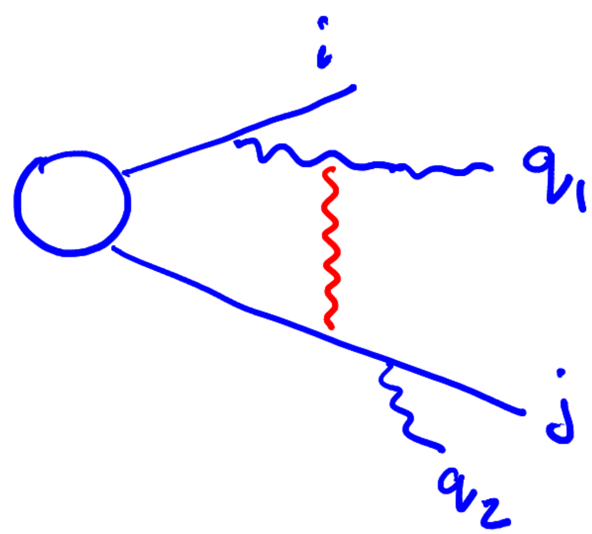
A remarkable result?

The ordering variable



$$q_{1\perp}^{ij} = \left[\frac{2\, p_i\cdot q_1\; p_j\cdot q_1}{p_i\cdot p_j}\right]^{1/2}$$

$$T_j \frac{p_j}{p_j\cdot k} \int_{\tilde{q}_2}^{\tilde{q}_1} \frac{dk_\perp}{k_\perp} \{\dots\}\; T_i \frac{p_i}{p_i\cdot k}$$



$$T_j \frac{p_j}{p_j\cdot k} \int_{\tilde{q}_2}^{q_{1\perp}^{ij}} \frac{dk_\perp}{k_\perp} \{\dots\}\; T_i \left(\frac{p_i}{p_i\cdot k} - \frac{p_j}{p_j\cdot k}\right)$$

$\sum_i$  $= \sum_{i\neq j} \frac{i f^{abc}}{C_A} T_i^b T_j^c \left(\frac{p_i}{p_i\cdot k} - \frac{p_j}{p_j\cdot k}\right)$

Bierenbaum, Czakon & Mitov arXiv: 1107.4384

Thanks to Mike Seymour & René Angeles Martinez

Highly non-trivial

- full 3 & 4 gluon vertices
- exact Θ functions

R. Angeles Martinez
JF & Seymour
arXiv: 1510.07998
1602.00623

"
$$\int_0^{2\pi} \frac{d\phi}{2\pi} \frac{1 + \alpha \cos\phi}{1 + 2\alpha\cos\phi + \alpha^2} = \Theta(1 - |\alpha|)$$
"

$$\alpha = k_\perp / (q_\perp^{ij})$$

A bit more detail….

Double emission & one-loop case

- Limit 1: Both emissions are at wide angle but one gluon is much softer than the other, i.e. $(q_1^\pm \sim q_{1T}) \gg (q_2^\pm \sim q_{2T})$. Specifically, we take $q_2 \to \lambda q_2$ and keep the leading term for small λ.

- Limit 2: One emission (q_2) collinear with p_i by virtue of its small transverse momentum and the other (q_1) at a wide angle, i.e. $q_2^+ \gg q_{2T}$ and $q_1^+ \sim q_{1T} \gg q_{2T}$. Specifically, we take $q_2 \to (q_2^+, \lambda^2 q_{2T}^2/(2q_2^+), \lambda \mathbf{q}_{2T})$ and keep the leading term for small λ.

- Limit 3: One emission (q_1) collinear with p_i by virtue of its high energy and the other (q_2) at a wide angle, i.e. $q_1^+ \gg q_{1T}$ and $q_{1T} \gg q_{2T} \sim q_2^+$. Specifically, we take[3] $q_1 \to (q_1^+/\lambda, \lambda q_{1T}^2/(2q_1^+), \mathbf{q}_{1T})$ and $q_2 \to \lambda q_2$, and keep the leading term for small λ.

Limit-1

Limit-2

Limit-3

René Ángeles-Martínez: PhD thesis

Ángeles-Martínez , JRF, Seymour: JHEP 1512 (2015) 091; Physical Review Letters 116 (2016) 21 212003

Eikonal cuts

e.g. 1st row of graphs

$$G_{11} = \frac{q_1^-}{(q_2^- + q_1^-)} \int_0^{Q^2} \frac{\mathrm{d}k_T^2}{k_T^2} \qquad G_{13} = \frac{q_2^-}{(q_1^- + q_2^-)} \int_0^{Q^2} \frac{\mathrm{d}k_T^2}{k_T^2}$$

$$G_{11} + G_{13} = \int_0^{Q^2} \frac{\mathrm{d}k_T^2}{k_T^2}$$

as expected

$$G_{12} = -\left[\int_0^{2q_1^- q_2^+} \frac{\mathrm{d}k_T^2}{k_T^2} + \frac{q_2^- - q_1^-}{q_2^- + q_1^-} \int_0^{2(q_1^+ + q_2^-)^2 q_2^+ / q_1^-} \frac{\mathrm{d}k_T^2}{k_T^2}\right]$$

$$G_{12} + G_{14} = -\frac{1}{(q_1^- + q_2^-)} \left[q_2^- \int_0^{q_{2T}^2} \frac{\mathrm{d}k_T^2}{k_T^2} + q_1^- \int_0^{q_{1T}^2} \frac{\mathrm{d}k_T^2}{k_T^2}\right]$$

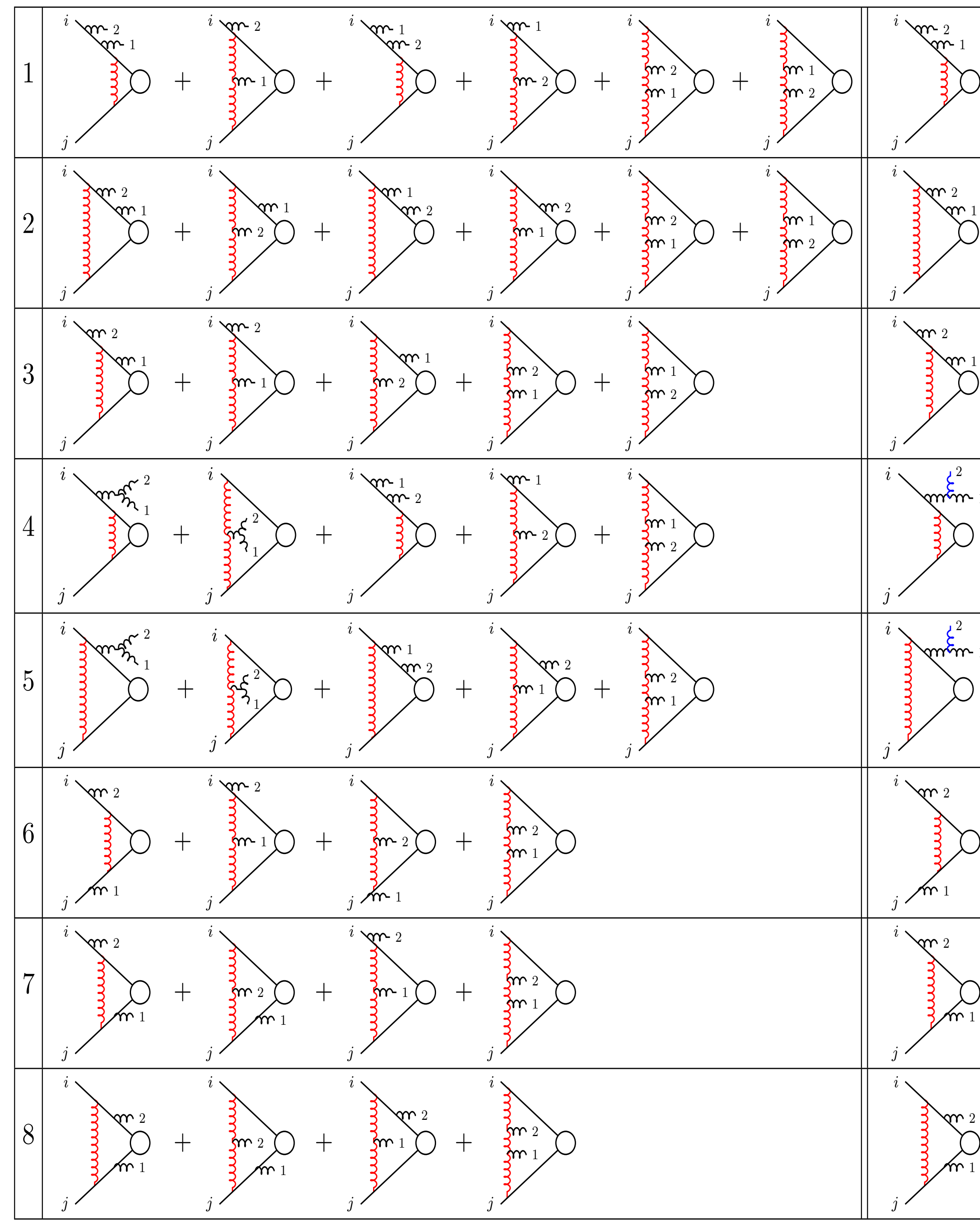
subleading in limits 1 & 2

$$G_{15} + G_{16} \approx -\frac{q_2^-}{(q_1^- + q_2^-)} \int_{q_{2T}^2}^{q_{1T}^2} \frac{\mathrm{d}k_T^2}{k_T^2}$$

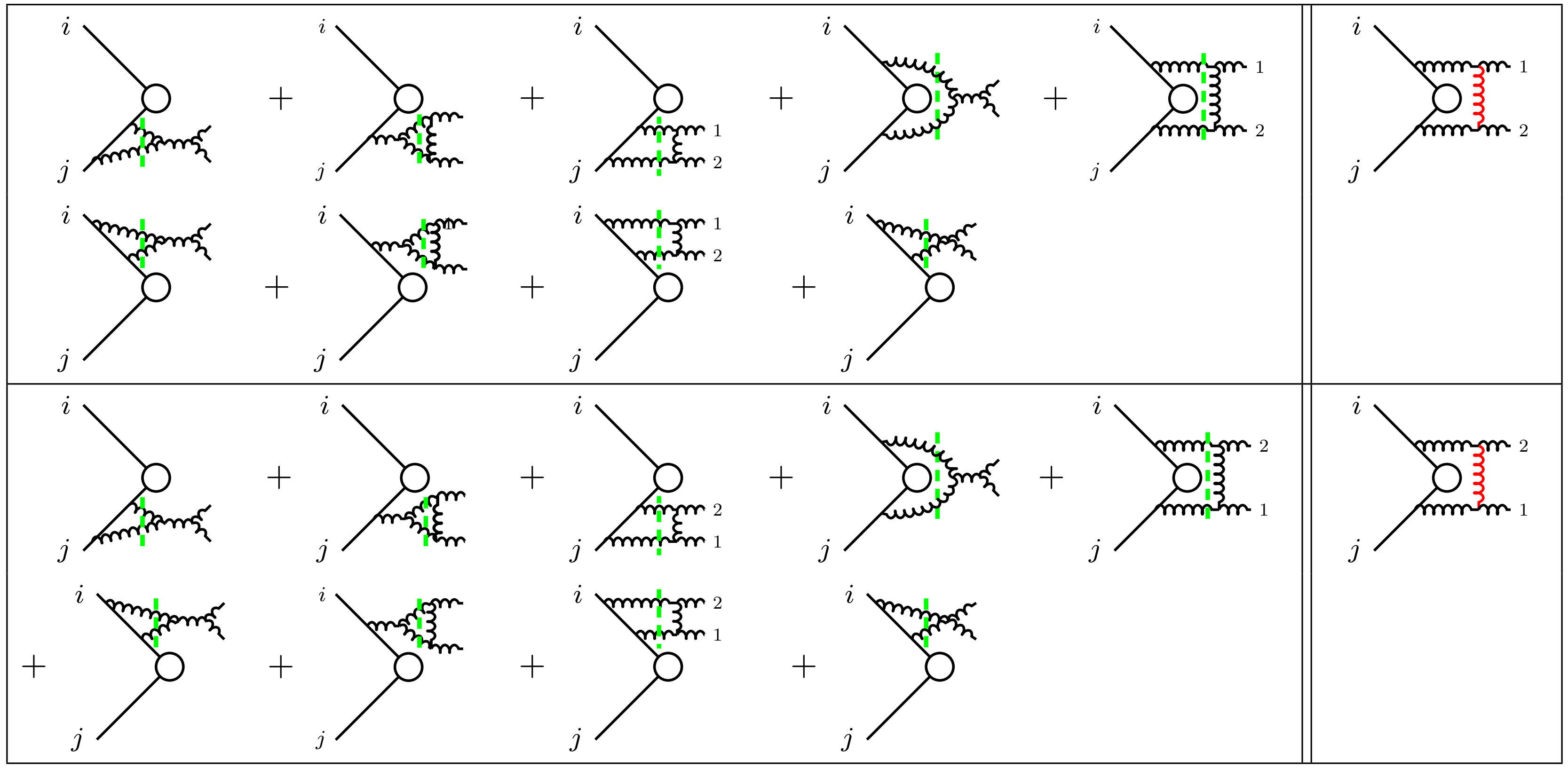
only leading in limit 3

In all 3 limits the sum over all graphs =

$$\boxed{\int_{q_{1T}^2}^{Q^2} \frac{\mathrm{d}k_T^2}{k_T^2}}$$



Soft-gluon cuts



$$\text{e.g.} \quad -\frac{i\pi}{8\pi^2}\frac{p_j\cdot\varepsilon_1}{p_j\cdot q_1}\frac{q_1\cdot\varepsilon_2}{q_1\cdot q_2} \quad \text{in limit 1}$$

$$G_{1c} = -\frac{3}{2}\int_{p_j\cdot q_1}^{p_j\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2} - \frac{3}{2}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2},$$

$$G_{1d} = \frac{3}{2}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2},$$

$$G_{1e} = -\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2} + \frac{1}{2}\int_{p_j\cdot q_1}^{p_j\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2}.$$

$$\text{sum} = \boxed{-\int_0^{(q_2^{(1j)})^2}\frac{\mathrm{d}l_T^2}{l_T^2}}$$

$$G_{2c} = \frac{3}{4}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2},$$

$$G_{2d} = -\frac{3}{2}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2},$$

$$G_{2e} = \frac{7}{4}\int_0^{2q_1\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2} + \int_{p_i\cdot q_1}^{p_i\cdot q_2}\frac{\mathrm{d}l_T^2}{l_T^2}.$$

$$\text{sum} = \boxed{\int_0^{(q_2^{(1i)})^2}\frac{\mathrm{d}l_T^2}{l_T^2}}$$

- Note this is *NOT* the dipole ordering that has previously appeared in the literature.*

- This is occurring at amplitude level.

- No statement on the ordering of the real emissions.

- Originally proved for imaginary part of loops and Drell-Yan but now proved for real part too and for general hard processes at one-loop with any number of real emissions.

Ángeles Martínez, JF, Seymour, in preparation
"A new aspect of QCD coherence"

* e.g. Caron-Huot, Neill and Vaidya, Höche and Prestel

Conclusions

- Hopefully we will soon have a publicly available sub-leading colour parton shower

- Coulomb/Glauber exchanges make factorisation highly non-trivial

- Loop integrals are limited by real emissions in an interesting way