Conformal symmetry in QCD

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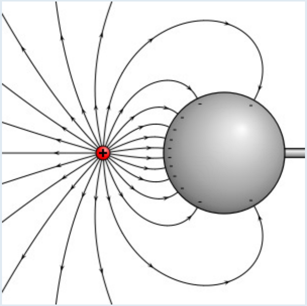
Uni Hamburg and DESY theory colloquium, 13.05.2020

Prologue

L.I. Magnus (1831):
Inversion transformation

$$x^\mu \to \frac{x^\mu}{x^2}$$



General motivation

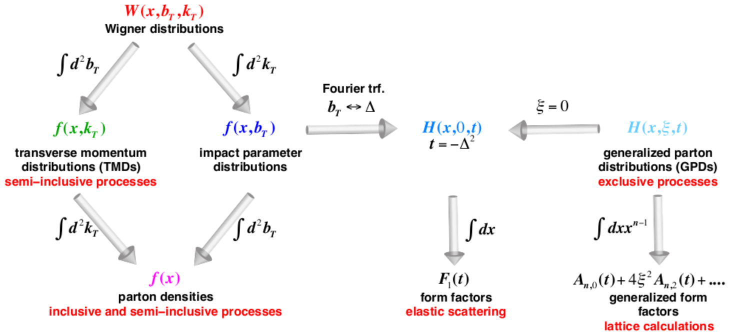
- **New quality in hadron physics on the 10 years scale**
  - Very high luminosity
  - Kinematic range
  - Particle identification, energy resolution
  - Polarized beams and targets, nuclear beams

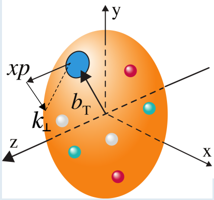
    $\Rightarrow$ high statistics, smaller systematics, rare processes
- **Must be matched by adequate theory development**

> **Aim: Deeper understanding of strong interactions:**
>
> **— Novel hadron physics agenda: 3D imaging, . . .**
>
> **— Benefit particle physics at the energy frontier**

- Strong support in the community

Nucleon Tomography



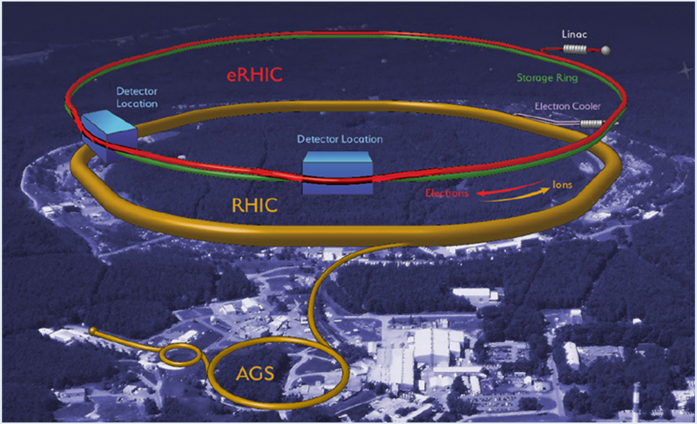


access to the second level:

- transverse momentum imaging
- impact parameter imaging

The Electron-Ion Collider Project

EIC facility concepts



— Brookhaven National Lab (BNL) — Jefferson Lab (JLAB)

Begin Physics Program > 2028

In this talk:

How to find a representation, for a generic quantity $\mathcal{Q}$

$$\mathcal{Q} = \mathcal{Q}^{\text{conf}} + \frac{\beta(g)}{g}\Delta\mathcal{Q}\,, \quad \text{where} \quad \Delta\mathcal{Q} = \text{power series in } \alpha_s$$

Proposal: $\mathcal{Q}^{\text{conf}}$ is conformal QCD at the Wilson-Fischer fixed point in $d{\neq}4$ dimensions

V.Braun, A.Manashov, Eur. Phys. J. C ***73*** *(2013) 2544*

Conformal field theories

$$\langle O_1(x_1)\, O_2(x_2)\rangle = \frac{\text{const}}{|x_1-x_2|^{2\Delta_1}}\,\delta_{\Delta_1\Delta_2}$$

$$\langle O_1(x_1)\, O_2(x_2)\, O_3(x_3)\rangle = \frac{\text{const}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}\,|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}\,|x_2-x_3|^{\Delta_3+\Delta_2-\Delta_1}}$$

- $\leftarrow$ Δ_k is a scaling dimension (canonical + anomalous)
- $\leftarrow$ Can be taken as a definition

alternatively:

$$\mathsf{T}\{O_1(x)\, O_2(0)\} = \sum_N \Big[C_N^{(0)}(x,\mu)\mathcal{O}_N^{(\mu)}(0) + C_N^{(1)}(x,\mu)\partial\mathcal{O}_N^{(\mu)}(0) + C_N^{(2)}(x,\mu)\partial^2\mathcal{O}_N^{(\mu)}(0) + \ldots\Big]$$

- (!) Coeff. functions of operators with total derivatives are fixed by conformal algebra
- (!) Matrix elements of $\partial_\mu \mathcal{O}_N^{\mu\mu_1\ldots\mu_N}$ vanish on free quarks

Conformal Transformations

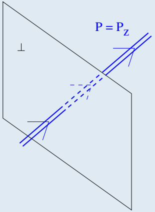
- Translations
- Rotations and Lorentz boosts
- **Dilatation** (global scale transformation) $x^\mu \to x'^\mu = \lambda x^\mu$
- **Inversion** $x^\mu \to x'^\mu = x^\mu / x^2$

Special conformal transformation

$$x^\mu \to x'^\mu = \frac{x^\mu + a^\mu x^2}{1 + 2a\cdot x + a^2 x^2}$$

= inversion, translation $x^\mu \to x^\mu + a^\mu$, inversion

Collinear Subgroup



- Special conformal transformations

$$z \to z' = \frac{z}{1+2az}$$

- translations $z \to z' = z + c$
- dilatations $z \to z' = \lambda z$

form the so-called collinear subgroup $SL(2,R)$

$$z \to z' = \frac{a\,z+b}{c\,z+d}, \quad ad - bc = 1$$

$$p_+ = \frac{1}{\sqrt{2}}(p_0 + p_z) \to \infty$$
$$p_- = \frac{1}{\sqrt{2}}(p_0 - p_z) \to 0$$
$$px \to p_+ x_-$$

write

$$x_- = zn\,, \qquad n^2 = 0\,, \qquad z \in \mathbb{R}$$

SL(2) algebra

$SL(2,R)$ algebra

$$S_- = -\frac{d}{dz}$$
$$S_+ = z^2\frac{d}{dz} + 2j\, z$$
$$S_0 = z\frac{d}{dz} + j$$

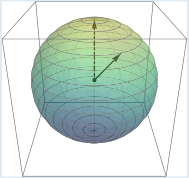
$SL(2)$ commutation relations

$$[S_+, S_-] = 2S_0 \qquad [S_0, S_\pm] = \pm S_+$$

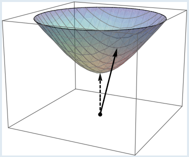
Conformal spin

$$j = \frac{1}{2}(l+s)$$

Non-compact spin



$SO(3)$



$SL(2,\mathbb{R}) \sim O(2,1)$

Separation of variables

In Quantum Mechanics: **O(3) rotational symmetry** $\Rightarrow$ **Angular vs. radial dependence**

$$\left[-\frac{\hbar^2}{2m}\Delta + V(|r|)\right]\Psi = E\Psi \quad \Rightarrow \quad \Psi(\vec{r}) = R(r)Y_{lm}(\theta,\phi)$$

$Y_{lm}(\theta,\phi)$ **are eigenfunctions of** $L^2 Y_{lm} = l(l+1)Y_{lm}$, $[\mathcal{H}, L^2] = 0$.

In high-energy scattering: **SL(2,R) conformal symmetry** $\Rightarrow$ **Longitudinal vs. transverse dependence**

"longitudinal" dependence in terms of eigenfunctions of S^2
"transverse" dependence through RG equations $[\mathcal{H}, S^2] = 0$.

. . . Unfortunately, conformal symmetry is broken . . .

Consider gauge theory with a hard cutoff M, integrate out the fields with frequencies above μ

$$S_{\rm eff} = -\frac{1}{4}\int d^4x \left(\frac{1}{(g^{(0)})^2} - \frac{\beta_0}{16\pi^2}\ln M^2/\mu^2\right)\left[F^a_{\mu\nu}F^{a\mu\nu}\right]_{\rm slow}(x) + \ldots$$

Under the scale transformation $x^\mu \to \lambda x^\mu$, $A_\mu(x) \to \lambda A_\mu(\lambda x)$, $\psi(x) \to \lambda^{3/2}\psi(\lambda x)$ and $\mu \to \mu/\lambda$ with the fixed cutoff

$$\delta S = -\frac{1}{32\pi^2}\beta_0 \ln\lambda \int d^4x\, F^a_{\mu\nu}F^{a\mu\nu}(x) + \ldots$$

Thus dilatation symmetry (scale invariance) is broken. Inversion as well.

Idea:

V.Braun, A.Manashov, Eur. Phys. J. C **73** *(2013) 2544*

Consider QCD at Wilson-Fisher critical point in $d \neq 4$ dimensions

- **This is a conformal theory**
- **Anomalous dimensions in MS schemes in this theory coincide with physical QCD**
- **For a generic observable $\mathcal{Q}$ one can write**

$$\mathcal{Q} = \mathcal{Q}^{\text{conf}} + \frac{\beta(g)}{g}\Delta\mathcal{Q}\,, \quad \text{where} \quad \Delta\mathcal{Q} = \text{power series in } \alpha_s$$

- All statements valid within perturbation theory

Wilson-Fisher fixed points

- Consider a QFT at noninteger number of dimensions, here $d = 4 - 2\epsilon$, $\epsilon \ll 1$

$$\int d^4x\, \mathcal{L}(x) \longrightarrow \int d^{4-2\epsilon}x\, \mathcal{L}(x)$$

- In this world, the coupling constant becomes dimensionful

$$g_0 = g\mu^\epsilon = Z_g(\mu)g(\mu)\mu^\epsilon$$

- and the beta-function receives an extra term

$$\mu\frac{d}{d\mu}g(\mu) = g(-\epsilon + \gamma_g) \qquad \gamma_g = -\mu\frac{d}{d\mu}\ln Z_g$$

- One can fine-tune $g \to g^*(\epsilon)$ such that the coupling becomes scale-independent

Minimal subtraction

changing

$$\int d^4x\, \mathcal{L}(x) \longrightarrow \int d^{4-2\epsilon}x\, \mathcal{L}(x)\,, \qquad \epsilon > 0$$

suppresses small distances/large momenta — alternative to a hard cutoff

- UV divergences show up as poles in $1/\epsilon$
- UV regularization prescription — subtraction of poles in $1/\epsilon^k$ — "minimal subtraction" (MS)
- This procedure is the same for theories in $d = 4$ and $d = 4 - 2\epsilon$;

  The difference is that in $d = 4$ (physical theory) we send $\epsilon \to 0$ after subtraction of poles, and in $d{\neq}4$ theory we keep ϵ finite.
- Thus

  — The renormalization constants (anomalous dimensions) are the same
  — Renormalized amplitudes differ by terms $\sim \epsilon = \beta_{\rm QCD}(\alpha_s^*)$

Deformed generators

- Generators of symmetry transformations in conformal theory are modified by quantum corrections

Conformal Ward Identities:
exploit invariance under change of variables $\Phi \to \Phi + \delta_{D,K,\ldots}\Phi$ in the path integral

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\rangle = \int \mathcal{D}\Phi\, \mathcal{O}_1(x)\mathcal{O}_2(y)\, e^{S(\Phi)}$$

$$\langle \delta_C \mathcal{O}_1(x)\mathcal{O}_2(y)\rangle + \langle \mathcal{O}_1](x)\delta_C \mathcal{O}_2](y)\rangle + \langle \delta_C S\, \mathcal{O}_1(x)\mathcal{O}_2(y)\rangle$$

one can show that

$$\delta_D S = -\frac{1}{4}\frac{\beta(a_s)}{a_s}\int d^d z\, [F^a_{\mu\nu}F^{a,\mu\nu}] + \ldots\,, \qquad \delta^\mu_K S = -\frac{1}{4}\frac{\beta(a_s)}{a_s}\int d^d z\, (2z_\mu)[F^a_{\mu\nu}F^{a,\mu\nu}] + \ldots\,,$$

The last term does not disappear for $\beta(a^*) = 0$ because of divergencies at $z \to x$ and $z \to y$
but extra terms can be included in the redefinition of $\delta_D \mathcal{O}$, $\delta_K \mathcal{O}$

— for $\mathbf{D}$, the effect is modification of scaling dimension $\ell \to \ell + \gamma(\alpha_s^*)$

— for $\mathbf{K}_\mu$ extra terms are allowed that do not spoil the algebra

— in a certain convenient representation

$$S_- = S_-^{(0)}$$
$$S_0 = S_0^{(0)} - \epsilon + \frac{1}{2}\mathbb{H}(a_s^*)$$
$$S_+ = S_+^{(0)} + (z_1 + z_2)\Big(- \epsilon + \frac{1}{2}\mathbb{H}(\alpha_s^*)\Big) + (z_1 - z_2)\Delta_+(\alpha_s^*)$$

$\leftarrow\ \mathbb{H}(a_s^*) = a_s^*\,\mathbb{H}^{(1)} + (a_s^*)^2\,\mathbb{H}^{(2)} + \ldots$ — “Hamiltonian”

$\leftarrow\ \Delta_+(a_s^*) = a_s^*\,\Delta_+^{(1)} + (a_s^*)^2\,\Delta_+^{(2)} + \ldots$ — “Conformal anomaly”

- “hidden” conformal invariance of QCD RG equations

$$[S_k, \mathbb{H}] = 0$$
$$[S_+, S_-] = 2S_0 \qquad [S_0, S_+] = S_+ \qquad [S_0, S_-] = -S_-$$

- True to all orders in perturbation theory (in MS-like schemes)
- Complete RG kernel in $d = 4$, a digression to $d = 4 - \epsilon$ is an intermediate step

Conformal symmetry in gauge theories

- **In a scalar theory** e.g. Vasiliev, '04

Let $\mathcal{O}_{\Delta_\alpha}$ be a set of local composite operators that diagonalize the anomalous dimension matrix; i.e. they have definite scaling dimensions $\Delta_\alpha = \Delta + \gamma_\alpha$

— Scale and conformal Ward identities imply

$$\delta_D \mathcal{O}_{\Delta_\alpha}(x) = D_{\Delta_\alpha}(x)\mathcal{O}_{\Delta_\alpha}(x), \qquad \delta_{K^\mu} \mathcal{O}_{\Delta_\alpha}(x) = K^\mu_{\Delta_\alpha}(x)\mathcal{O}_{\Delta_\alpha}(x) + \mathcal{O}^\mu_\alpha(x)$$

where

$$D_\Delta(x) = x\partial_x + \Delta, \qquad K^\mu_\Delta(x) = 2x^\mu(x\partial) - x^2\partial^\mu + 2\Delta x^\mu - 2x_\nu\Sigma^{\mu\nu},$$

and $\mathcal{O}^\mu_\alpha$ are local operators with scaling dimension $\Delta_\alpha - 1$

— The set $\mathcal{O}_{\Delta_\alpha}, \mathcal{O}^\mu_\alpha, \ldots$ defines a Verma module. (inf.dim. rep. of conformal algebra)

— Applying D, K_μ sequentially to $\mathcal{O}_{\Delta_\alpha}, \mathcal{O}^\mu_\alpha, \ldots$ one eventuallly will nullify the addenda $\mathcal{O}^\mu_\alpha$

— The resulting operator transforms homogeneously under conformal trafos (lowest weight vector of the rep.)

— Conformal operator

- **In a (nonabealian) gauge theory** VB, A.Manashov, S.Moch, M.Strohmaier, 2019

— Problem: Gauge-noninvariant operators in Verma module?

- Variation of the QCD action

$$\delta_D S_R = \int d^d x\, \mathcal{N}(x)\,, \qquad \delta_K^\mu S_R = \int d^d x\, 2x^\mu \Big(\mathcal{N}(x) - (d-2)\partial^\rho \mathcal{B}_\rho(x)\Big)$$

where

$$\mathcal{N}(x) = 2\epsilon\, \mathcal{L}_R^{YM+gf} = 2\epsilon \left(\frac{1}{4} Z_A^2 F^2 + \frac{1}{2\xi}(\partial A)^2\right),$$

$$\mathcal{B}_\rho(x) = Z_c^2 \bar{c} D^\rho c - \frac{1}{\xi} A^\rho (\partial A) = \delta_{\mathsf{BRST}}(\bar{c}^a A_\mu^a)$$

- Conformal symmetry can be expected only for correlation functions of gauge-invariant operators

Gauge-noninvariant operators in RG equations

- From the analysis of BRST Ward identities — Joglekar, Lee; '76, '77

— For dilatations

— Gauge invariant operators, $\mathcal{O}$, mix under renormalization with BRST operators $\mathcal{B} = s\mathcal{B}'$, and EOM operators, $\mathcal{E} = F(\Phi)\delta S_R/\delta\Phi$.

— The mixing matrix has a triangular structure

$$\begin{bmatrix}\mathcal{O}\\ \mathcal{B}\\ \mathcal{E}\end{bmatrix} = \begin{pmatrix} Z_{\mathcal{O}\mathcal{O}} & Z_{\mathcal{O}\mathcal{B}} & Z_{\mathcal{O}\mathcal{E}}\\ 0 & Z_{\mathcal{B}\mathcal{B}} & Z_{\mathcal{B}\mathcal{E}}\\ 0 & 0 & Z_{\mathcal{E}\mathcal{E}}\end{pmatrix}\begin{pmatrix}\mathcal{O}\\ \mathcal{B}\\ \mathcal{E}\end{pmatrix}$$

so that renormalized gauge-invariant operators take the following generic form

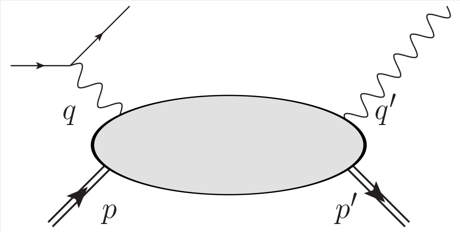
$$[\mathcal{O}] = Z_{\mathcal{O}\mathcal{O}}\mathcal{O} + Z_{\mathcal{O}\mathcal{B}}\mathcal{B} + Z_{\mathcal{O}\mathcal{E}}\mathcal{E}\,,$$

— The renormalization factor $Z_{\mathcal{O}\mathcal{O}}$ in the $\overline{\mathrm{MS}}$ scheme does not depend on gauge parameter ξ

— For special conformal trafos — VB, A.Manashov, S.Moch, M.Strohmaier, 2019

Conformal variation of a gauge-inv. operator is a gauge-inv. operator up to terms that vanish in all correlation functions of gauge-inv. operators

Deeply-virtual Compton scattering



- Access to 3D quark/gluon distributions — longitudinal and transverse plane
- Requires high precision both in theory and experiment — NNLO is a goal
- Off-forward kinematics — operators with total derivatives contribute
  — need complete off-forward mixing matrix and coefficient function

RG equations from operator algebra

- **Expanding the commutation relations in powers of a_s^***

$$
\begin{aligned}
&[S_+^{(0)}, \mathbb{H}^{(1)}] = 0\,,\\
&[S_+^{(0)}, \mathbb{H}^{(2)}] = [\mathbb{H}^{(1)}, S_+^{(1)}]\,,\\
&[S_+^{(0)}, \mathbb{H}^{(3)}] = [\mathbb{H}^{(1)}, S_+^{(2)}] + [\mathbb{H}^{(2)}, S_+^{(1)}]\,, \qquad \text{etc.}
\end{aligned}
$$

- A nested set of inhomogenious first order differential equations for $\mathbb{H}^{(k)}$
  Their solution determines $\mathbb{H}^{(k)}$ up to an $SL(2)$-invariant term
- The r.h.s. involves $\mathbb{H}^{(k)}$ and $S_+^{(m)}$ at one oder less compared to the l.h.s.

$\longrightarrow$ $\Delta_+^{(2)}$: 1601.05937

$\longrightarrow$ $\mathbb{H}^{(3)}$: 1703.09532

Vladimirov correspondence

Different geometries of Wilson lines

jet physics **multiparton interactions**

are related by a conformal transformation

$$\{x_+, x_-, x_t\} \mapsto \left\{-\frac{1}{2x_+}, x_- - \frac{x_t^2}{2x_+}, \frac{x_t}{\sqrt{2}x_+}\right\}$$

Implies exact relation between soft and rapidity anomalous dimensions

$$\gamma_{\text{soft}}(v_1,\dots,v_n) = \gamma_{\text{rapidity}}(b_1,\dots,b_n;\epsilon^*)\,, \quad b_k = \frac{1}{\sqrt{2}}\frac{v_t}{v_+}$$

- checked to NNLO

A. Vladimirov, 1610.05791

Vladimirov correspondence

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jet physics **multiparton interactions**

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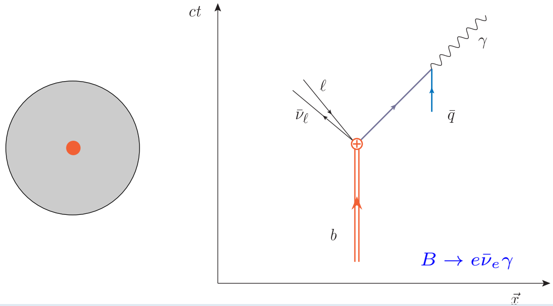
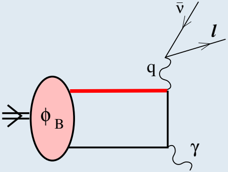
A. Vladimirov, 1610.05791

Light-cone distribution amplitide of the B-meson)



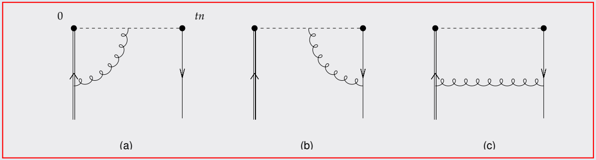
- Decay rate depends on the probability amplitude to find a light antiquark at a given light-like distance from the heavy quark — the B-meson distribution amplitude



$$= \int_0^\infty \frac{d\omega}{\omega} T(\omega, E_\gamma; \mu_F) \Phi_+(\omega, \mu_F)$$

Lange-Neubert evolution equation

Lange, Neubert '03



$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \frac{\alpha_s C_F}{\pi} H_{LN}\right)\Phi_+(\omega,\mu) = 0\,,$$

$$[H_{LN}f](\omega) \quad = \quad -\int_0^\infty d\omega' \left[\frac{\omega}{\omega'}\frac{\theta(\omega'-\omega)}{\omega'-\omega} + \frac{\theta(\omega-\omega')}{\omega-\omega'}\right]_+ + \left[\ln\frac{\mu}{\omega} - \frac{5}{4}\right] f(\omega)$$

V. M. Braun, Y. Ji and A. N. Manashov, PRD **100** *(2019) 014023*

- To all orders in perturbation theory

$$H_{LN}(a) = \Gamma_{\rm cusp}(a)\ln(iK(a)\mu e^{2\gamma_E}) + \text{const}$$

— $\Gamma_{\rm cusp}(a)$ is the cusp anomalous dimension

— $K(a) = v^\mu K_\mu(\epsilon(a), a)$ is the generator of special conformal trafos in $d - 2\epsilon$

- Two-loop result available (NNLO)

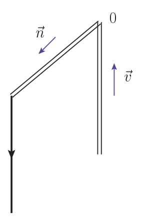
- Geometry is invariant under $v^\mu K_\mu$

$$(*) \quad [K, H_{LN}] = 0$$

- Dilatation invariance broken by the cusp

$$(**) \quad [D, H_{LN}] = \Gamma_{\text{cusp}}$$

- $(*) \mapsto H_{LN}$ is a function of K
- This function can be found from $(**) \mapsto$ answer



Heavy-light factorization

V.Braun, Yao Ji, A. Manashov JHEP **1806**, *017 (2018)*

- SL(2) symmetry for light quarks

$$S_+ = z^2\partial_z + 2js, \qquad S_0 = z\partial_z + s, \qquad S_- = -\partial_z\,,$$

$\mapsto$ one-loop RG kernels can be written in terms $S_{12}^2 = (\vec{S}_1 + \vec{S}_2)^2$ *Bukhvostov, Frolov, Lipatov, Kuraev, '85*

$$H_{\bar{q}q} = 4C_F\left\{2\Big[\psi(J_{\bar{q}q}) - \psi(2)\Big] - \frac{1}{J_{\bar{q}q}(J_{\bar{q}q}-1)} + \frac{1}{2}\right\}, \qquad S_{\bar{q}q}^2 = J_{\bar{q}q}(J_{\bar{q}q}-1)\,,$$

- For very heavy quarks

$$S_-^{(h)} \to \lambda S_-^{(h)}\,, \qquad S_+^{(h)} \to \lambda^{-1} S_+^{(h)}\,, \qquad \lambda \sim m_Q/\Lambda_{QCD} \to \infty$$

The two-particle heavy-light quadratic Casimir operator

$$S_{qh}^2 = S_+^{(qh)} S_-^{(qh)} + S_0^{(qh)}(S_0^{(qh)} - 1) \mapsto \lambda S_+^{(q)} S_-^{(h)} + \mathcal{O}(1)$$

The the hamiltonian also factorizes

$$\mathcal{H}_{qh} \mapsto \ln\left(i\mu S_+^{(q)}\right) + \ln\left(-i\mu^{-1}\lambda S_-^{(h)}\right)$$

where μ is arbitrary mass scale. HQET keeps only light degrees of freedom; thus reproduce the same result

— Can be extended to the correspondence on the level of eigenfunctions

— Novel integrable models

V.Braun, Yao Ji, A. Manashov JHEP **1806** *(2018), 017*

— Application to B mesons LCDAs of higher twist

V.Braun, Yao Ji, A. Manashov JHEP ***1705*** *(2017), 022*

Venturing into the transverse plane

All discussion so far was about operators "living" on the light-cone:

$$\Phi(0)\partial_+^n\Phi(0) \qquad \Longleftrightarrow \qquad \Phi(0)\Phi(zn)$$

What happens if we include transverse or "minus" derivatives?

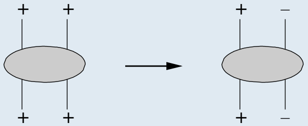
$$\Phi(0)(\partial_\perp^2)\partial_+^n\Phi(0) \qquad \Phi(0)(\partial_-\partial_+)\partial_+^n\Phi(0)$$

Embedding $SL(2,\mathbb{R})$ in $SO(4,2)$?

Applications:

- Complete results for operator renormalization in QCD up to twist four
  VB, A. Manashov and J. Rohrwild, Nucl. Phys. B **826** (2010), 235
- Kinematic corrections in hard exclusive reactions (2011-2014, work in progress)

$SL(2,\mathbb{R}) \to SO(4,2)$



$$SL(2,\mathbb{R}): \qquad \mathbb{C}_2^{SL(2,R)} = J(J-1)$$

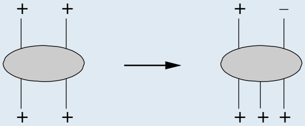
$$SO(4,2): \qquad \mathbb{C}_2^{SO(4,2)} = \mathbf{J}(\mathbf{J}-1)$$

$\mathbb{H}(J) \quad \to \quad \mathbb{H}(\mathbf{J})$

the same function !

- This works because two–particle representations are not degenerate w.r.t. both groups

$2 \to 3$ transitions



E.g. $\psi_-\psi_+,\ \psi_+\psi_- \to \psi_+\psi_+\bar{f}_{++}$

Idea:

- Infinitesimal translation in transverse plane $P_{\mu\bar{\lambda}}$

$$i[\mathbf{P}_{\mu\bar{\lambda}}, \psi_+] \quad = \quad 2\partial_z\psi_- + igA_{\mu\bar{\lambda}}\psi_+ + \text{ EOM}\,,$$

- Lorentz Rotation $M_{\mu\mu}$

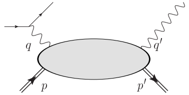
$$i[\mathbf{M}_{\mu\mu}, \psi_+] \quad \sim \quad (z\partial_z + 1)\psi_- + \frac{1}{2}igzA_{\mu\bar{\lambda}}\psi_+ + \text{ EOM}\,,$$

↪ Exact relations between renormalized operators containing “plus” and “minus” fields
↪ The counterterms on the LHS and RHS must coincide

Evolution equations for operators of arbitrary twist are reduced to leading-twist kernels

— no new Feynman diagrams

Power corrections $\sim t/Q^2$ and m^2/Q^2 in non-planar kinematics



- Identification of longitudinal and transverse directions not unique
- Leading twist approximation for helicity amplitudes ambiguos
- On the top of it, violation of electromagnetic Ward identities

What is missing?

$$T\{j_\mu(x)j_\nu(0)\} = \sum_N \Big[a_N\mathcal{O}_N + b_N(\partial\mathcal{O})_N + c_N\partial^2\mathcal{O}_N + \ldots + \text{qqG-operators}\Big]$$

- Matrix elements $\langle(\partial\mathcal{O})_N\rangle$ vanish on free quarks
- b_N cannot be calculated directly(?) even at tree level, but...
- b_N and c_N are related to a_N by conformal algebra

PRL107(2011)202001; PRL109(2012)242001;
JHEP1201(2012)085; PRD89(2014)074022

M. Defurne *et al.* [Hall A Collaboration] arXiv:1504.05453

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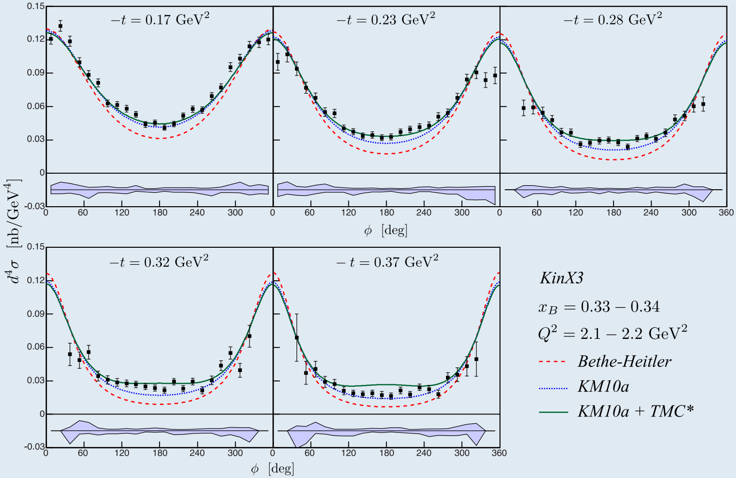


- **TMC* refers to the calculation that includes full kinematic twist-4 corrections**

GPD model: KM10a (Kumericki, Mueller, Nucl. Phys. B **841** (2010) 1)

to summarize:

- **QCD in $d = 4$ and conformal QCD at $d = 4 - 2\epsilon$ at fine-tuned coupling have the same RG equations in $\overline{\text{MS}}$-scheme**
- **The difference, terms $\mathcal{O}(\epsilon)$, can be reexpanded in terms of QCD β-function**

$$\mathcal{Q}^{d=4} = \mathcal{Q}^{d=4-2\epsilon^*} + \frac{\beta(g)}{g}\Delta\mathcal{Q}$$

V.Braun, A.Manashov, Eur. Phys. J. C **73** *(2013) 2544*

- — Save one loop in the calculation of off-forward RG kernels and coef. functions
- — Heavy-light systems
- — Relation between soft and rapidity anomalous dimensions
- — Higher twists
- — Regge limit, BFKL and beyond, nonglobal logarithms . . .

Using hidden symmetries . . .

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + \bar\psi i \not{D}\psi$$

- conformal symmetry — broken on quantum level
- integrability — revealed on quantum level

. . . to reveal the structure and as a calculational tool

Supplementary slides

Leading-order QCD Evolution Equations

ERBL evolution equation

$$\mu^2 \frac{d}{d\mu^2} \phi_\pi(u,\mu) = \int_0^1 dv\, V(u,v;\alpha_s(\mu))\, \phi_\pi(v,\mu)$$

$$V_0(u,v) = C_F \frac{\alpha_s}{2\pi} \left[\frac{1-u}{1-v} \left(1 + \frac{1}{u-v}\right) \theta(u-v) + \frac{u}{v} \left(1 + \frac{1}{v-u}\right) \theta(v-u) \right]_+$$

How to make maximum use of conformal symmetry?

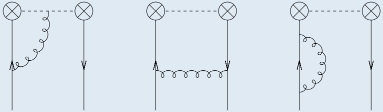
Bukhvostov, Frolov, Kuraev, Lipatov ‘85

Where is the symmetry?

RG equation for the nonlocal operator $\mathcal{Q}(\alpha_1,\alpha_2) = \bar\psi(\alpha_1)\gamma_+\,\psi(\alpha_2)$

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)\mathcal{Q}(\alpha_1,\alpha_2) = -\frac{\alpha_s C_F}{4\pi}\left[\mathbb{H}\cdot\mathcal{Q}\right](\alpha_1,\alpha_2)$$

$$\mathbb{H} = 2\mathcal{H}_{\rm v}^{(12)} - 2\mathcal{H}_{\rm e}^{(12)} + 1$$



(a) (b) (c)

Explicit calculation

$$[\mathcal{H}_{\rm v}^{(12)}\cdot\mathcal{Q}](\alpha_1,\alpha_2) = -\int_0^1 \frac{du}{u}(1-u)\left\{\mathcal{Q}(\alpha_{12}^u,\alpha_2) + \mathcal{Q}(\alpha_1,\alpha_{21}^u) - 2\mathcal{Q}(\alpha_1,\alpha_2)\right\},$$

$$[\mathcal{H}_{\rm e}^{(12)}\cdot\mathcal{Q}](\alpha_1,\alpha_2) = \int_0^1 [du]\,\mathcal{Q}(\alpha_{12}^{u_1},\alpha_{21}^{u_2}) \qquad \alpha_{12}^u \equiv \alpha_1(1-u)+\alpha_2 u$$

Balitsky, Braun '88

Now verify

$$[\mathbb{H}\cdot S_a\,\mathcal{Q}](\alpha_1,\alpha_2) = S_a\,[\mathbb{H}\cdot\mathcal{Q}](\alpha_1,\alpha_2)\,, \qquad S_a\mathcal{Q}(\alpha_1,\alpha_2) = (S_{1,a}+S_{2,a})\mathcal{Q}(\alpha_1,\alpha_2)$$

$$[\mathbb{H},S_+] = [\mathbb{H},S_-] = [\mathbb{H},S_0] = 0$$

Balitsky, Braun, 1989

General expression

$$\mathbb{H}[\mathcal{O}](z_1, z_2) = \int_0^1 d\alpha \int_0^1 d\beta\, h(\alpha,\beta)\, [\mathcal{O}](z_{12}^\alpha, z_{21}^\beta)$$

$$z_{12}^\alpha \equiv z_1\bar\alpha + z_2\alpha \qquad \bar\alpha = 1-\alpha$$

$$\boxed{[S_+, \mathbb{H}] = 0} \qquad \Longrightarrow \qquad h(\alpha,\beta) = h\left(\frac{\alpha\beta}{\bar\alpha\bar\beta}\right) = h^{(1)}(\tau)$$

i.e. function of two variables reduces to a function of one variable
⟶ can be restored from anomalous dimensions

$$\gamma_N = \int_0^1 d\alpha \int_0^1 d\beta\, (1-\alpha-\beta)^{N-1} h(\alpha,\beta)$$

Braun, Korchemsky, Manashov, 1999

$$h(\alpha,\beta) = -4C_F\left[\delta_+(\tau) + \theta(1-\tau) - \frac{1}{2}\delta(\alpha)\delta(\beta)\right]$$

- **Combined LO DGLAP, ERBL and GPD evolution equations in the most compact form**

Covariant Representation

if $[\mathbb{H}, S_k] = 0$, $\mathbb{H}$ must be a function of the two-particle Casimir operator

$$S_{12}^2 = -\partial_{\alpha_1}\partial_{\alpha_2}(\alpha_1 - \alpha_2)^2 ,$$

To find this function, compare action of $\mathbb{H}$ and S_{12}^2 on the set of functions $(\alpha_1 - \alpha_2)^n$

$$\mathcal{H}_{\rm v}^{(12)} = 2\left[\psi(J_{12}) - \psi(2)\right] , \qquad S_{12}^2 = J_{12}(J_{12}-1)$$
$$\mathcal{H}_{\rm e}^{(12)} = 1/[J_{12}(J_{12}-1)] = 1/S_{12}^2$$

where $\psi(x)$ is the Euler's digamma function

$$\boxed{\mathbb{H}_{\rm ERBL} = 4\left[\psi(J_{12}) - \psi(2)\right] - 2/[J_{12}(J_{12}-1)] + 1}$$

For local operators take S_{12}^2 in the adjoint representation $\widetilde{S}_{12}^2$

Bukhvostov, Frolov, Kuraev, Lipatov '85