Nonperturbative Mellin Amplitudes

EPFL

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based on: • arXiv: 1912.11100 with Joao Silva and Alexander Zhiboedov
• in progress with Dean Carmi, " "

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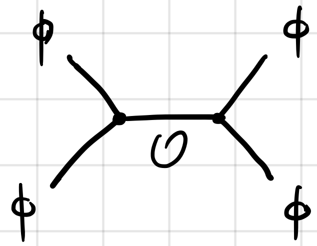
Motivation

- Conformal Bootstrap in Mellin space ?

$$\langle \phi(x_1) \dots \phi(x_4) \rangle = \sum_{\mathcal{O}} C^2_{\mathcal{O}\phi\phi}$$



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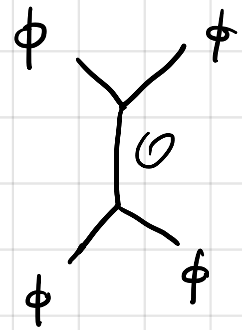

$\rightarrow$ More efficient Bootstrap ?

Mellin-Polyakov Bootstrap, Holographic Correlators, ...

- When does the Mellin representation of the 4pt-function exist ?
- How to impose crossing, OPE and unitarity in Mellin space ?

- S-matrix from flat space limit of QFT in AdS

Outline

* Motivation
* Mellin Amplitudes Basics
* Existence and Analyticity
* Polyakov conditions
* Regge boundedness
* Dispersion relations
* Sum rules
* Open Questions

Mellin Amplitudes basics

Scaling dimension of ϕ

$$\sum_{\substack{j=1 \\ j\neq i}}^{4} \gamma_{ij} = \Delta$$

$$\langle \phi(x_1) \dots \phi(x_4) \rangle = \int [d\gamma] \, \underbrace{\boxed{M(\gamma_{ij})}}_{\text{Mellin Amplitude}} \prod_{1\le i<j\le 4} \frac{\Gamma(\gamma_{ij})}{(x_i - x_j)^{2\gamma_{ij}}}$$

Mellin Amplitude

$$= \frac{1}{(x_{13}^2 \; x_{24}^2)^{\Delta}} \int_{-i\infty}^{i\infty} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} \underbrace{\boxed{M(\gamma_{12},\gamma_{14})} \, \Gamma^2(\gamma_{12}) \, \Gamma^2(\gamma_{14}) \, \Gamma^2(\Delta - \gamma_{12} - \gamma_{14})}_{\hat{M}(\gamma_{12},\, \gamma_{14})} \, u^{-\gamma_{12}} \, v^{-\gamma_{14}}$$

cross ratios

$$u = \frac{x_{12}^2 \, x_{34}^2}{x_{13}^2 \, x_{24}^2}$$

$$v = \frac{x_{14}^2 \; x_{23}^2}{x_{13}^2 \; x_{24}^2}$$

Crossing : $\gamma_{12} + \gamma_{13} + \gamma_{14} = \Delta$

$$M(\gamma_{12}, \gamma_{14}) = M(\gamma_{14}, \gamma_{12}) = M(\gamma_{12}, \gamma_{13})$$

OPE $\Rightarrow$ **Poles**

$$\phi \times \phi = \sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}} \, \mathcal{O}$$

twist

$\tau = \Delta_{\mathcal{O}} - J_{\mathcal{O}}$

Mack polynomial

$$M(\gamma_{12}, \gamma_{14}) \approx \frac{C^2_{\phi\phi\mathcal{O}} \; Q_{J,m}(\gamma_{14})}{\gamma_{12} - \Delta + \frac{\tau}{2} + m}$$

$m = 0, 1, 2, 3, \dots$

$\rightarrow$ Analogy :

Mellin Amplitudes $\longleftrightarrow$ Scattering Amplitudes

$\gamma_{ij} \longleftrightarrow P_i \cdot P_j$

Existence and Analyticity

$$F(u,v) \equiv \left(x_{13}^2 \, x_{24}^2 \right)^{\Delta} \langle \phi(x_1) \dots \phi(x_4) \rangle = \int_{-i\infty}^{i\infty} \frac{d\gamma_{12} \, d\gamma_{14}}{(2\pi i)^2} \hat{M}(\gamma_{12}, \gamma_{14}) \, u^{-\gamma_{12}} \, v^{-\gamma_{14}}$$

$$\Rightarrow \qquad \hat{M}(\gamma_{12}, \gamma_{14}) = \int_0^\infty \frac{du \, dv}{u \, v} \, u^{\gamma_{12}} \, v^{\gamma_{14}} \, F(u,v)$$

but inverse Mellin does not converge !

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but inverse Mellin does not converge!

Idea: split the integral

$$\boxed{K(\gamma_{12}, \gamma_{14}) \equiv \int_0^{1} \frac{du\, dv}{u\, v} \, u^{\gamma_{12}} \, v^{\gamma_{14}} \, F(u,v)}$$

Analytic for $\operatorname{Re} \gamma_{12} > \Delta$ and $\operatorname{Re} \gamma_{14} > \Delta$.

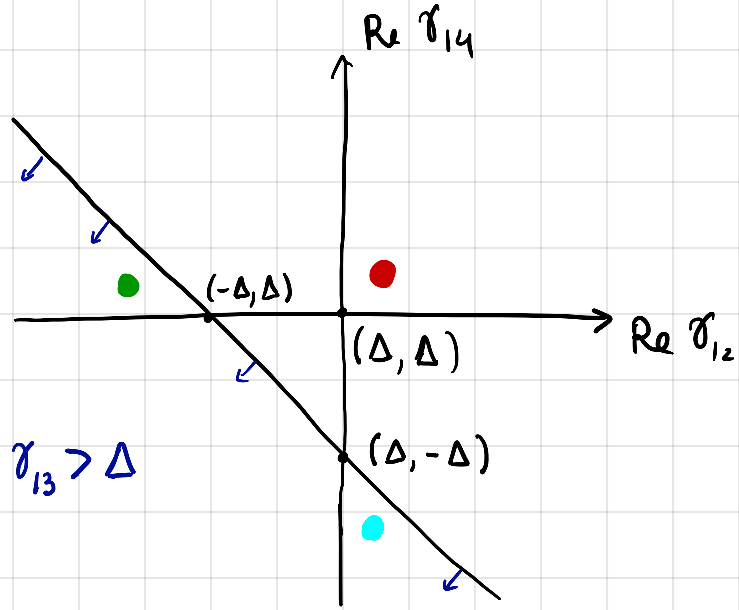
$$F(u,v) = \int_{\mathrm{Re}(\gamma_{12},\gamma_{14}) > \Delta} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} K(\gamma_{12}, \gamma_{14})\, u^{-\gamma_{12}} v^{-\gamma_{14}}$$

$$+ \int_{\mathrm{Re}(\gamma_{13},\gamma_{14}) > \Delta} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} K(\gamma_{13}, \gamma_{14})\, u^{-\gamma_{12}} v^{-\gamma_{14}}$$

$$+ \int_{\mathrm{Re}(\gamma_{12},\gamma_{13}) > \Delta} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} K(\gamma_{12}, \gamma_{13})\, u^{-\gamma_{12}} v^{-\gamma_{14}}$$

where we used crossing.

$$\gamma_{12} + \gamma_{13} + \gamma_{14} = \Delta$$



If we can bring the 3 integrals to the same contour then

$$\boxed{\hat{M}(\gamma_{12}, \gamma_{14}) \equiv K(\gamma_{12}, \gamma_{14}) + K(\gamma_{12}, \gamma_{13}) + K(\gamma_{13}, \gamma_{14})}$$

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Example : GFF

$$F(u, v) = 1 + u^{-\Delta} + v^{-\Delta}$$

$$K(\gamma_{12}, \gamma_{14}) = \frac{1}{\gamma_{12}\gamma_{14}} + \frac{1}{\gamma_{12}(\gamma_{14}-\Delta)} + \frac{1}{(\gamma_{12}-\Delta)\gamma_{14}}$$

$$\hat{M} = 0$$

→ Disconnected part of the correlator does not contribute to M

Contour deformation requires analytic continuation of K.

$$\boxed{K(\gamma_{12}, \gamma_{14}) \equiv \int_0^1 \frac{du\, dv}{u\, v}\, u^{\gamma_{12}}\, v^{\gamma_{14}}\, F(u,v)}$$

$$F(u,v) = \sum_{\tau \geq 0} u^{\tau/2 - \Delta} f_\tau(v)$$

$$\Downarrow$$

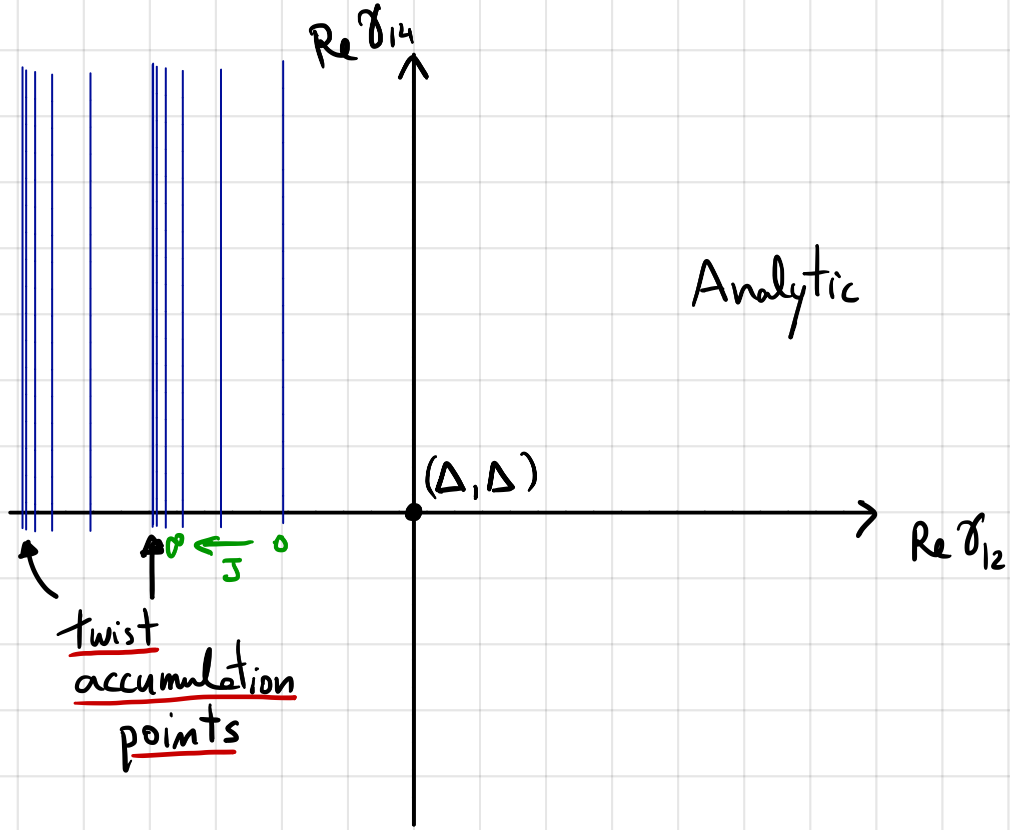
$$\boxed{\text{poles at } \gamma_{12} = \Delta - \frac{\tau}{2}}$$

τ = twist of $\mathcal{O} \subset \phi \times \phi$

$\exists$ double twist operators

$$\tau = 2\Delta + 2n + \gamma(n,J)$$

$$\gamma(n,J) \xrightarrow{J \to \infty} 0$$



Twist spectrum discrete or continuous?

OPE :

$$F(u,v) = \sum_{\tau,\ell} \underbrace{a_{\tau,\ell}}_{\geq 0} \, u^{\tau/2-\Delta} \left(z^{\ell} + \bar{z}^{\ell} \right)$$

$u = z\bar{z}$
$v = (1-z)(1-\bar{z})$

$$K(\gamma_{12},\gamma_{14}) = \underbrace{\int_0^1 \frac{du\,dv}{uv} \, u^{\gamma_{12}} v^{\gamma_{14}} \left[F_{sub}(u,v) \right.}_{\text{Analytic for } \operatorname{Re}\gamma_{14} > \Delta,\ \operatorname{Re}\gamma_{12} > \Delta - \frac{\tau_{max}}{2}} + \underbrace{\left. \sum_{\tau<\tau_{max}} \sum_{\ell} a_{\tau,\ell} \, u^{\tau/2-\Delta} \left(z^{\ell} + \bar{z}^{\ell} \right) \right]}_{\text{Integrate term by term}}$$

Analytic for
$\operatorname{Re}\gamma_{14} > \Delta$
$\operatorname{Re}\gamma_{12} > \Delta - \frac{\tau_{max}}{2}$

Integrate term by term

$\Downarrow$

poles at $\gamma_{12} = \Delta - \tau/2$

analytic for $\operatorname{Re}\gamma_{14} > \Delta$

OPE :

$$F(u,v) = \sum_{\tau,\ell} \underbrace{a_{\tau,\ell}}_{\geq 0} u^{\tau/2-\Delta} \left(z^\ell + \bar{z}^\ell\right)$$

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Integrate term by term

$\Downarrow$

poles at $\gamma_{12} = \Delta - \tau/2$

analytic for $\text{Re}\,\gamma_{14} > \Delta$

Double-Light cone limit bound

$$F_{sub} \lesssim u^{-\Delta + \frac{\tau_{max}}{2}} v^{-\Delta}$$

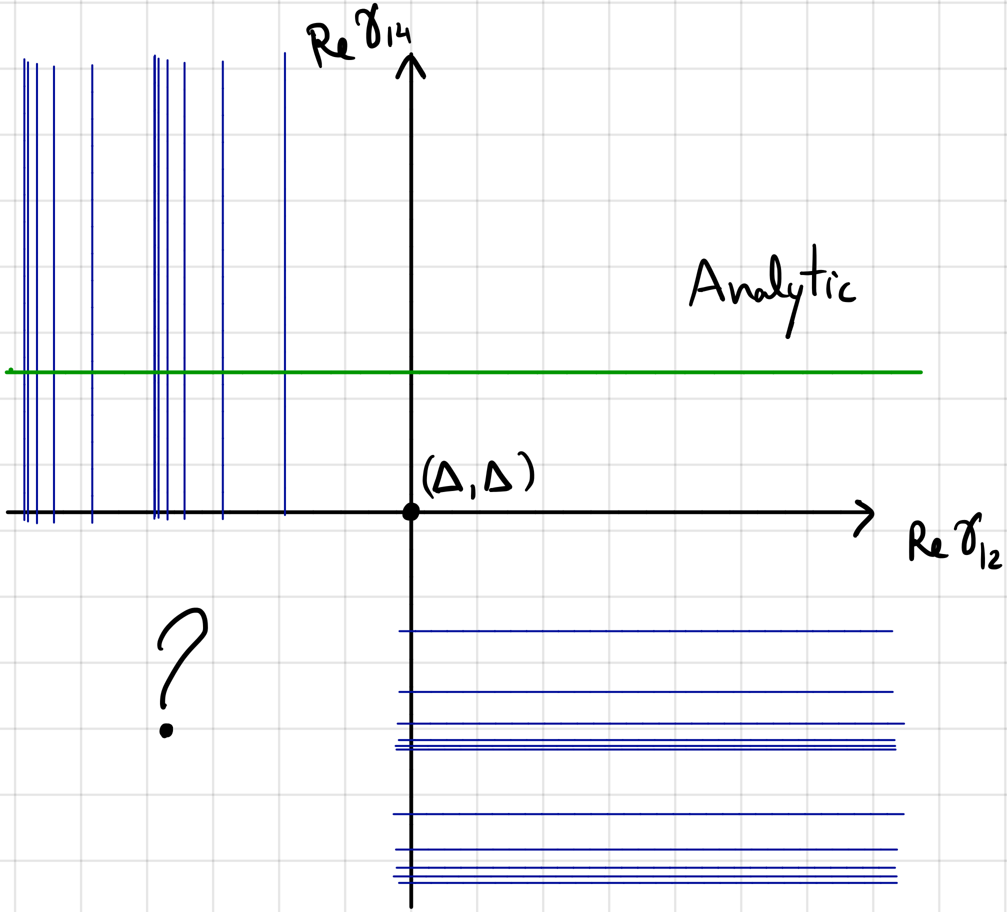
$$\boxed{u \sim v \to 0}$$

This follows from :

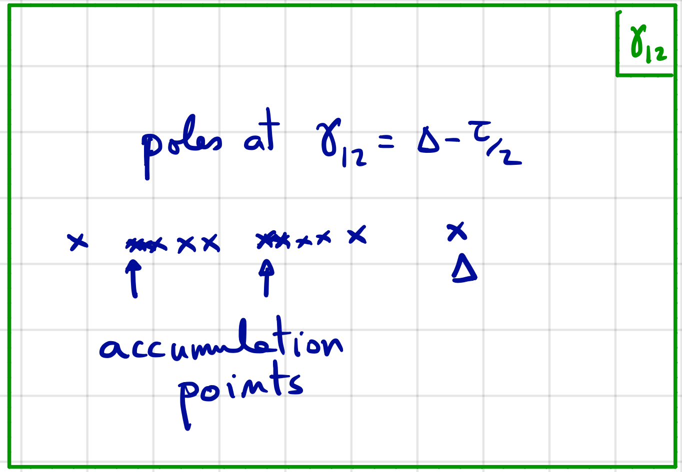
$$F_{sub} = \sum_{\tau \geq \tau_{max}} \sum_{\ell} a_{\tau,\ell} (z\bar{z})^{\tau/2-\Delta} \left(z^\ell + \bar{z}^\ell\right) \Rightarrow (z_1\bar{z})^{\Delta - \frac{\tau_{max}}{2}} F_{sub}(z_1,\bar{z}) < (z_2\bar{z})^{\Delta - \frac{\tau_{max}}{2}} F_{sub}(z_2,\bar{z})$$

$$0 < z_1 < z_2 < 1 \quad , \quad 0 < \bar{z} < 1$$

Using the OPE, we showed that



fixed γ_{14} with $\mathrm{Re}\,\gamma_{14} > \Delta$



Maximal Mellin Analyticity Conjecture : The OPE poles are the only singularities

[If no accumulation points $\Longrightarrow$ No more singularities]

Bochner's theorem

If Δ = minimal twist, then we proved MMA in the blue region that contains the crossing symmetric point $\gamma_{12} = \gamma_{14} = \gamma_{13} = \frac{\Delta}{3}$.

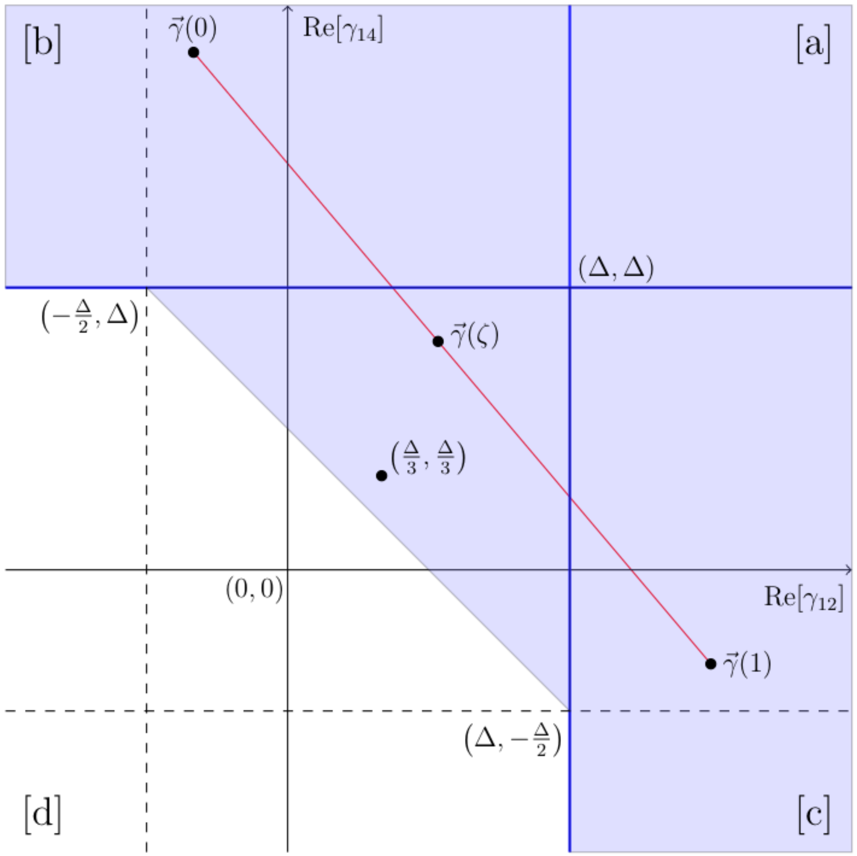


Figure 19: Construction that leads to equations (317) and (318). We represent the case $\Delta = \tau_{lightest}$ and therefore $\tau_* = 2\Delta$. We can establish analyticity in the shaded domain without crossing the accumulation point of accumulation points of triple-twist operators (marked with dashed lines).

Straight Contour

$$F(u,v) = \int_{\mathrm{Re}(\gamma_{12}) = \mathrm{Re}(\gamma_{14}) = \frac{\Delta}{3}} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} \hat{M}(\gamma_{12}, \gamma_{14})\, u^{-\gamma_{12}}\, v^{-\gamma_{14}}$$

$$+ \sum_{0 \leq \tau < \frac{4\Delta}{3}} c_\tau^2 \left[u^{\frac{\tau}{2} - \Delta} g_{\tau, \ell(\tau)}(v) + v^{\frac{\tau}{2} - \Delta} g_{\tau, \ell(\tau)}(u) + v^{-\tau/2} g_{\tau, \ell(\tau)}\left(\frac{u}{v}\right) \right]$$

↳ collinear block

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↳ collinear block

If $\boxed{\Delta < \frac{3}{4} \tau_{gap}}$ then $F - (\text{subtractions}) = F_{\text{connected}}$.

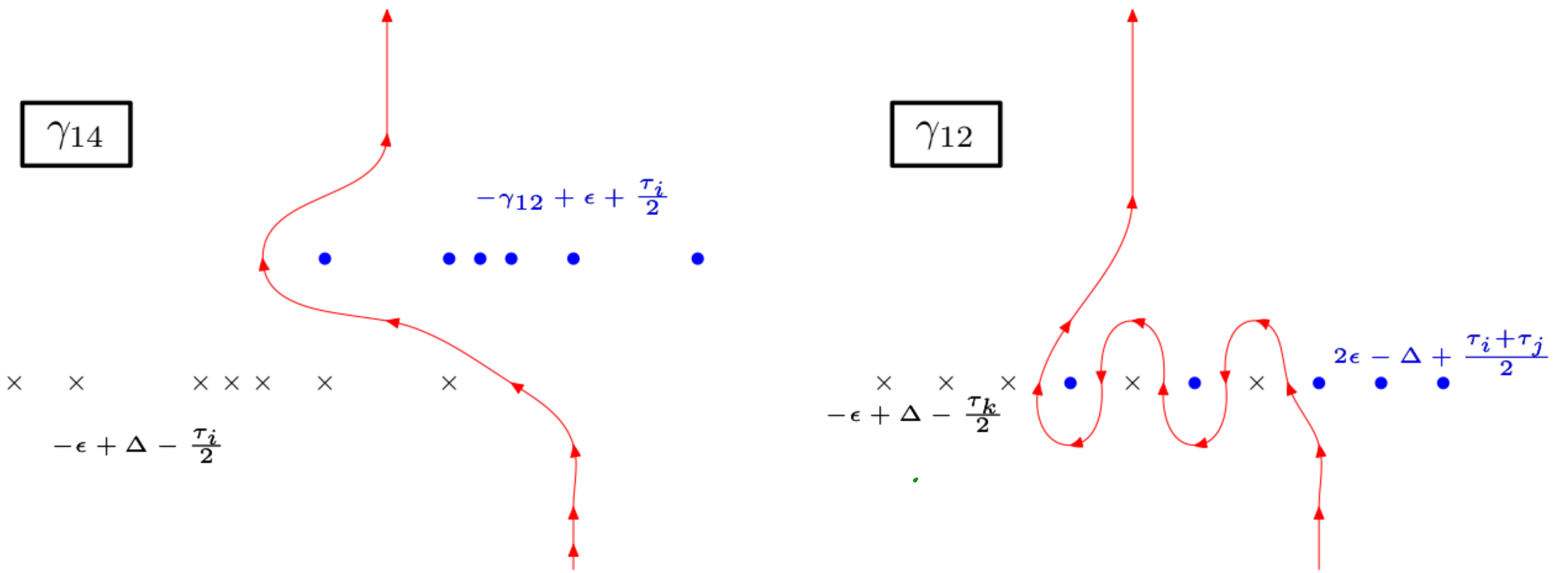
Example : 3D Ising $\Delta = 0.518$ $\tau_{gap} = 1$

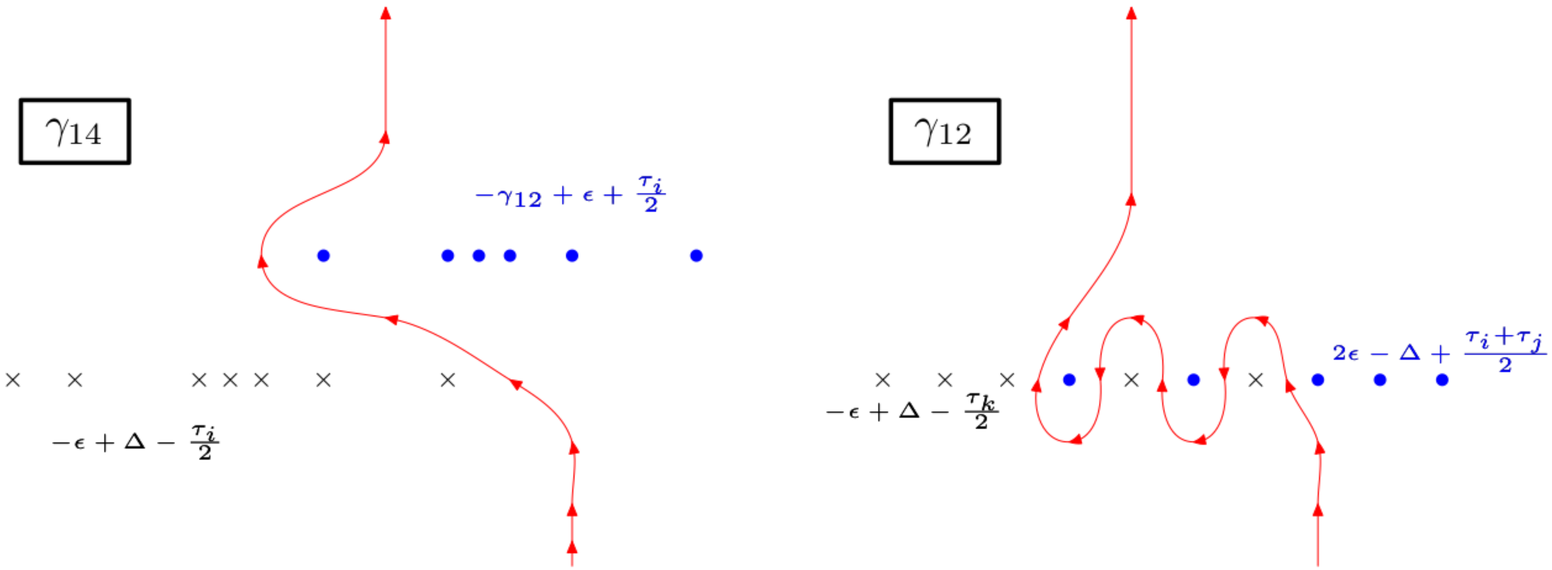
$$\boxed{(x_{13}^2\, x_{24}^2)^{\Delta} \langle \sigma(x_1) \cdots \sigma(x_4) \rangle_{\text{conn}} = \int_{\mathrm{Re}(\gamma_{12}) = \mathrm{Re}(\gamma_{14}) = \frac{\Delta}{3}} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} \hat{M}(\gamma_{12}, \gamma_{14})\, u^{-\gamma_{12}}\, v^{-\gamma_{14}}}$$

Deformed Contour

$$F(u,v) = \int_C \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2}\, \hat{M}(\gamma_{12}, \gamma_{14})\, u^{-\gamma_{12}}\, v^{-\gamma_{14}}$$

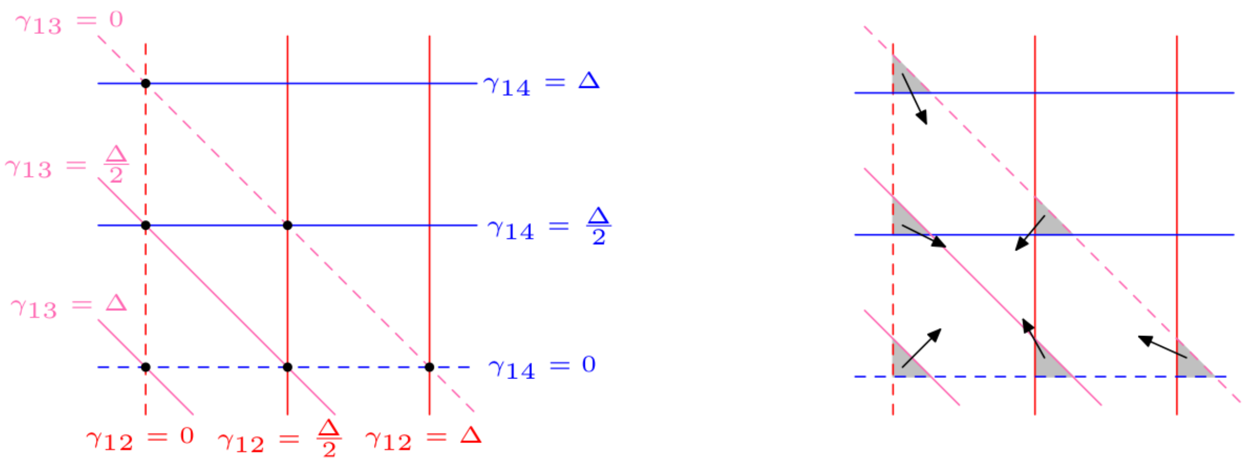
$$\left.\begin{aligned} &+ 1 + u^{-\Delta} + v^{-\Delta} \\ &+ C^2_{\phi\phi\phi}\left[u^{-\Delta/2} g_{\Delta,0}(u) + v^{-\Delta/2} g_{\Delta,0}(v) + v^{-\Delta/2} g_{\Delta,0}\left(\frac{u}{v}\right)\right] \end{aligned}\right\} \text{"pinches"}$$





<u>Deformed contour</u> pinched if $\tau_i + \tau_j + \tau_k = 4\Delta$

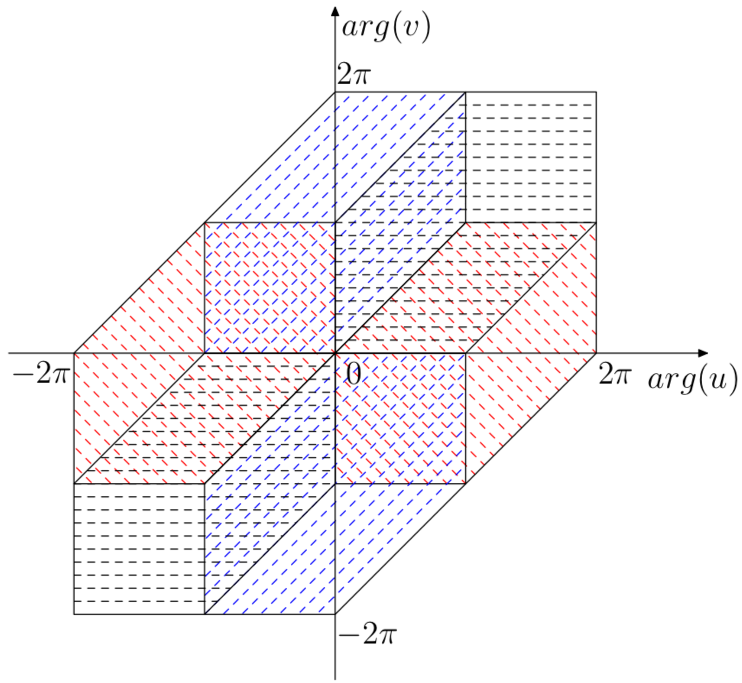
Generic CFT $\rightarrow$ 6 pinches.



Sectorial Domain of Analyticity

$F\left(|u|e^{i\arg u}, |v|e^{i\arg v}\right)$ analytic $\forall\ |u|>0,\ |v|>0,\ (\arg u, \arg v) \in \Theta_{CFT}$

OPE in 3 channels $\Rightarrow$ Θ_{CFT} $=$



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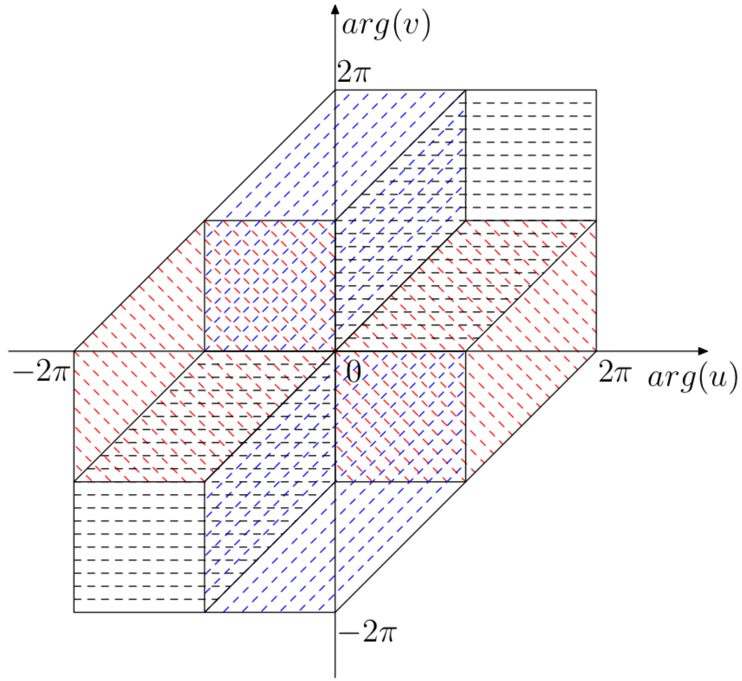

$$F(u,v) = \int \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2}\, |u|^{-\gamma_{12}}\, |v|^{-\gamma_{14}}\, \hat{M}(\gamma_{12},\gamma_{14})\, \exp\left(\arg u\ \mathrm{Im}\,\gamma_{12} + \arg v\ \mathrm{Im}\,\gamma_{14}\right)$$

$$\Rightarrow \quad \underset{\Gamma^2(\gamma_{12})\,\Gamma^2(\gamma_{14})\,\Gamma^2(\Delta-\gamma_{12}-\gamma_{14})\,M(\gamma_{12},\gamma_{14})}{\underset{\shortparallel}{\hat{M}(\gamma_{12},\gamma_{14})}} \sim \exp\left[-\operatorname{Sup}_{\Theta_{CFT}}\left(\arg u\ \mathrm{Im}\,\gamma_{12} + \arg v\ \mathrm{Im}\,\gamma_{14}\right)\right]$$

$\Rightarrow$ <u>M is polynomially bounded</u>!

Polyakov Conditions

Naively $\quad M(\gamma_{12},\gamma_{14}) \xrightarrow[\gamma_{12}\to -n]{} 0 \quad$ and $\quad \dfrac{M(\gamma_{12},\gamma_{14})}{\gamma_{12}+n} \xrightarrow[\gamma_{12}\to -n]{} 0 \qquad n = 0, 1, 2, \dots$

because

$$M(\gamma_{12},\gamma_{14}) = \frac{\hat{M}(\gamma_{12},\gamma_{14})}{\Gamma^2(\gamma_{12})\,\Gamma^2(\gamma_{14})\,\Gamma^2(\Delta-\gamma_{12}-\gamma_{14})} .$$

However, $\quad \gamma_{12} = -n \quad$ are accumulation points of poles !

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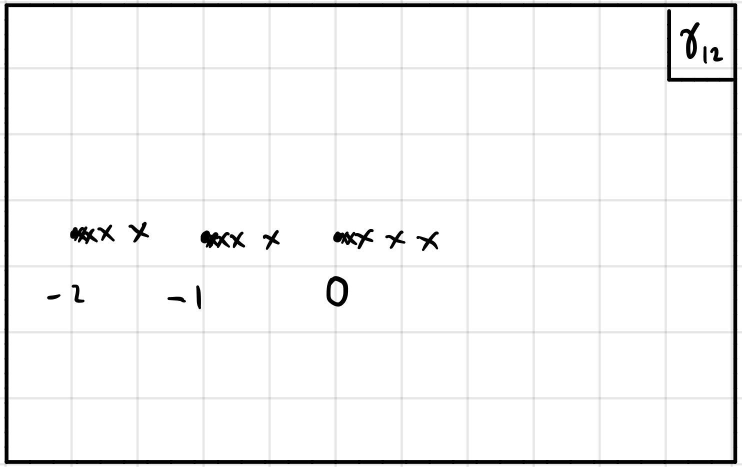

double-twist Regge trajectories

$$\tau = 2\Delta + 2n + \gamma(n,J)$$

$$\Rightarrow \text{poles at } \gamma_{12} = -n - \tfrac{1}{2}\gamma(n,J)$$

$$\gamma(n,J) \sim \frac{f(n)}{J^{\tau_{gap}}} , \quad J \to \infty$$

Analogous to $s \to \infty$ with fixed t for Veneziano amplitude.

$$A(s,t) \sim s^{\alpha(t)}\,\beta(t) \qquad s \to \infty\, e^{i\theta},\ \theta \neq 0, \pi$$

Near the accumulation point $\boxed{\gamma_{12} = 0}$

$$\gamma_{12}\,\hat{M}(\gamma_{12},\gamma_{14}) \sim \sum_{\substack{J=0\\ J\,\text{even}}}^{\infty} \frac{1}{\gamma_{12} + \gamma(J)/2} \operatorname*{Res}_{\gamma_{12}=-\gamma(J)/2} \hat{M}(\gamma_{12},\gamma_{14})\,\gamma_{12}$$

$$\left[\frac{\gamma(J)}{2} \sim -\frac{a}{J^{\tau_{gap}}}\right] \sim \int_1^\infty dJ\, \frac{1}{\gamma_{12} - \frac{a}{J^{\tau_{gap}}}} \underbrace{\left[C_J^2\right]^{GFF} Q_J(\gamma_{14})\,\gamma(J)}_{J^{2\gamma_{14}-\tau_{gap}-1}}$$

$$f(\gamma_{14}) + \left[\gamma_{14} \longleftrightarrow \Delta - \gamma_{14}\right]$$

Near the accumulation point $\gamma_{12} = 0$

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$$\left[\frac{\gamma(J)}{2} \sim -\frac{a}{J^{\tau_{gap}}}\right] \qquad \sim \int_1^\infty dJ \, \frac{1}{\gamma_{12} - \frac{a}{J^{\tau_{gap}}}} \underbrace{[C_J^2]^{GFF}\, Q_J(\gamma_{14})\, \gamma(J)}_{J^{2\gamma_{14}-\tau_{gap}-1}}$$

$$f(\gamma_{14}) + [\gamma_{14} \leftrightarrow \Delta-\gamma_{14}]$$

converges for $\boxed{\Delta - \tau_{gap}/2 < \gamma_{14} < \tau_{gap}/2}$

$$\gamma_{12}\,\hat{M}(\gamma_{12},\gamma_{14}) \sim (-\gamma_{12})^{-\frac{2\gamma_{14}}{\tau_{gap}}}\, g(\gamma_{14}) + (\gamma_{14} \leftrightarrow \Delta-\gamma_{14})$$

"Regge behavior"

Polyakov condition : $\boxed{M(\gamma_{12}=0,\gamma_{14}) = 0 \,, \quad \Delta - \frac{\tau_{gap}}{2} < \gamma_{14} < \frac{\tau_{gap}}{2}}$

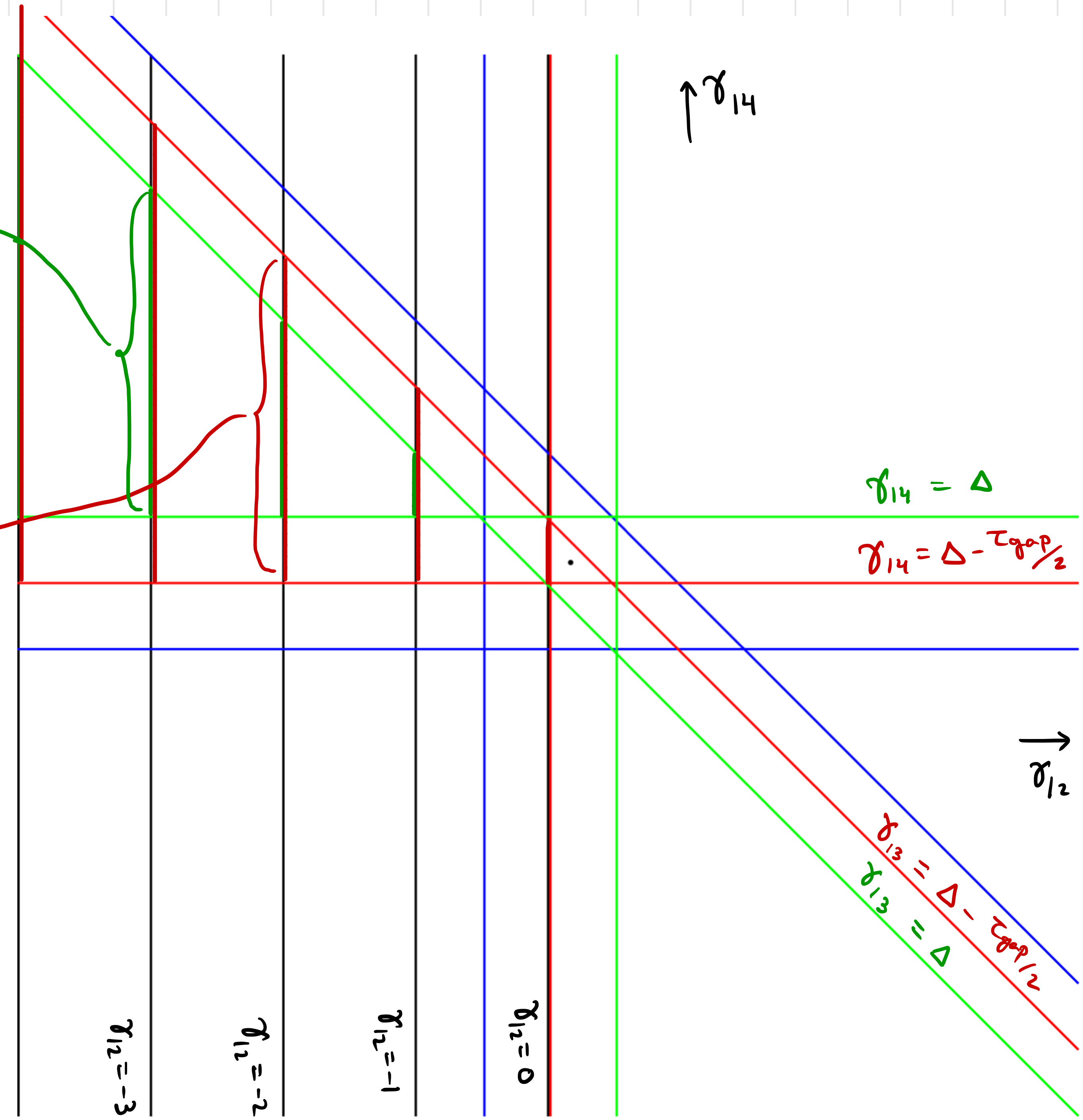
[Preliminary]

$$\boxed{\frac{M(\gamma_{12},\gamma_{14})}{\gamma_{12}+n} \xrightarrow[\gamma_{12}\to -n]{} 0}$$

$$\boxed{M(\gamma_{12},\gamma_{14}) \xrightarrow[\gamma_{12}\to -n]{} 0}$$

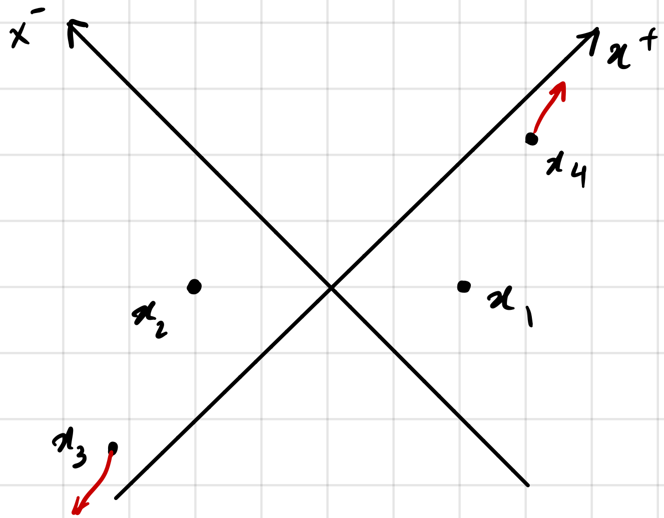
For $\gamma_{12} \to -n$:

$$M(\gamma_{12},\gamma_{14}) \sim (\gamma_{12}+n)^{1+\frac{2(\gamma_{14}-\Delta)}{\tau_{gap}}} + (\gamma_{12}+n)^{1+\frac{2(\gamma_{13}-\Delta)}{\tau_{gap}}}$$



Regge limit

$$\begin{cases} x_1^\pm = \pm 1 \\ x_2^\pm = \mp 1 \\ x_3^\pm = \mp e^{\rho \pm t} \\ x_4^\pm = \pm e^{\rho \pm t} \end{cases}$$



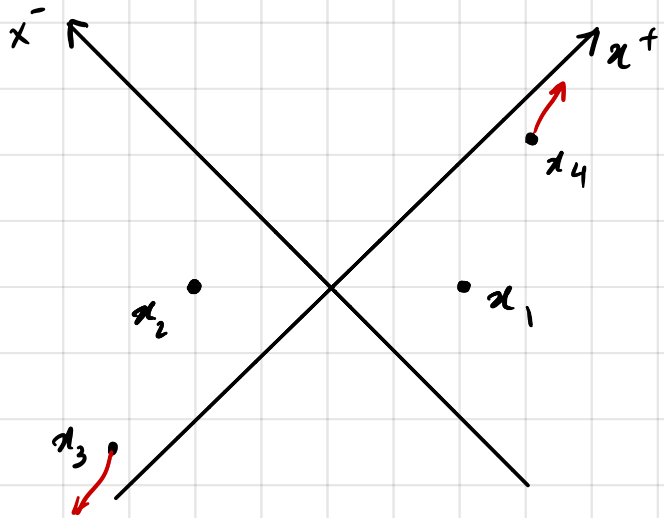
$t \to \infty \iff \gamma_{14} \to \infty$

$$u = 16\, e^{-2t} \xrightarrow[t \to \infty]{} 0$$
$$v \simeq 1 - 8\, e^{-t} \cosh \rho \xrightarrow{} 1$$

Unitarity + OPE $\Rightarrow$ $\displaystyle \lim_{t \to \infty} \left| \frac{F(u,v)}{F_{disc}(u,v)} \right| \leq 1$

Regge limit

$$\begin{cases} x_1^{\pm} = \pm 1 \\ x_2^{\pm} = \mp 1 \\ x_3^{\pm} = \mp e^{\rho \pm t} \\ x_4^{\pm} = \pm e^{\rho \pm t} \end{cases}$$



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Unitarity + OPE $\Rightarrow$ $\displaystyle \lim_{t\to\infty} \left| \frac{F(u,v)}{F_{disc}(u,v)} \right| \leq 1$

From time ordering

$$F(u,v) = \int \frac{d\gamma_{12}}{2\pi i}\, u^{-\gamma_{12}}\, \Gamma^2(\gamma_{12}) \int \frac{d\gamma_{14}}{2\pi i} \underbrace{\Gamma^2(\gamma_{14})\,\Gamma^2(\Delta-\gamma_{12}-\gamma_{14})\,(v\, e^{2\pi i})^{-\gamma_{14}}}_{\approx\ (\gamma_{14})^{2(\Delta-\gamma_{12}-1)}\, e^{-\gamma_{14}\log v}} \underbrace{M(\gamma_{12},\gamma_{14})}_{\approx\ \gamma_{14}^{j_0}\, m(\gamma_{12})}$$

Regge boundedness

$$\lim_{\gamma_{14} \to i\infty} M(\gamma_{14}, \gamma_{12}) \leq c\,|\gamma_{14}| \qquad , \qquad \mathrm{Re}\,\gamma_{12} > \Delta - \frac{\tau_{gap}}{2}$$

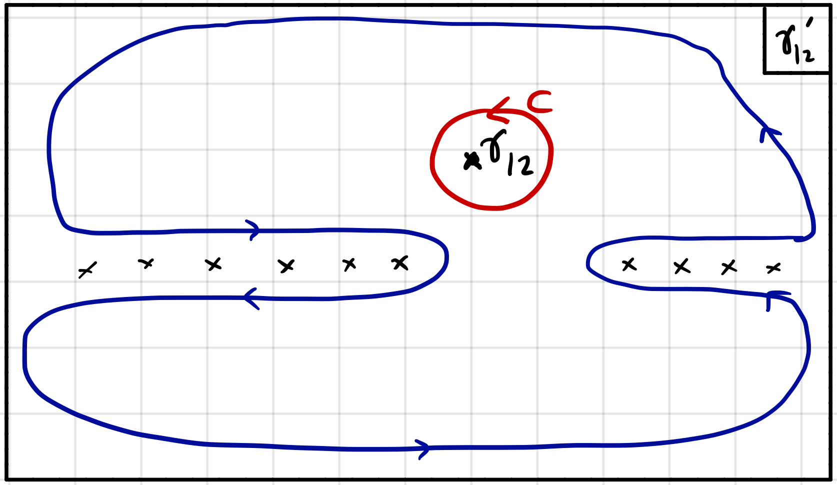
Conjecture:

$$\boxed{\lim_{|\gamma_{14}| \longrightarrow \infty} M(\gamma_{14}, \gamma_{12}) \leq c\,|\gamma_{14}| \qquad , \qquad \mathrm{Re}\,\gamma_{12} > \Delta - \frac{\tau_{gap}}{2}}$$

Validity at the crossing symmetric point $\Rightarrow$ $\Delta < \frac{3}{4}\tau_{gap}$

$$\gamma_{12} = \gamma_{13} = \gamma_{14} = \frac{\Delta}{3}$$

Dispersion Relations



$$\frac{M(\gamma_{12},\gamma_{14})}{\gamma_{12}\,\gamma_{13}} = \oint_C \frac{d\gamma_{12}'}{2\pi i}\,\frac{1}{\gamma_{12}'-\gamma_{12}}\,\frac{M(\gamma_{12}',\gamma_{14})}{\gamma_{12}'\,(\Delta-\gamma_{14}-\gamma_{12}')} =$$

$$\boxed{\Delta-\frac{\tau_{gap}}{2} < \gamma_{14} < \frac{\tau_{gap}}{2}}$$

$$= \sum_{\tau} \frac{\underset{\gamma_{12}=\Delta-\tau/2}{\mathrm{Res}}\, M(\gamma_{12},\gamma_{14})}{\left(\Delta-\tau/2\right)\left(\frac{\tau}{2}-\gamma_{14}\right)} \left[\frac{1}{\gamma_{12}-\Delta+\tau/2} + \frac{1}{\gamma_{13}-\Delta+\tau/2}\right]$$

with $\gamma_{13} = \Delta - \gamma_{12} - \gamma_{14}$

Dispersion Relations

$$\frac{M(\gamma_{12},\gamma_{14})}{\gamma_{12}\,\gamma_{13}} = \oint \frac{d\gamma_{12}'}{2\pi i}\,\frac{1}{\gamma_{12}'-\gamma_{12}}\,\frac{M(\gamma_{12}',\gamma_{14})}{\gamma_{12}'\,(\Delta-\gamma_{14}-\gamma_{12}')} =$$

$$\boxed{\Delta-\frac{\tau_{gap}}{2}<\gamma_{14}<\frac{\tau_{gap}}{2}}$$

$$= \sum_{\tau} \frac{\operatorname*{Res}_{\gamma_{12}=\Delta-\tau/2} M(\gamma_{12},\gamma_{14})}{(\Delta-\tau/2)\,(\frac{\tau}{2}-\gamma_{14})}\left[\frac{1}{\gamma_{12}-\Delta+\tau/2}+\frac{1}{\gamma_{13}-\Delta+\tau/2}\right]$$

with $\gamma_{13}=\Delta-\gamma_{12}-\gamma_{14}$

$$\boxed{M(\gamma_{12},\gamma_{14}) = \sum_{\tau,J} C^2_{\tau,J}\,Q_J(\gamma_{14})\,\frac{\gamma_{12}\,\gamma_{13}}{(\Delta-\tau/2)(\frac{\tau}{2}-\gamma_{14})}\left[\frac{1}{\gamma_{12}-\Delta+\tau/2}+\frac{1}{\gamma_{13}-\Delta+\tau/2}\right]}$$

Crossing: $\gamma_{12}\longleftrightarrow\gamma_{13}$ manifest ✓

$\gamma_{12}\longleftrightarrow\gamma_{14}$ $\Rightarrow$ sum rules $\boxed{M(\gamma_{12},\gamma_{14})-M(\gamma_{14},\gamma_{12})=0}$

Sum rules

For example, $\dfrac{\partial}{\partial y} M\left(\dfrac{\Delta}{3}+y, \dfrac{\Delta}{3}-y\right)\Big|_{y=0} = 0$

$\Delta < \frac{3}{4}\tau_{gap}$

$$\Rightarrow \quad \boxed{\sum_{\tau,J} C^2_{\tau,J}\, \alpha_{\tau,J} = 0}$$

Mack polynomials

$$\alpha_{\tau,J} = -\frac{(\tau-\Delta)\, Q_J\left(\frac{\Delta}{3}\right)}{\left(\tau-\frac{4\Delta}{3}\right)^2\left(\tau-\frac{2\Delta}{3}\right)^2} + \frac{\Delta}{3}\,\frac{Q'_J\left(\frac{\Delta}{3}\right)}{\left(\tau-\frac{4\Delta}{3}\right)\left(\tau-\frac{2\Delta}{3}\right)\left(\tau-2\Delta\right)}$$

Sum rules

For example, $\dfrac{\partial}{\partial y} M\left(\frac{\Delta}{3}+y, \frac{\Delta}{3}-y\right)\Big|_{y=0} = 0$

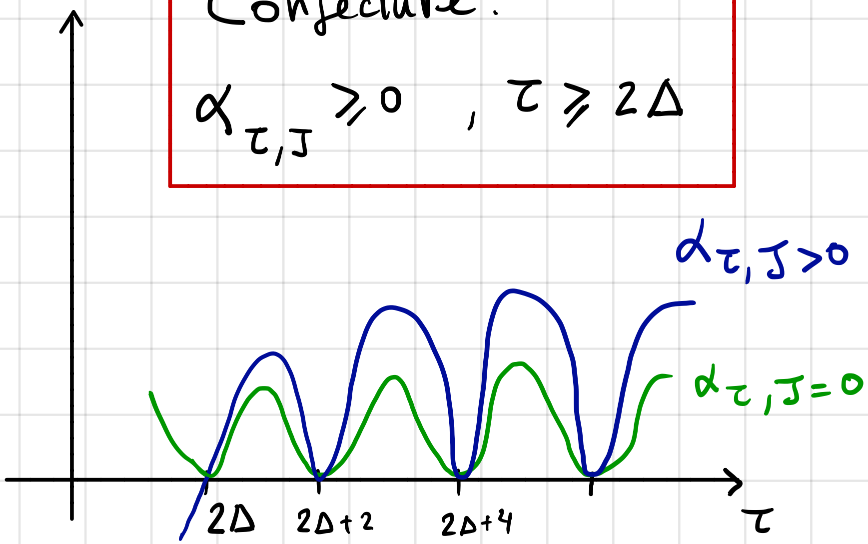
$$\Rightarrow \quad \boxed{\sum_{\tau,J} C^2_{\tau,J}\, \alpha_{\tau,J} = 0}$$

Mack polynomials $\times \dfrac{1}{\Gamma^2(\Delta-\tau/2)}$

$$\alpha_{\tau,J} = -\frac{(\tau-\Delta)\, Q_J\left(\frac{\Delta}{3}\right)}{\left(\tau-\frac{4\Delta}{3}\right)^2\left(\tau-\frac{2\Delta}{3}\right)^2} + \frac{\Delta}{3}\,\frac{Q'_J\left(\frac{\Delta}{3}\right)}{\left(\tau-\frac{4\Delta}{3}\right)\left(\tau-\frac{2\Delta}{3}\right)(\tau-2\Delta)}$$

Conjecture:

$\alpha_{\tau,J} \geq 0 \quad , \quad \tau \geq 2\Delta$



$\Rightarrow$

Extremal Functional

for $\max \tau_{gap} = \,?$

Solution : GFF $\quad \tau_{gap} = 2\Delta$

Example : 3D Ising

$$\underbrace{\underset{\text{Leading Regge traj.}}{\sum} C^2_{\tau,J}\,\alpha_{\tau,J}}_{<0} + \underbrace{\underset{\text{Rest}}{\sum} C^2_{\tau,J}\,\alpha_{\tau,J}}_{>0} = 0$$

$$\begin{aligned}
<0 &= -0.028968 \quad (T_{\mu\nu}) \\
&\quad -0.012122 \quad (J=4) \\
&\quad -0.029107 \quad (6 \le J \le 30) \\
&\quad -0.0222 \quad (J>30) \\
&= -0.0924
\end{aligned}$$

$$\begin{aligned}
>0 &= +0.084569 \quad (\epsilon) \\
&\quad +0.0018 \quad \left([\sigma,\sigma]_{n=1}^{0\le J\le 30}\right) \\
&\quad +0.0016 \quad \left([\epsilon,\epsilon]_{n=0}^{4\le J\le 30}\right) \\
&\quad +0.0014 \quad \left([\epsilon,\epsilon]_{n=0}^{J>30}\right) \\
&\quad + \cdots \\
&= 0.0894 + \cdots
\end{aligned}$$

They cancel up to 3% error !

Example : Large N CFTs

single-trace : $C^2_{\phi\phi\mathcal{O}_{st}} \sim \frac{1}{N^2}$

double-trace : $C^2_{\phi\phi\mathcal{O}_{dt}} \sim N^0$ $\quad \tau\left[\mathcal{O}_{dt} = :\phi\, \partial^{2n}\partial^J\phi:\right] = 2\Delta + 2n + \frac{1}{N^2}\gamma(n,J)$

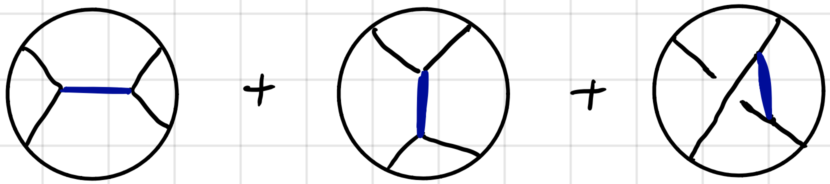
Sum rule at order $\frac{1}{N^2}$:

$$\boxed{\underbrace{\sum_{\mathcal{O}_{st}} C^2_{\phi\phi\mathcal{O}_{st}}\, \alpha_{\tau,J}}_{\text{single-trace exchanges}} + \underbrace{\sum_{J=2}^{\infty} C^2_{\phi\phi[\phi\phi]^J_{n=0}} \frac{\gamma(0,J)}{N^2} \left.\frac{\partial\alpha_{\tau,J}}{\partial\tau}\right|_{\tau=2\Delta}}_{\text{Leading double-Trace trajectory}} + \underbrace{\left(\mathrm{Rest}_{UV} \geq 0\right)}_{\text{Heavy operators}} = O\left(\frac{1}{N^4}\right)}$$

$$\Rightarrow \quad F\left[\{\tau, J, C^2\}_{st}, \{\gamma(0,J)\}_{dt}\right] \leq 0$$

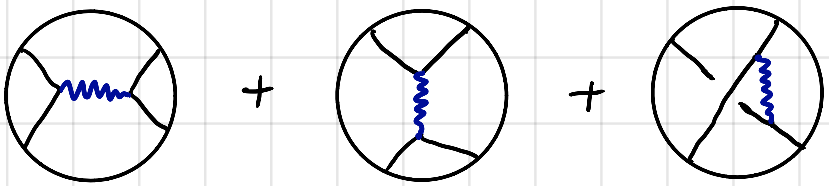
EFTs in AdS

- $\lambda \varphi^2 \chi \qquad \lambda \sim \frac{1}{N} \qquad \Delta < \frac{3}{4} \Delta_\chi$



$\rightarrow \quad F = 0$

- Minimally coupled scalar $\qquad G_N \sim \frac{1}{N^2}$



$\rightarrow \quad F < 0$

$$\frac{d-2}{2} < \Delta < \frac{3}{4}(d-2)$$

Open Questions

- Prove the Maximal Mellin Analyticity conjecture
- Prove Regge bound
- Construct a basis of extremal functionals
- More efficient numerical bootstrap ?
- Extend beyond $\boxed{\Delta < \frac{3}{4}\tau_{gap}}$
- Extend to spinning and non-identical operators
- Higher point functions
- BCFT
- Connection to S-matrix via Flat Space limit
- Connection to Polyakov-Mellin bootstrap

Thank you!