Localization in supersymmetric field theories

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Tentative program

Lecture One:

- Introduction on Euclidean path-integrals and exact results
- Operators: order, disorder & defect theories
- Supersymmetry on curved manifolds

Lecture Two:

- Supersymmetric localization
- S^2 partition function of 2d $\mathcal{N} = (2, 2)$ gauge theories
- One-loop determinants: harmonics, cohomological argument, index theorems

Lecture Three:

- Zamolodgikov metric of CFTs and S^2 partition function
- Higgs branch localization
- Vortex moduli space in Ω-background, through "ADHM" construction

1 Lecture One

In these lectures I will talk about recent developments in supersymmetric quantum field theories—in particular about exact non-perturbative results, such as certain correlators, expectation values of operators and protected indices of states and operators, which can be obtained with localization techniques. The latter are powerful techniques to treat the path-integral, and reduce it to a tractable problem. We will start with some general considerations, and then we will focus on a specific example to be concrete.

1.1 Euclidean path-integrals and exact results

The full information about a local quantum field theory is encoded in the Euclidean Feynman path-integral:

$$\int \mathcal{D}\varphi \, e^{-\frac{S}{\hbar}} \;, \tag{1.1}$$

which is an integral over all possible field configurations in Euclidean spacetime. Unfortunately this is in general too hard to compute, because the integration is over an infinite-dimensional space of functions. The standard approximation scheme is to expand the action around free fields and to work perturbatively. This work very well at weak coupling, but it does not, in general, when the couplings are of order 1—that we call "strong coupling". Because even if we were able to compute all perturbative orders, the resulting series would only be an asymptotic series with zero radius of convergence, and a finite result at strong coupling would only follow after including all non-perturbative corrections.

Thus, one approach is to study theories for which certain path-integrals can be evaluated. Until a few years ago, besides Gaussian free theories, the only available examples where some cohomological and topological field theories on compact manifolds, such as Chern-Simons theory. However, after the works of Nekrasov and Pestun on 4d supersymmetric theories, we have considerably enlarged the class of quantum field theories for which we can compute various path-integrals, to include physical supersymmetric gauge theories.

Let me remark that, even restricting to supersymmetric theories, we cannot compute generic path-integrals with *generic* sources: that would mean that we can solve the theory. What we know how to compute is path-integrals that preserve some supersymmetry, from which we can extract non-trivial information. Besides, notice that there exist other approaches to strong coupling: for instance the bootstrap method to CFTs (see Rychkov's lectures), and integrability to integrable theories (see Göhmann's and Volin's lectures) such as 4d $\mathcal{N} = 4$ SYM in the planar limit.

So, the objective of these lectures is to compute the Euclidean partition functions, *i.e.* path-integrals such as

$$Z_{\mathcal{M}}(t) = \int \mathcal{D}\varphi \, e^{-S[\varphi;\,t]} \;, \tag{1.2}$$

of supersymmetric gauge theories on compact manifolds $\mathcal{M}$ (because this provides a convergent integral), and the expectation values of local and non-local operators that preserve some supersymmetry. A technique that allows us—in favorable circumstaces—to compute such path-integrals *exactly* (in the sense of reducing them to finite-dimensional integrals or series) is "supersymmetric localization".

It turns out that it is a profitable exercise to explore in a systematic way on what manifolds $\mathcal{M}$ it is possible to preserve some supersymmetry, and what is the most general SUSY background on them, for two reasons:

1. On some manifolds and backgrounds, localization reduces the path-integral to a simpler

problem than on others. In other words, we would like to encounter moduli spaces that are simple and tractable.

2. Different manifolds and SUSY backgrounds grant access to different sectors of operators and correlators, including those between holomorphic and anti-holomorphic operators and conserved currents.

The topological twist, which has been extensively studied in the past, is one example. It usually gives access to holomorphic correlators between supersymmetric operators, and it usually reduces the path-integral to interesting but complicated moduli spaces of solutions to partial differential equations (such as moduli spaces of instatons, or of holomorphic maps, etc. . .).

Operator insertions. We will not discuss operators insertions very much, but they are important and can be included. In fact, one can equally well study both local operators (located at a point in spacetime) and non-local operators (located along a submanifold, such as a line as in Wilson and 't Hooft line operators, or a surface, etc. . .).

Operator insertions in the path-integral can be defined in different ways:

- Order operators: they are defined as functions of the fundamental fields in the theory. E.g.: Wilson line operators:

$$W_R[\gamma] = \mathrm{Tr}_R\, \mathcal{P} \exp \oint_\gamma A \,.$$

  In the path-integral, we insert such functions in the integrand.

- Disorder operators: they are defined through (singular) boundary conditions along the submanifold. E.g.: 't Hooft line operators in 4d (or monopole operators in 3d):

$$\text{fix the conjugacy class of } \int_{S^2} F \,.$$

  In the path-integral, we integrate over singular field configurations satisfying the conditions.

- Defect operators: they are defined by introducing extra degrees of freedom on the submanifold, with their own action and coupled to the bulk. E.g.:

$$S_D = \int_\gamma d\tau\, \bar\psi(\partial_\tau - iA_\tau)\psi \,.$$

  In the path-integral, we integrate both over the bulk and defect theories.

These classes are not disjoint: for instance, Wilson line operators can also be described as defect operators (see appendix B).

1.2 SUSY on curved spacetime

The first step to compute the partition function of a QFT on a compact manifold $\mathcal{M}$ is to define the theory on that manifold. In order to be able to apply localization techniques, we need to preserve supersymmetry on the curved manifold $\mathcal{M}$, and this is non-trivial. So, first of all, let us see how to do that.

On a Lorentzian flat spacetime, a supersymmetry algebra is an algebra of symmetries that enlarges the Poincaré group of spacetime symmetries with fermionic generators (anti-commuting with half-integer spin). For instance, in 4d the minimal supersymmetry algebra adds[1]

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \,, \qquad \{Q_\alpha, Q_\beta\} = \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0 \,. \tag{1.3}$$

In a local quantum field theory[2] the supercharges can be written in terms of the *supersymmetry current* $S_{\alpha\mu}$, and the theory is supersymmetric if the supersymmetry current is conserved:

$$Q_\alpha = \int d^{d-1}x\, S_\alpha{}^0 \,, \qquad \partial^\mu S_{\alpha\mu} = 0 \,. \tag{1.5}$$

For theories with a Lagrangian description, the Lagrangian is invariant under the supercharges, up to total derivatives:[3] schematically

$$\delta\,\mathcal{L} = \partial_\mu(\dots)^\mu \,, \tag{1.6}$$

where $\delta = \epsilon^\alpha Q_\alpha$ is anticommuting scalar.

Starting with an Euclidean theory with Lagrangian $\mathcal{L}_{\mathbb{R}^d}$ on flat space $\mathbb{R}^d$, we want to place the theory on a smooth curved manifold $\mathcal{M}$. We require that the theory is not modified at short distances—compared with the characteristic curvature radius of the manifold—because at short distances the manifold is essentially flat. In other words, we only admit deformations of the Lagrangian by *relevant* operators.

This procedure is ambiguous: given a Lagrangian on $\mathcal{M}$, we can always add relevant deformations whose couplings are written in terms of the curvature tensors or powers of the "curvature radius" R, since such terms respect our conditions.

[1] To this algebra one could add 1-brane and 2-brane charges Z_μ, $Z_{\mu\nu}$, which commute with the supercharges but not with the Poincaré generators: $\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}(P_\mu + Z_\mu)$, $\{Q_\alpha, Q_\beta\} = \sigma^{\mu\nu}_{\alpha\beta} Z_{\mu\nu}$.

[2] Every local quantum field theory has a real, symmetric, conserved energy-momentum tensor $T_{\mu\nu}$, which integrates to the momentum:

$$P_\mu = \int d^{d-1}x\, T_\mu{}^0 \,, \qquad \partial^\mu T_{\mu\nu} = 0 \,. \tag{1.4}$$

The existence of a not-necessarily-symmetric $\widehat{T}_{\mu\nu}$ is guaranteed by Noether's theorem, and the fact that it can be improved to a symmetric one is guaranteed by Lorentz invariance. The energy-momentum tensor is not unique: it can be modified by improvement transformations. The energy-momentum tensor and the supersymmetry current are the only operators with spin higher than 1.

[3] In the off-shell formulation of supersymmetry, the algebra closes and the action is invariant. In the on-shell formulation the algebra only closes up to the equations of motion, while the action is still invariant.

We also require that the theory on $\mathcal{M}$ possesses some supersymmetry. The superalgebra in the UV must reduce to a subalgebra of the flat-space superalgebra (since we add relevant deformations), therefore the supercharges on $\mathcal{M}$ are a subset of the ones on $\mathbb{R}^d$. However their algebra can be deformed with respect to the flat-space one. As we will see, given a manifold $\mathcal{M}$, it is not always possible to preserve some supersymmetry. On the other hand, when it is possible, the Lagrangian $\mathcal{L}_\mathcal{M}$ is still ambiguous: requiring some supersymmetry does not fix all the ambiguities in general.

As a first attempt, we can try to simply substitute the flat with the curved metric:

$$\eta_{\mu\nu} \to g_{\mu\nu} \;, \qquad \partial_\mu \to \nabla_\mu \qquad\qquad \text{in } \mathcal{L} \text{ and } \delta \;. \tag{1.7}$$

However this does not work, because

$$\delta_{g,\nabla}\, \mathcal{L}_{g,\nabla} = \nabla_\mu (\dots)^\mu + \dots \;, \tag{1.8}$$

unless it exists on $\mathcal{M}$ some covariantly constant spinor ϵ (then never vanishing): $\nabla_\mu \epsilon = 0$. This, however, imposes drastic constraints on the topology and metric of $\mathcal{M}$ that we do not want.[4]

We can follow two strategies.

1. Trial and error, order by order in a characteristic length scale of $\mathcal{M}$. We can introduce a length scale R rescaling the metric as

   $$g_{\mu\nu} = R^2 g^{(0)}_{\mu\nu} \;. \tag{1.9}$$

   Then we can expand[5]

   $$\delta = \delta^{(0)}_{g,\nabla} + \sum_{n\geq 1} \frac{1}{R^n} \delta^{(n)} \;, \qquad\qquad \mathcal{L} = \mathcal{L}^{(0)}_{g,\nabla} + \sum_{n\geq 1} \frac{1}{R^n} \mathcal{L}^{(n)} \;. \tag{1.10}$$

   Here $\delta^{(0)}_{g,\nabla}$, $\mathcal{L}^{(0)}_{g,\nabla}$ are the flat-space supersymmetry variations and Lagrangian with curved metric, while the correction terms have an extra explicit dependence on R. Since we restrict ourselves to relevant deformations, there can be only a finite number of corrections.

   This procedure is correct (and is widely used in the literature), however it has a few drawbacks. 1) It is not guaranteed to succeed because not all manifold $\mathcal{M}$ admit supersymmetry. 2) It is tedious because one has to deform the Lagrangian, the supersymmetry variations of fields and the supersymmetry algebra. 3) The underlying structure is not manifest and it is not clear what manifolds will admit supersymmetry.

[4] The existence of a parallel spinor $\nabla_\mu \epsilon = 0$ implies first of all that the manifold is spin. Then it also implies that it is Ricci-flat. We have $0 = [\nabla_\mu, \nabla_\nu]\epsilon = R_{\mu\nu}{}^{ab}\gamma_{ab}\epsilon$. Then contract with another gamma and use $\gamma^b\gamma^{cd} = \gamma^{bcd} + \delta^{bc}\gamma^d - \delta^{bd}\gamma^c$ and the first Bianchi identity $R_{a[bcd]} = 0$. This gives $R_{ab}\gamma^b\epsilon = 0$, which implies $R_{\mu\nu} = 0$ because $\gamma^\mu\epsilon$ are independent. In 2d and 3d, this implies that the metric is flat therefore the manifolds are T^2 and T^3. In 4d we can have T^4 and K3.

[5] In this notation all terms are written in terms of g, ∇, but the terms of order (n) have an extra explicit dependence on R.

2. Use a systematic method, explored initially by Festuccia and Seiberg [3], which consists in coupling the theory to off-shell supergravity and taking a *rigid limit* $M_{Pl} \to \infty$. I will now describe this method.

To place a theory on a curved manifold $\mathcal{M}$ we can proceed as follows. We couple the theory to gravity, *i.e.* we add a field $g_{\mu\nu}$—the metric—coupled in a generally covariant way, and then we give an expectation value to such a field. The new theory has more EOMs coming from varying the metric—Einstein's equations—which limit our choices of $\mathcal{M}$. But we take a "rigid limit" in which we send Newton constant $G_N \to 0$, keeping the metric fixed to some background $g_{\mu\nu}$. In the limit, gravity becomes non-dynamical and we do not impose the gravitational equations anymore, therefore we can place the theory on any $\mathcal{M}$. The procedure gives us an action on the curved background.

Our theory is supersymmetric, though, so we should couple it to supergravity. It turns out that, to make the method efficient, we should use *off-shell supersymmetry* which includes auxiliary fields such that the supersymmetry algebra closes with no need to impose the EOMs.

How do we couple to off-shell SUGRA? (see Vandoren's lectures). In a supersymmetric theory the stress tensor $T_{\mu\nu}$ and the supersymmetry current $S_{\alpha\mu}$ sit in the same supermultiplet, called a "supercurrent multiplet", together with other operators of spin less or equal to one. In general they sit in the so-called $\mathcal{S}$-multiplet, which is relatively long.[6] However if the theory has some extra properties, the $\mathcal{S}$-multiplet is decomposable and they sit in a shorter multiplet. For instance:

- if the theory has a continuous non-anomalous R-symmetry, one can reduce (improve) to the $\mathcal{R}$-multiplet;
- if, roughly speaking, the target space for scalars does not have 2-cycles and there is no FI term, one can reduce to the Ferrara-Zumino multiplet;
- if the theory is superconformal, one can reduce to the standard superconformal multiplet.

To each off-shell formulation of supergravity corresponds a supercurrent multiplet. The gravity multiplet contains the metric $g_{\mu\nu}$, the gravitino $\Psi_{\alpha\mu}$, as well as other auxiliary fields of spin less or equal to one. At the linearized level, the supergravity multiplet is coupled to

[6] In 4d $\mathcal{N}=1$ the $\mathcal{S}$-multiplet has 16+16 real independent components, in 3d $\mathcal{N}=2$ it has $12+12$ components, in 2d $\mathcal{N}=(2,2)$ it has 8+8 components. The components are counted as the number of independent fields, minus the number of conservation laws. In 4d, the standard FZ multiplet (12+12) contains $(T_{\mu\nu}, J_\mu, X, \widetilde{X}, S_{\alpha\mu}, \widetilde{S}_{\alpha\mu})$ where J_μ is *not* conserved (more generally X is replaced by a closed 1-form Y, while here $Y = dX$). The $\mathcal{R}$-multiplet (12+12) contains $(T_{\mu\nu}, J^R_\mu, F_{\mu\nu})$ with $0 = \partial^\mu J^R_\mu = dF$. The superconformal multiplet has 8+8 components: $(T_{\mu\nu}, S_{\alpha\mu}, \overline{S}_{\dot\alpha\mu}, J^R_\mu)$ where they are all conserved, $T^\mu_\mu = 0$ and $\gamma^\mu S_\mu = 0$. Recall that a closed n-form in d dimensions has $\binom{d-1}{n-1}$ independent components.

the supercurrent. Thus there is a paring between fields and operators:

$$\begin{array}{l|ccccccc} \text{gravity multiplet} & g_{\mu\nu} & \Psi_{\alpha\mu} & \dots & V_\mu & \psi_\alpha & H & \dots \\ \text{supercurrent} & T_{\mu\nu} & S_{\alpha\mu} & \dots & J_\mu & j_\alpha & \sigma & \dots \end{array} \tag{1.11}$$

and the coupling looks like

$$\mathcal{L}^{\text{lin}}_{\text{SUGRA}} \sim g^{\mu\nu} T_{\mu\nu} + \Psi^{\alpha\mu} S_{\alpha\mu} + \ldots + V^\mu J_\mu + \psi^\alpha j_\alpha + H\sigma + \ldots \tag{1.12}$$

Now we can give an expectation value to all bosonic fields in the gravity multiplet, including the auxiliary fields. We take a rigid limit $G_N \to 0$ keeping fixed the background for the metric and the bosonic auxiliary fields. We do not impose the EOMs, nor we integrate out the auxiliary fields. We only impose supersymmetry, *i.e.* we impose the vanishing of the gravitino variation:[7]

$$\delta\Psi_\mu = 2\nabla_\mu \epsilon + M_\mu(\text{SUGRA fields})\, \epsilon = 0 \; . \tag{1.13}$$

The crucial point is that, in the off-shell formulation, the gravitino variation does not contain the matter fields! Therefore the conditions for supersymmetry are *independent* of the particular matter theory we are discussing (they only depend on which supercurrent multiplets exist). We call

$$\delta\Psi_\mu = 0 \tag{1.14}$$

the "generalized Killing spinor" (GKS) equation. We should solve it for the background supergravity fields and for the spinors ϵ. The number of solutions for ϵ is the number of preserved supercharges. The equation also lead to integrability conditions on $\mathcal{M}$, which tell us what manifolds can admit supersymmetry.

We stress that the same supergravity theory can have different off-shell formulations, and depending on which supercurrent multiplets exist in the matter theory, we can couple the theory to different off-shell formulations: this can lead to different classes of supersymmetric backgrounds for the same theory. For instance in 4d, the FZ multiplet can be coupled to "old minimal SUGRA" while the $\mathcal{R}$-multiplet to the "new minimal SUGRA".

Finally, from the supergravity transformations of matter fields and the Lagrangian, we can immediately read off:

- the deformed SUSY algebra on $\mathcal{M}$;
- the deformed matter theory.

In the full non-linear supergravity, the auxiliary fields enter at most linearly in the SUSY transformations and at most quadratically in the action. In particular $\mathcal{L}^{(1)}$ is the linear coupling between the auxiliary fields and the supercurrent multiplet, schematically

$$\mathcal{L}^{(1)} = \ldots + V^\mu J_\mu + H\sigma + \ldots \; , \tag{1.15}$$

[7] If the gravity multiplet contains also spinors, such as gauginos, we set their variation to zero as well.

while $\mathcal{L}^{(2)}$ are seagull terms.

Remark. In Euclidean signature the fields are complexified, and

$$\widetilde{Q}_{\dot\alpha} \neq Q_\alpha^\dagger \; . \tag{1.16}$$

It might happen that, to preserve supersymmetry, an auxiliary field which is real in Lorentzian signature must take a complex background. On spheres, this means that the matter theory is not reflection positive (the Euclidean version of unitarity). If the theory is superconformal, the complex auxiliary fields couple to a redundant operator and reflection positivity is restored (possibly in the IR).

As a final remark, the coupling to external off-shell multiplets works for other types of multiplets as well. For instance, if the theory has a continuous global symmetry G, the current j_μ sits in a current multiplet.[8] We can couple it to an external off-shell vector multiplet, by first coupling to SYM and then taking a rigid limit $e \to 0$, keeping a background value for the vector and the auxiliary scalar fields. In the limit we do not solve the EOMs, but impose the vanishing of the gaugino variation:

$$\delta\lambda = 0 \tag{1.17}$$

In the off-shell formulation, this variation does not depend on the matter fields and it gives theory-independent conditions on the flavor vector bundle.

2 Lecture Two

2.1 The localization argument

Supersymmetric localization is a very powerful tool that, in favorable circumstances, allows us to *exactly* compute the partition function and the expectation values of certain operators in supersymmetric theories. It has been used for a long time in cohomological and topological field theories, which often can be realized as topological twists of supersymmetric theories. More recently it has been applied directly to physical theories, for instance to 4d $\mathcal{N} = 2$ theories in the so-called Ω-background by Nekrasov [1] and on S^4 by Pestun [2]. Supersymmetric localization can be thought of as an infinite-dimensional version of the Duistermaat-Heckman and Atiyah-Bott-Berline-Vergne localization formulæ in equivariant cohomology. Let me describe how it works.

Suppose we have a fermionic symmetry $\mathcal{Q}$ of the action:

$$\mathcal{Q}\, S = 0 \; . \tag{2.1}$$

[8]In 4d $\mathcal{N} = 1$, the current multiplet is a linear multiplet $\mathcal{J} = (j, \psi, \overline{\psi}, j_\mu)$ with 4+4 independent components.

Since $\mathcal{Q}$ is fermionic, its square is either zero or a bosonic symmetry δ_B of the action (a composition of translations with Lorentz, R-symmetry and flavor symmetry rotations). We are interested in the path-integral $Z = \int \mathcal{D}\varphi \, e^{-S[\varphi]}$, where we have set $\hbar = 1$. Consider, instead, the following path-integral

$$Z(t) = \int \mathcal{D}\varphi \; e^{-S[\varphi] - t\,\mathcal{Q}V[\varphi]} \qquad\qquad \text{with} \qquad \delta_B V = 0 \;, \tag{2.2}$$

which depends on a parameter t and where V is some functional. The dependence on t is

$$\frac{\partial Z}{\partial t} = -\int \mathcal{D}\varphi \; \mathcal{Q}V \; e^{-S-t\mathcal{Q}V} = -\int \mathcal{D}\varphi \; \mathcal{Q}\Big(V \, e^{-S-t\mathcal{Q}V} \Big) \;. \tag{2.3}$$

If the measure is $\mathcal{Q}$-invariant, *i.e.* that the fermionic symmetry is non-anomalous (this requires that also δ_B is non-anomalous), by a field redefinition this is zero:

$$\frac{\partial Z}{\partial t} = 0 \;. \tag{2.4}$$

More precisely, $\mathcal{Q}$ is the generator of shifts along a fermionic direction on the supermanifold of fields, in other words it is a derivative. If there are no boundary terms at infinity in field space, the integral of a total derivative is zero. In some cases there are boundary terms,[9] however if $e^{-\mathcal{Q}V}$ falls-off fast enough at infinity in field space, such boundary terms are absent.

Then the partition function is independent of t. Clearly the argument still holds if we insert $\mathcal{Q}$-invariant operators, $\mathcal{Q}\,\mathcal{O}[\varphi] = 0$. The argument also shows three things:

- the partition function or VEV does not depend on coupling constants in front of $\mathcal{Q}$-exact terms in the action;
- VEVs only depend on the $\mathcal{Q}$-cohomology class of the operators;
- the partition function or VEV is not modified by the deformation term $\mathcal{Q}V$.

We can use the last fact at our advantage.

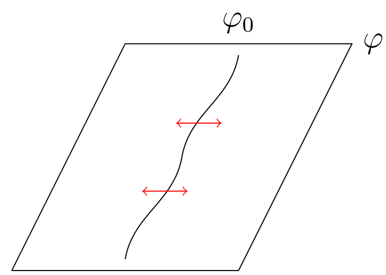
Before proceeding, let me remind you that, when going to Euclidean signature, all fields get complexified. This means that real fields such as A_μ become complex, and complex conjugate fields such as $\psi, \widetilde{\psi}$ become independent.[10] However, when studying the path-integral, we want to compute an analytic continuation of the Lorentzian path-integral (since ultimately we want to learn about the physical theory), therefore we do not want to integrate

[9] One example is in section 11.3 of [8]. Another example is in the cigar SCFT of [9], see section 6.1.

[10] One reason is that that spinor representations are different in Lorentzian and Euclidean signature, therefore if we want both local rotation symmetry and supersymmetry, the fields have to be complexified. If we do that, all supersymmetry variations look the same in Euclidean signature. Another reason is that, to preserve SUSY on a curved space, in general we *have* to allow for complexified backgrounds.

over complexified fields, which means twice as many fields compared with the Lorentzian path-integral. Moreover the path-integral would not be convergent. Instead, we need to specify a contour in field space on which the path-integral is performed, and the contour must be such that the path-integral is convergent for all values of t.[11]

At $t = 0$, $Z(0)$ is the original path-integral we want to compute. Suppose we can find some V such that the bosonic part of $\mathcal{Q}V$ is ≥ 0 along the contour. Then, in the limit $t \to +\infty$ all field configurations for which $\mathcal{Q}V[\varphi] > 0$ are infinitely suppressed!



Therefore the path-integral *localizes* to the bosonic zeros φ_0 of $\mathcal{Q}V$ (which are also stationary points). Let us parametrize the fields around φ_0 as[12]

$$\varphi = \varphi_0 + t^{-1/2}\hat{\varphi} \ . \tag{2.5}$$

We can then Taylor expand the action around φ_0:

$$S + t\mathcal{Q}V = S[\varphi_0] + \big(\mathcal{Q}V\big)^{(2)}[\hat{\varphi}] + \mathcal{O}(t^{-1/2}) \ , \tag{2.6}$$

therefore only the on-shell action S_0 and the quadratic expansion of $\mathcal{Q}V$ around the fixed points matters. We obtain the localization formula:

$$Z = \int_{\text{BPS}} \mathcal{D}\varphi_0 \; e^{-S[\varphi_0]} \; \frac{1}{\text{SDet}'(\mathcal{Q}V)^{(2)}_{\varphi_0}} \tag{2.7}$$

by Gaussian integration. The superdeterminant is the ratio of the bosonic and fermionic determinants, and is called “one-loop determinant”; it can be thought of as a measure on the subspace of fixed points. If $\{\varphi_0\}$ is a bosonic moduli space, SDet has zeros: we remove them and integrate over the bosonic zero-modes. If $\{\varphi_0\}$ has fermion zero-modes, one has to absorb them—either inserting operators or expanding S in the fermions.

We stress that this formula is *exact*. If the space of fixed points $\{\varphi_0\}$ is finite-dimensional, then we have reduced the path-integral to an ordinary integral and we may be able to solve it!

[11]If more than one contour makes the path-integral convergent, than they correspond to different quantizations of the theory.

[12]We choose the specific power $t^{-1/2}$ because when the deformation term dominates at large t, the kinetic term should be canonically normalized with no powers of t.

It turns out that one localizes on some BPS configurations. We can make a canonical choice for V:

$$V = \sum_{\text{fermions } \psi} (\mathcal{Q}\psi)^{\ddagger}\psi , \tag{2.8}$$

where $\ddagger$ is some anti-linear operator. If we can make a choice for $\ddagger$ such that the bosonic part of $\mathcal{Q}V$ is non-negative along the contour and $\delta_B V = 0$, then the fixed points are essentially

$$\mathcal{Q}\psi = 0 , \tag{2.9}$$

which are the BPS equations.[13]

Notice that the computation of SDet in general might seem exceedingly hard: we don't know how to compute the spectrum of the Laplace and Dirac operators on a generic compact manifold. However there is supersymmetry, and so we expect huge cancellations among the eigenvalues in SDdet. In fact in most cases the computation reduces to a much simpler cohomological problem, and sometimes index theorems can be applied.

2.2 Example: S^2 partition function of 2d $\mathcal{N} = (2,2)$ gauge theories

As a concrete example, we consider two-dimensional theories with $\mathcal{N} = (2,2)$ supersymmetry (4 supercharges). This is the dimensional reduction of 4d $\mathcal{N} = 1$. The dimensionality is low enough that we can easily do the computation, at yet it contains all the key ingredients that one finds in higher-dimensional theories. We want to study those theories on S^2.

2.2.1 The untwisted background on S^2

In Lorentzian signature, there are two complex supercharges, one left-moving and one right-moving. The biggest R-symmetry that these theories can have is

$$U(1)_{\text{left}} \times U(1)_{\text{right}} \;\simeq\; U(1)_R \times U(1)_A ,$$

a vector-like and an axial R-symmetry. This is an outer automorphism of the algebra, so a supersymmetric theory does not need to possess R-symmetry. But we require that the vector-like $U(1)_R$ is present. In this case, the algebra admits a complex central charge.[14]

On Euclidean flat space, the supersymmetry algebra is

$$\{Q_\alpha, \widetilde{Q}_\beta\} = \big[2\gamma^\mu P_\mu + 2i\mathbb{P}_+ Z + 2i\mathbb{P}_- \widetilde{Z}\big]_{\alpha\beta} , \qquad \{Q_\alpha, Q_\beta\} = \{\widetilde{Q}_\alpha, \widetilde{Q}_\beta\} = 0 , \tag{2.10}$$

[13] Here we are assuming that the supersymmetric configurations have $\psi_0 = 0$, *i.e.* that there are no fermion zero-modes. When there are, the localization formula includes an integral over the fermion zero-modes, which is the same as a derivative and extra care has to be used. One example is the elliptic genus, and the treatment of zero-modes in that case is done in [10–12]. Another example is the Coulomb-branch localization formula for the two-dimensional A-twist in [13, 14].

[14] With no R-symmetries, two complex central charges are possible. They break the corresponding R-symmetry, because they are charged. A superconformal theory cannot have central charges.

where $\mathbb{P}_\pm$ are the projectors on positive/negative chirality spinors.[15] Here $Z, \widetilde{Z}$ are complex central charges, which are complex conjugate in Lorentzian signature. All tilded quantities we will use are complex conjugate in Lorentzian signature, but are independent complexified fields in Euclidean signature. To $Q_\pm$, $\widetilde{Q}_\pm$ we assign R-charges -1 and $+1$ respectively.

Because of $U(1)_R$, the class of theories we look at have an $\mathcal{R}$-multiplet with $4+4$ independent components.[16] It contains the following operators:

$$\mathcal{R}_\mu : \quad \big(T_{\mu\nu},\, S_{\alpha\mu},\, \widetilde{S}_{\alpha\mu},\, j_\mu^R,\, j_\mu^Z,\, j_\mu^{\widetilde{Z}}\big) \;. \tag{2.12}$$

They are: the stress tensor, the supersymmetry currents, and the currents for the R-symmetry and the central charges; they are all conserved. Correspondingly, there exist an off-shell 2d supergravity, which is the dimensional reduction of the new minimal SUGRA in 4d discussed in [5] (found in [6, 7]), whose gravity multiplet couples to the $\mathcal{R}$-multiplet. In a Wess-Zumino gauge it contains

$$\mathcal{G}_\mu : \quad \big(g_{\mu\nu},\, \Psi_{\alpha\mu},\, \widetilde{\Psi}_{\alpha\mu},\, V_\mu,\, C_\mu,\, \widetilde{C}_\mu\big) \;. \tag{2.13}$$

The gauge fields $V_\mu, C_\mu, \widetilde{C}_\mu$ appear in the covariant derivatives

$$D_\mu = \nabla_\mu - irV_\mu + \frac{1}{2} z \widetilde{C}_\mu - \frac{1}{2}\widetilde{z} C_\mu \;, \tag{2.14}$$

as well as through their field strengths. It is convenient to introduce the dual field strengths

$$\mathcal{H} = -i\epsilon^{\mu\nu}\partial_\mu C_\nu \;, \qquad\qquad \widetilde{\mathcal{H}} = -i\epsilon^{\mu\nu}\partial_\mu \widetilde{C}_\nu \;. \tag{2.15}$$

In the full non-linear supergravity theory, the gravitino variation—which is our generalized

---

[15] In components:

$$\begin{aligned} \{Q_+, \widetilde{Q}_+\} &= 4P_z \;, \qquad & \{Q_+, \widetilde{Q}_-\} &= -2iZ \\ \{Q_-, \widetilde{Q}_-\} &= -4P_{\bar{z}} \;, \qquad & \{Q_-, \widetilde{Q}_+\} &= 2i\widetilde{Z} \end{aligned} \tag{2.11}$$

and all other vanishing.

[16] The bottom component is j_μ^R. Taking into account the conservation equations, there are 4+4 independent components. This has been studied in [4]. More in details. The $\mathcal{S}$-multiplet has $T_{\mu\nu}$, j_μ^R, j_μ^Z, $j_\mu^{\widetilde{Z}}$, Y_μ, $\widetilde{Y}_\mu$, A, $S_{\alpha\mu}$, $\widetilde{S}_{\alpha\mu}$, ψ_α, $\widetilde{\psi}_\alpha$, where j_μ^R is *not* conserved but $dY = d\widetilde{Y} = 0$: this gives 8+8 independent components. In the $\mathcal{R}$-multiplet we can set $Y_\mu = \widetilde{Y}_\mu = A = \psi_\alpha = \widetilde{\psi}_\alpha = 0$, as well as $\partial^\mu j_\mu^R = 0$: this gives 4+4 independent components. In the superconformal case we further set $j_\mu^Z = j_\mu^{\widetilde{Z}} = T_\mu^\mu = \gamma^\mu S_\mu = 0$, as well as $\varepsilon^{\mu\nu}\partial_\mu j_\nu^R = 0$ (we get two chiral currents out of j_μ^R): this gives 0+0 independent components.

Killing spinor (GKS) equation—is:[17]

$$\begin{aligned} \tfrac{1}{2}\delta\Psi_\mu &= (\nabla_\mu - iV_\mu)\epsilon - \frac{1}{2}\begin{pmatrix} \mathcal{H} & 0 \\ 0 & \widetilde{\mathcal{H}} \end{pmatrix}\gamma_\mu\epsilon + \dots \\ \tfrac{1}{2}\delta\widetilde{\Psi}_\mu &= (\nabla_\mu + iV_\mu)\widetilde{\epsilon} - \frac{1}{2}\begin{pmatrix} \widetilde{\mathcal{H}} & 0 \\ 0 & \mathcal{H} \end{pmatrix}\gamma_\mu\widetilde{\epsilon} + \dots \ . \end{aligned} \tag{2.17}$$

I am using conventions in which

$$\gamma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) ,$$

therefore the last term can be written as a combination of $\mathbb{1}$ and γ_3. Here $\epsilon, \widetilde{\epsilon}$ are the two supersymmetry parameters, which are complex Dirac spinors (in the Lorentzian theory they are charge conjugate). They have R-charges $1, -1$ respectively, and no central charges. The dots represent terms that vanish when we set $\Psi_\mu = \widetilde{\Psi}_\mu = 0$ (they are present in the full non-linear supergravity theory). Defining the $U(1)$ spin connection

$$\omega_\mu = -\frac{1}{2}\omega_\mu^{ab}\epsilon_{ab} , \tag{2.18}$$

the metric-covariant derivative on fields of definite spin is

$$\nabla_\mu^{(s)} = \partial_\mu - is\omega_\mu . \tag{2.19}$$

The full analysis of these equations has been done in [15]. Here we consider two simple solutions.

Twist. A simple solution for *any* manifold and metric is[18]

$$V_\mu = \frac{1}{2}\omega_\mu , \qquad \epsilon = \begin{pmatrix} 0 \\ \epsilon_- \end{pmatrix} , \qquad \widetilde{\epsilon} = \begin{pmatrix} \widetilde{\epsilon}_+ \\ 0 \end{pmatrix} , \qquad \mathcal{H} = 0 , \qquad \widetilde{\mathcal{H}} = 0 \tag{2.20}$$

where $\epsilon_-, \widetilde{\epsilon}_+$ are constant. This is called the topological *A-twist*. There are two Killing spinors of opposite R-charge and chirality, and $(g-1)$ units of R-symmetry flux. In this

[17] To derive the GKS equation it is enough to know the linearized supergravity. First, after imposing WZ gauge, $\mathcal{H}_\mu$ still has a residual gauge redundancy that for the gravitini consists of *local* supersymmetry transformations:

$$\delta\Psi_\mu = \partial_\mu\varepsilon , \qquad \delta\widetilde{\Psi}_\mu = \partial\widetilde{\varepsilon} . \tag{2.16}$$

From these expressions it seems that Ψ_μ, $\widetilde{\Psi}_\mu$ are invariant under constant (global) SUSY transformations $\epsilon, \widetilde{\epsilon}$. However the latter bring out of WZ gauge, and to restore the gauge one has to compensate with gauge transformations (that we used to impose WZ gauge). These give an expression, linear in and with no derivatives of $\epsilon, \widetilde{\epsilon}$. Then there is a unique way to make the transformations covariant, which also introduces the derivatives of $\epsilon, \widetilde{\epsilon}$. This procedure gives (2.17).

Alternatively, one can perform the dimensional reduction from the 4d new minimal supergravity.

[18] More generally, $\widetilde{\mathcal{H}}$ can be an arbitrary function. We cannot turn on holonomies for V_μ because $\epsilon, \widetilde{\epsilon}$ (and the supercharges) would no longer be periodic and there would not be solutions.

background the spinors behave as scalars because the R-symmetry background has, in a sense, twisted their spin:

$$D_\mu \epsilon_- = \partial_\mu \epsilon_- = 0 \; . \tag{2.21}$$

The deformed algebra is simply $\delta_\epsilon^2 = \delta_{\widetilde{\epsilon}}^2 = 0$ and $\{\delta_\epsilon, \delta_{\widetilde{\epsilon}}\} = 0$. For $g > 1$ this is essentially the only solution.

Untwisted S^2. With round metric, one finds the following *untwisted* background:

$$V_\mu = 0 \; , \qquad \mathcal{H} = \widetilde{\mathcal{H}} = \frac{i}{R} \; , \qquad \nabla_\mu \epsilon = \frac{i}{2R} \gamma_\mu \epsilon \; , \qquad \nabla_\mu \widetilde{\epsilon} = \frac{i}{2R} \gamma_\mu \widetilde{\epsilon} \; . \tag{2.22}$$

The spinors solve the Killing spinor equation.[19] On the round S^2 there are *four* solutions (two for ϵ and two for $\widetilde{\epsilon}$), so the number of preserved supersymmetries is maximal.

With no central charges, the deformed supersymmetry algebra is

$$\{\delta_\epsilon, \delta_{\widetilde{\epsilon}}\} = i\mathcal{L}_K - \frac{\epsilon\widetilde{\epsilon}}{2R} R_V \; , \qquad \{\delta_{\epsilon_1}, \delta_{\epsilon_2}\} = \{\delta_{\widetilde{\epsilon}_1}, \delta_{\widetilde{\epsilon}_2}\} = 0 \; , \tag{2.23}$$

and the Killing vectors K^μ generate the $SO(3)$ isometry algebra of S^2. In fact the superalgebra is

$$\mathfrak{su}(2|1)_A \;\supset\; \mathfrak{su}(2) \times \mathfrak{u}(1)_R \; . \tag{2.24}$$

Notice that the background is not the analytic continuation of a real background in Lorentzian signature. However, if the theory is superconformal, the auxiliary fields $\mathcal{H}, \widetilde{\mathcal{H}}$ couple to redundant operators and reflection positivity is not broken.[20]

2.2.2 Localization for gauge theories

We will now focus on a class of theories, namely gauge theories, for which the S^2 partition function can be explicitly evaluated with localization techniques. We will consider a simple class of gauge theories, composed of vector and chiral multiplets only (there are many other multiplets one can use) as the result is interesting enough.

Multiplets. The 2d vector and chiral multiplets are just the dimensional reduction of the 4d vector and chiral multiplets.

The chiral multiplet Φ and the antichiral multiplet $\widetilde{\Phi}$ are complex scalar multiplets satisfying the constraints

$$\widetilde{D}_\alpha \Phi = 0 \; , \qquad D_\alpha \widetilde{\Phi} = 0 \; . \tag{2.25}$$

[19]The Killing spinor equation in dimension d is $\nabla_\mu \epsilon = \lambda \gamma_\mu \epsilon$ for some constant λ. By manipulations, one gets $\lambda^2 = -\frac{R_s}{4d(d-1)}$, in particular R_s has to be constant. On S^2 one gets $\lambda = \pm \frac{i}{2R}$. Notice that even without assuming that λ is constant, such a condition follows from the initial equation.

[20]The action becomes the conformal transformation from flat space, with the correct conformal couplings of scalars. In particular, scalars of R-charge zero have no extra curvature couplings.

Their components are

$$\Phi = (\phi, \psi, F) \ , \qquad\qquad \widetilde{\Phi} = (\widetilde{\phi}, \widetilde{\psi}, \widetilde{F}) \ .$$

The vector multiplet $\mathcal{V}$ is a real multiplet: $\mathcal{V} = \mathcal{V}^\dagger$. In Wess-Zumino gauge[21] its components are

$$\mathcal{V} = (A_\mu, \sigma, \widetilde{\sigma}, \lambda, \widetilde{\lambda}, D) \ .$$

We parametrize

$$\sigma = \sigma_1 - i\sigma_2 \ , \qquad\qquad \widetilde{\sigma} = \sigma_1 + i\sigma_2 \ . \tag{2.26}$$

Out of the vector multiplet, we can construct a twisted chiral multiplet Σ, with

$$\widetilde{D}_+ \Sigma = D_- \Sigma = 0 \tag{2.27}$$

and its "conjugate" $\widetilde{\Sigma}$, whose components are

$$\begin{aligned} \Sigma &= (\sigma,\, \lambda_+,\, \widetilde{\lambda}_-,\, D - iF_{12} + i\widetilde{\mathcal{H}}\sigma) \\ \widetilde{\Sigma} &= (\widetilde{\sigma},\, \lambda_-,\, \widetilde{\lambda}_+,\, -D + iF_{12} - i\mathcal{H}\widetilde{\sigma}) \ . \end{aligned}$$

Gauge theory data.

- Gauge group G.
- Matter is in chiral multiplets, and they transform as some representation $\mathfrak{R}$ of G, in general reducible. In other words, the target space for chiral multiplets is a vector space V, and each component transform as a weight $\rho \in \mathfrak{R}$.
- SUSY transformations and the $\mathfrak{su}(2|1)$ algebra contain R_V, therefore we must specify the R-charges. Contrary to flat space, the action will depend on the R-charges: they control some "curvature couplings".
- Interactions. There are 2 types of them: superpotential and twisted superpotential.
- After including interactions, there can be some residual flavor symmetry G_F. Therefore V gives a representation of $G \times G_F \times U(1)_R$. A background for an external vector multiplet coupled to G_F produces the so-called twisted masses.

Summarizing:

$$(G, V, r, W, \mathcal{W}, m) \ . \tag{2.28}$$

[21] The parameter for gauge transformations Λ is promoted to a chiral multiplet, therefore there is more gauge freedom. We can fix the extra freedom going in the so-called Wess-Zumino gauge. The residual symmetry is the standard gauge symmetry. Recall that when performing a SUSY transformation, we go out of WZ gauge and we must compensate with a super-gauge transformation: this is what brings the vector multiplet in the transformations of the chiral multiplet.

Actions. Then the SYM action is

$$\begin{aligned}\mathcal{L}_{SYM} = \frac{1}{2}\operatorname{Tr}\Big[\Big(F_{12}-\frac{\sigma_2}{R}\Big)^2 + (D_\mu\sigma_1)^2 + (D_\mu\sigma_2)^2 - [\sigma_1,\sigma_2]^2 + \Big(D-\frac{\sigma_1}{R}\Big)^2\Big] \\ + \operatorname{Tr}\Big[i\widetilde{\lambda}\not{D}\lambda - i\widetilde{\lambda}[\sigma_1,\lambda] + \widetilde{\lambda}\gamma_3[\sigma_2,\lambda]\Big] .\end{aligned} \tag{2.29}$$

The matter action is

$$\begin{aligned}\mathcal{L}_{\text{mat}} = D_\mu\widetilde{\phi}D^\mu\phi + \widetilde{\phi}\Big[-iD + \sigma_1^2 + \sigma_2^2 + \frac{ir}{R}\sigma_1 + \frac{r(2-r)}{4R^2}\Big]\phi + F\widetilde{F} \\ + i\widetilde{\psi}\not{D}\psi + \widetilde{\psi}\Big[-i\sigma_1 + \sigma_2\gamma_3 + \frac{r}{2R}\Big]\psi + i\sqrt{2}\,\widetilde{\psi}\widetilde{\lambda}\phi + i\sqrt{2}\,\widetilde{\phi}\lambda\psi .\end{aligned} \tag{2.30}$$

As promised, these actions explicitly depend on the R-charges.

If we choose a "real" contour, the real bosonic parts of the actions are non-negative and the path-integral is convergent. We choose a supercharge[22]

$$\delta_\mathcal{Q} = \delta_\epsilon + \delta_{\widetilde{\epsilon}} \tag{2.31}$$

for some choice of ϵ and $\widetilde{\epsilon}$, and we use it for localization. These two actions are $\mathcal{Q}$-exact:

$$\mathcal{L}_{SYM} = \delta_\mathcal{Q}\big(\dots\big) , \qquad\qquad \mathcal{L}_{\text{mat}} = \delta_\mathcal{Q}\big(\dots\big) . \tag{2.32}$$

Thus we localize at their fixed points.

Then we can write interactions. Superpotential interactions are given by the F-terms of gauge-invariant chiral multiplets of R-charge 2:

$$\mathcal{L}_W = i\big(F_W + \widetilde{F}_W\big) . \qquad\qquad F_W = \partial_i W\, F_i + \frac{i}{2}\partial^2_{ij}W\,\psi_i\psi_j \tag{2.33}$$

where $W(\Phi)$ is a holomorphic function of R-charge 2. These interactions are $\mathcal{Q}$-exact, therefore the partition function does not depend on the coefficients of W. But the R-charges are constrained.

Twisted superpotential interactions are the G-terms of twisted chiral multiplets, and are controlled by a gauge-invariant holomorphic function $\mathcal{W}(\Omega)$.[23] In the special case of a linear twisted superpotential we get the FI term and the theta angle:

$$\mathcal{W} = \frac{i}{2}\Big(\xi + i\frac{\theta}{2\pi}\Big)\operatorname{Tr}\Sigma \qquad \Rightarrow \qquad \mathcal{L}_{\text{FI}} = \operatorname{Tr}\Big(i\xi D + i\frac{\theta}{2\pi}F_{12}\Big) . \tag{2.34}$$

[22]Such a supercharge satisfies $\delta_\mathcal{Q}^2 = i\mathcal{L}_K - \frac{\epsilon\widetilde{\epsilon}}{2R}R_V$ and it defines a superalgebra $\mathfrak{su}(1|1) \subset \mathfrak{su}(2|1)$, whose bosonic part $\mathfrak{u}(1)_{\text{rot}+R_V}$ is a mix of rotation and R-trasformation.

[23]We have

$$\mathcal{L}_\mathcal{W} = G_\mathcal{W} - i\widetilde{\mathcal{H}}\mathcal{W}(\omega) + \widetilde{G}_\mathcal{W} + i\mathcal{H}\widetilde{\mathcal{W}}(\widetilde{\omega})$$

and the formula for $G_\mathcal{W}$ is the same as for F_W but sending back $F \to -iF$.

This term is *not* $\mathcal{Q}$-exact.

Finally, we can add twisted masses. Whenever the theory has a continuous flavor symmetry G_F, we can couple to an external vector multiplet and give a background to the bosonic fields. As long as the SUSY condition $\delta\lambda_{\text{ext}} = 0$ is satisfied, this provides supersymmetric deformations: mass terms and magnetic flux (see matter action). We will discuss the conditions in a moment.

What G_F is depends on the superpotential. Eventually, the twisted masses can be rotated by a flavor rotation to the Cartan, and so in general they break G_F to its maximal torus.

BPS configurations. Setting to zero the real part of the bosonic action (and imposing the real contour) we get:

$$\begin{aligned} F_{12} = \frac{\sigma_2}{R} \;, \qquad D = \frac{\sigma_1}{R} \;, \qquad 0 = D_\mu \sigma_1 = D_\mu \sigma_2 = [\sigma_1, \sigma_2] \\ \phi = \widetilde{\phi} = 0 \;, \qquad F = \widetilde{F} = 0 \;. \end{aligned} \tag{2.35}$$

If we simultaneously diagonalize σ_1, σ_2, F_{12} and the connection A_μ, then $\sigma_{1,2}$ are simply constant. The constraint $\phi = \widetilde{\phi} = 0$ is imposed whenever $\sigma_1^2 + \sigma_2^2 + \frac{r(2-r)}{4R^2} > 0$, which we will *assume* is the case (we choose $0 < r < 2$). This is the same as the solutions to the BPS equations along the real contour (there are more and interesting solutions using complexified fields, *i.e.* different contours).

This is also the condition for external vector multiplets, therefore we can always rotate the twisted masses to the Cartan subalgebra of the flavor group.

The key point is that the localization locus is finite-dimensional, and it extremely simple! This will allow us to get a simple formula. The locus is

$$\begin{aligned} D = \frac{\sigma_1}{R} = const = \frac{a}{R^2} \;, \qquad F_{12} = \frac{\sigma_2}{R} = const = \frac{\mathfrak{m}}{2R^2} \;, \qquad [\sigma_1, \sigma_2] = 0 \\ \phi = \widetilde{\phi} = F = \widetilde{F} = 0 \;. \end{aligned} \tag{2.36}$$

Since $\sigma_{1,2}$ commute, we can rotate them to the Cartan subalgebra. The flux is GNO quantized, *i.e.*

$$\frac{1}{2\pi} \int F = \mathfrak{m} \qquad\qquad e^{2\pi i \mathfrak{m}} = \mathbb{1}_G \;. \tag{2.37}$$

For instance for $U(1)$: $\mathfrak{m} \in \mathbb{Z}$.

On-shell action. The only contribution to the classical action comes from the twisted superpotential. For an FI term:

$$S_{FI}^{(0)} = 4\pi i \xi \, R \operatorname{Tr} \sigma_1 + i\theta \operatorname{Tr} \mathfrak{m} \;. \tag{2.38}$$

2.2.3 1-loop determinants

The 1-loop determinants are obtained from the linearized actions around the BPS backgrounds. Since they are decoupled, we can compute separately the contributions from chiral and vector multiplets.

There are two strategies.

- Compute the full spectrum by hand. This is straightforward, but feasible only with a lot of symmetry (such as on the round S^2) because essentially we ask the spectrum of the Laplacian and the Dirac operator on the manifold.
- Use a cohomological argument that reduces to the modes that don't cancel.

Full spectrum. Consider first the chiral multiplet. The 1-loop determinant is

$$Z_{\text{1-loop}} = \frac{\text{Det}\,\mathcal{O}_\psi}{\text{Det}\,\mathcal{O}_\phi}\ ,$$

and the determinant of $F, \widetilde{F}$ is trivial and can be set to 1. On the round S^2 we can compute these determinants exactly.

The way to compute the determinants is to decompose the wavefunctions into spin spherical harmonics

$$Y^s_{j,j_3} \qquad \text{with} \qquad j, j_3 \in \mathbb{Z}+s\ , \qquad |j_3|, |s| \leq j\ . \tag{2.39}$$

Let me consider for simplicity fields of charge 1. Then s is the effective spin:

$$s = s_z - \frac{\mathfrak{m}}{2} \tag{2.40}$$

Recall that the spin connection is an Abelian gauge field.[24] The spin spherical harmonics Y^s_{j,j_3} are eigenfunctions of $D_\mu D^\mu$:

$$R^2\, D_\mu D^\mu Y^s_{j,j_3} = \big[-j(j+1) + s^2 \big]\, Y^s_{j,j_3}\ . \tag{2.43}$$

[24]The scalar spherical harmonics Y^0_{j,j_3} are eigenfunctions of the Laplacian $R^2\,\nabla^\mu\partial_\mu Y^0_{j,j_3} = -j(j+1)Y^0_{j,j_3}$ and are parametrized by $j, j_3 \in \mathbb{Z}$ with $|j_3| \leq j$. This generalizes to fields with spin and moving in a magnetic background. In fact if φ has spin s_z and it transforms as a weight ρ of a representation $\mathfrak{R}$, its covariant derivative is

$$D_\mu = \partial_\mu - i\big(s_z - \tfrac{\rho(\mathfrak{m})}{2}\big)\omega_\mu \tag{2.41}$$

since $\frac{1}{2\pi}\int d\omega = -2$ and we can choose a gauge $A_\mu = -\frac{\mathfrak{m}}{2}\omega_\mu$. Thus φ behaves as a field of effective spin $s = s_z - \frac{\rho(\mathfrak{m})}{2}$. Moreover the holomorphic derivatives $D_\pm = \frac{D_1 \mp iD_2}{2}$ (using vielbein indices) behave as rasing/lowering operators for the spin, $s \to s \pm 1$. They also satisfy

$$2\{D_+, D_-\} = D_\mu D^\mu\ , \qquad [D_+, D_-] = -\frac{s}{2R^2}\ . \tag{2.42}$$

The scalar operator is

$$\mathcal{O}_\phi = -D_\mu D^\mu - iD + \sigma_1^2 + \sigma_2^2 + i\frac{r\sigma_1}{R} + \frac{r(2-r)}{4R^2} \;. \tag{2.44}$$

We decompose each field into spin spherical harmonics Y^s_{j,j_3} with $s = -\frac{\mathfrak{m}}{2}$. It is then immediate to compute

$$\operatorname{Det}\mathcal{O}_\phi = \prod_{j=\frac{|\mathfrak{m}|}{2}}^\infty \Big(j + \frac{r}{2} - ia\Big)^{2j+1} \Big(j + 1 - \frac{r}{2} + ia\Big)^{2j+1} \;, \tag{2.45}$$

where we discarded an infinite products of factors of R.

The fermionic operator is

$$\mathcal{O}_\psi = i\not{D} - i\sigma_1 + \sigma_2\gamma_3 + \frac{r}{2R} \;. \tag{2.46}$$

Using matrix notation, $\not{D}$ is written in terms of $D_\pm$. Then the eigenfunctions are

$$\begin{pmatrix} Y^{\frac12 - \frac{\mathfrak{m}}{2}}_{j,j_3} \\ Y^{-\frac12 - \frac{\mathfrak{m}}{2}}_{j,j_3} \end{pmatrix} \;.$$

One has to be careful because, in the presence of magnetic field, for the minimal possible value of j only one harmonic exists: those are the chiral zeromodes of the Dirac operator predicted by the index theorem. After taking that into account, one reaches a similar expression:

$$\operatorname{Det}\mathcal{O}_\psi = (-1)^{\frac{\mathfrak{m}+|\mathfrak{m}|}{2}} \prod_{k=\frac{|\mathfrak{m}|}{2}}^\infty \Big(k + \frac{r}{2} - iR\sigma_1\Big)^{2k} \Big(k + 1 - \frac{r}{2} + iR\sigma_1\Big)^{2k+2} \;. \tag{2.47}$$

Most terms cancel out in the ratio, and we are left with

$$\frac{\operatorname{Det}\mathcal{O}_\psi}{\operatorname{Det}\mathcal{O}_\phi} = \prod_{n=0}^\infty \frac{n + 1 - \frac{r}{2} + ia - \frac{\mathfrak{m}}{2}}{n + \frac{r}{2} - ia - \frac{\mathfrak{m}}{2}} \;. \tag{2.48}$$

We should expect such a cancelation. The theory is supersymmetric, therefore—following the argument of the Witten index—all "positive-energy modes" should come in pairs and simplify: only a fraction of the modes does not cancel, and this will be at the core of the cohomological computation that we will see later.

The expression above does not make sense, because the product does not converge, and it requires regularization. We can use ζ-function regularization. We use the Hurwitz zeta function $\zeta(z;q)$:

$$\zeta(z;q) = \sum_{n=0}^\infty (q+n)^{-z} \;, \qquad -\frac{\partial}{\partial z}\zeta(z;q)\big|_{z=0} = \log\frac{\sqrt{2\pi}}{\Gamma(q)} = \text{“}\log\prod_{n=0}^\infty (q+n)\text{”} \;. \tag{2.49}$$

After including the product over the weights ρ of the representation $\mathfrak{R}$, the one-loop determinant for chiral multiplets is

$$Z_{\text{1-loop}} = \prod_{\rho\in\mathfrak{R}} \frac{\Gamma\big(\frac{r}{2} - i\rho(a) - \frac{\rho(\mathfrak{m})}{2}\big)}{\Gamma\big(1 - \frac{r}{2} + i\rho(a) - \frac{\rho(\mathfrak{m})}{2}\big)} \; . \tag{2.50}$$

Notice that this is already enough to compute the partition function of Landau-Ginzburg models.

Vector multiplet. The vector multiplet can be treated in a similar way. The only complication is that we should fix the gauge. This is done with the standard Fadeev-Popov method. In fact we should perform gauge-fixing on a background. Separating the background from the oscillatory part,

$$A_\mu = A_\mu^{(0)} + \frac{1}{\sqrt{t}} \widehat{A}_\mu \; ,$$

the gauge-fixing action is

$$\mathcal{L}_{\text{g-f}} = -\widetilde{c}\big(D^\mu D_\mu c - i D^\mu [\widehat{A}_\mu, c]\big) - \frac{1}{2\xi} (D^\mu \widehat{A}_\mu)^2 \tag{2.51}$$

where covariant derivatives contain only the background $A_\mu^{(0)}$. The new fields $c, \widetilde{c}$ are anticommuting complex scalars in the adjoint representation. At this point we expand $\mathcal{L}_\mathcal{V} + \mathcal{L}_{\text{g-f}}$ at quadratic order around the background, decompose $\widehat{A}_\mu$ into two components of spin ± 1 and use spin spherical harmonics as before. There appear bosonic zero-modes corresponding to the moduli a of the background: one should remove the zero eigenvalues from the determinants, and integrate over the zero-modes.[25] The one-loop determinant is

$$\frac{\text{Det}' \, \mathcal{O}_c \; \text{Det} \, \mathcal{O}_\lambda}{\sqrt{\text{Det}' \, \mathcal{O}_{\text{gauge}}}} = \prod_{\alpha(\mathfrak{m})=0} \frac{1}{|\alpha(a)|} \cdot \prod_{\alpha>0} (-1)^{\alpha(\mathfrak{m})} \Big(\frac{\alpha(\mathfrak{m})^2}{4} + \alpha(a)^2 \Big) \; . \tag{2.52}$$

The zero-modes of a span the gauge subalgebra unbroken by the flux $\mathfrak{m}$: we integrate over the Cartan subalgebra with a Vandermonde determinant:

$$\int \text{zero-modes} = \frac{1}{|\mathcal{W}_\mathfrak{m}|} \int \prod_{n=1}^{\operatorname{rank} G} \frac{da_n}{2\pi} \prod_{\alpha(\mathfrak{m})=0} |\alpha(a)| \; . \tag{2.53}$$

$\mathcal{W}_\mathfrak{m}$ is the Weyl group of the unbroken subalgebra. We see that the Vandermonde cancels an equal term in the one-loop determinant. Finally we sum over the inequivalent gauge fluxes (divide by the Weyl transformations that act on the flux):

$$\sum_{\mathfrak{m}\in\Gamma_{\text{mag}}} \frac{1}{|\mathcal{W}/\mathcal{W}_\mathfrak{m}|} \; .$$

[25] There are also zero-modes for the ghosts. They should be removed including ghost-for-ghosts. This is done for instance in [2].

2.2.4 The final formula

Finally we put all pieces together:

$$Z_{S^2} = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m}\in\Gamma_{\text{mag}}} \int_{\mathbb{R}^{\text{rk}}} \prod_{n=1}^{\operatorname{rank} G} \frac{da_n}{2\pi} \cdot e^{-4\pi i\xi \operatorname{Tr} a - i\theta \operatorname{Tr}\mathfrak{m}}$$
$$\prod_{\alpha>0} (-1)^{\alpha(\mathfrak{m})} \Big(\frac{\alpha(\mathfrak{m})^2}{4} + \alpha(a)^2 \Big) \prod_{\rho\in\mathfrak{R}} \frac{\Gamma\big(\frac{r}{2} - i\rho(a) - \frac{\rho(\mathfrak{m})}{2}\big)}{\Gamma\big(1 - \frac{r}{2} + i\rho(a) - \frac{\rho(\mathfrak{m})}{2}\big)} \; . \tag{2.54}$$

The integration contour is along the real lines. Twisted masses and external fluxes are included by shifting $a, \mathfrak{m}$ in $Z^{\text{mat}}_{\text{1-loop}}$.

Comments:

- The final formula for the path-integral is very simple! Closing the contour in the complex a-plane and picking up the residues, it can be converted to a series. Each term of this series is an instanton (vortex) contribution to the path-integral. We will see this in the next lecture.
- Order operators, such as Wilson line operators, are easily included.
- The result for 3d $\mathcal{N} = 2$ gauge theories on S^3 is very similar [19]. In 4d it is more complicated, because one has to include instanton corrections that have to be computed separately (Ω-background). As we will see, our 2d formula already includes all instanton corrections, secretly.
- The partition function is independent of the gauge coupling, *i.e.* of the renormalization scale (there is a running of the FI, though). If the theory flows to an IR fixed point, the partition function is the same as the one of the IR fixed point. This is useful for the physics of NLSMs.

2.3 Tricks to compute one-loop determinants

There are various techniques to compute the one-loop determinants on spaces which are not as symmetric as the round sphere: they amount not to compute the full spectrum of the Laplacian and the Dirac operator, but rather to compute only the modes that do not cancel out.

Cohomological argument. Let us compute the chiral multiplet one-loop determinant using a cohomological argument [20]. Suppose we have a spinor eigenfunction:

$$\mathcal{O}_\psi \Psi = \frac{\lambda}{R}\Psi \; . \tag{2.55}$$

Then one can show that the scalar wavefunction $\Phi \equiv \epsilon\Psi$ is an eigenfunction:

$$\mathcal{O}_\phi\, \epsilon\Psi = \frac{1}{R^2}\, \lambda\big(\lambda + 1 - r + 2iR\sigma_1\big) \;. \tag{2.56}$$

A spinor eigenfunction with eigenvalue $-\frac{1}{R}(\lambda+1-r+2iR\sigma_1)$ would lead to the same scalar eigenvalue.

Then consider a scalar eigenfunction

$$\mathcal{O}_\phi \Phi = \frac{1}{R^2}\, \lambda\big(\lambda + 1 - r + 2iR\sigma_1\big)\, \Phi \;. \tag{2.57}$$

Then construct the two following spinor wavefunctions:

$$\Psi_1 = \widetilde{\epsilon}\, \Phi \;, \qquad\qquad \Psi_2 = i\gamma^\mu \widetilde{\epsilon}\, D_\mu \Phi + \Big(i\sigma_1 + \sigma_2\gamma_3 - \frac{r}{2R}\Big) \widetilde{\epsilon} \Phi \;. \tag{2.58}$$

These two expressions are suggested by the supersymmetry variations. Then one shows

$$\mathcal{O}_\psi \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} -2i\sigma_1 + \frac{r-1}{R} & 1 \\ \frac{1}{R^2}\lambda(\lambda+1-r+2iR\sigma_1) & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \;, \tag{2.59}$$

therefore on the subspace generated by $\Psi_{1,2}$, $\mathcal{O}_\psi$ has eigenvalues

$$\frac{\lambda}{R} \;, \qquad -\frac{1}{R}\big(\lambda + 1 - r + 2iR\sigma_1\big) \;.$$

The contributions from the scalar and spinor eigenfunctions cancel out, and there is no need to compute λ explicitly.

Thus the only contributions come from "unpaired" modes. Given the spinor eigenfunction Ψ in (2.55), the corresponding scalar wavefunction does not exists if $\epsilon\Psi = 0$, which happens if and only if $\Psi = \epsilon F$. This leads to an eigenvalue equation

$$\Big(\mathcal{O}_\psi - \frac{\lambda}{R} \Big)\, \epsilon F = 0 \tag{2.60}$$

which only gives contributing eigenvalues. The spectrum of this equation can be found without solving for the eigenfunctions explicitly, because it is two first order equations in the same function F and the spectrum if fixed by the regularity conditions around the zeros of ϵ_+, ϵ_-. Similarly, given the scalar eigenfunction Φ in (2.57), the corresponding spinor wavefunction $\Psi_1 = \widetilde{\epsilon}\,\Phi$ always exists, while Ψ_2 does not exist as an independent wavefunction if it is proportional to Ψ_1 (including the case that it vanishes): $\Psi_2 = -\frac{\alpha}{R}\Psi_1$. From the matrix above we conclude that the eigenvalues for Φ and Ψ_1 under $\mathcal{O}_\phi$, $\mathcal{O}_\psi$ respectively, are:

$$\Phi: \quad \frac{1}{R^2}\alpha\big(\alpha + 1 - r + 2iR\sigma_1\big) \;, \qquad\qquad \Psi_1: \quad -\frac{1}{R}(\alpha + 1 - r + 2iR\sigma_1) \;.$$

Therefore α is missing, and we should multiply by α in the denominator. The eigenvalue equation is now

$$\Psi_2 = -\frac{\alpha}{R}\Psi_1 \qquad \Rightarrow \qquad \mathcal{O}_\psi\, \widetilde{\epsilon}\Phi = -\frac{1}{R}\big(\alpha + 1 - r + 2iR\sigma_1\big)\, \widetilde{\epsilon}\Phi \;. \tag{2.61}$$

Again, this is a pair of first order differential equations in a single function Φ, and the spectrum follows from regularity conditions around the zeros of $\widetilde{\epsilon}_+, \widetilde{\epsilon}_-$ with no need to solve for the actual eigenfunctions.

Index theorems. First we divide the fields into two groups, as in cohomological field theory:

$$\text{bosonic: } \boldsymbol{X}\;, \mathcal{Q}\boldsymbol{\xi}\;, \qquad\qquad \text{fermionic: } \boldsymbol{\xi}\;, \mathcal{Q}\boldsymbol{X}\;. \tag{2.62}$$

The path-integral is then over $\mathcal{D}\boldsymbol{X}\,\mathcal{D}\boldsymbol{\xi}\,\mathcal{D}\mathcal{Q}\boldsymbol{X}\,\mathcal{D}\mathcal{Q}\boldsymbol{\xi}$. We choose the canonical deformation

$$\mathcal{Q}\mathcal{V} = \mathcal{Q}\big((\boldsymbol{X}, \mathcal{Q}\boldsymbol{X}) + (\boldsymbol{\xi}, \mathcal{Q}\boldsymbol{\xi})\big) = (\mathcal{Q}\boldsymbol{X}, \mathcal{Q}\boldsymbol{X}) + (\boldsymbol{X}, \mathcal{Q}^2\boldsymbol{X}) + (\mathcal{Q}\boldsymbol{\xi}, \mathcal{Q}\boldsymbol{\xi}) - (\boldsymbol{\xi}, \mathcal{Q}^2\boldsymbol{\xi})\;, \tag{2.63}$$

provided that $\mathcal{Q}^2\mathcal{V} = 0$ and $\mathcal{Q}\mathcal{V}$ has non-negative bosonic part. In these variables, the one-loop determinant is

$$Z_\text{1-loop} = \frac{\text{Det}_{\boldsymbol{\xi}}\, \mathcal{Q}^2}{\text{Det}_{\boldsymbol{X}}\, \mathcal{Q}^2}\;. \tag{2.64}$$

Still there are many cancelations. Suppose we can find a differential operator $D : \boldsymbol{X} \to \boldsymbol{\xi}$ which commutes with $\mathcal{Q}^2$: then the contribution to $Z_\text{1-loop}$ from modes $\boldsymbol{X}$ not in the kernel of D cancel with the contribution from modes $\boldsymbol{\xi}$ in the image of D. We then reduce to

$$Z_\text{1-loop} = \frac{\text{Det}_{\text{coker}\, D}\, \mathcal{Q}^2}{\text{Det}_{\ker D}\, \mathcal{Q}^2} \tag{2.65}$$

where $\ker D \subset \boldsymbol{X}$ while $\text{coker}\, D \subset \boldsymbol{\xi}$.

The eigenvalues can be extracted from the equivariant index of D (with respect to $\mathcal{Q}^2$), defined as

$$\text{ind}\, D(u) = \text{Tr}_{\ker D}\, e^{-iu\mathcal{Q}^2} - \text{Tr}_{\text{coker}\, D}\, e^{-iu\mathcal{Q}^2}\;. \tag{2.66}$$

By a Fourier transform of this object we can read off the eigenvalues and their multiplicities. In turn, if D is transversally elliptic with respect to $\mathcal{Q}^2$, we can compute the index with the Atiyah-Singer index theorem:

$$\text{ind}\, D(u) = \sum_{p\in\text{Fix}} \frac{\text{Tr}_{\boldsymbol{X}(p)}\, e^{-iu\mathcal{Q}^2} - \text{Tr}_{\boldsymbol{\xi}(p)}\, e^{-iu\mathcal{Q}^2}}{\text{Det}_{TM(p)}(1 - e^{-iu\mathcal{Q}^2})}\;, \tag{2.67}$$

where p run over the fixed points of $e^{-iu\mathcal{Q}^2}$, $\boldsymbol{X}(p)$ and $\boldsymbol{\xi}(p)$ are the fields at p, and $TM(p)$ is the tangent space to the manifold M at p. A full account of this method is reviewed in [21].

3 Lecture Three

3.1 Z_{S^2} and the Zamolodchikov metric

We would like to understand what physical information is contained in the S^2 partition function that we have computed.

Let us first do a first digression on CFTs. Given a CFT in d dimensions, consider its exactly marginal operators

$$O_i \qquad \text{with} \qquad \dim O_i = d \; . \tag{3.1}$$

They can be used to deform the theory:

$$\delta S = \int d^d x \, \lambda^i O_i(x) \tag{3.2}$$

where λ^i are coupling constants. At least in a neighbourhood of the original CFT, this gives a family of new CFTs parametrized by λ^i, called the *conformal manifold* $\mathcal{S}$. It admits a natural metric, called the Zamolodchikov metric

$$\langle O_i(x)\, O_j(0)\rangle_{p\in S} \equiv \frac{g_{ij}(p)}{x^{2d}} \; . \tag{3.3}$$

In a 2d $\mathcal{N} = (2,2)$ SCFT the R-symmetry group is $U(1)_V \times U(1)_A$, and exactly marginal operators are the superconformal descendants of operators in the chiral and twisted chiral rings with charges $(2,0)$ and $(0,2)$. The coupling constants λ^i are in chiral and twisted chiral multiplets of charges $(0,0)$, respectively. The conformal manifold $\mathcal{S}$ is Kähler, and the Zamolodchikov metric takes locally a factorized form

$$\mathcal{S} = \mathcal{S}_c \times \mathcal{S}_{tc} \; . \tag{3.4}$$

In the special case of a conformal NLSM, whose target is CY, marginal deformations correspond (at one-loop) to deformations of the metric that keeps it Ricci-flat. The moduli space of Ricci flat metrics is locally factorized into Kähler deformations and complex structure deformations:

$$\mathcal{S} = \mathcal{S}_K \times \mathcal{S}_{CS} \tag{3.5}$$

with dimensions $h^{1,1}$ and $h^{(d-1,1)}$. Complex structure deformations are already complex, while Kähler structure deformations have to be "complexified":

$$J_{\mathbb{C}} = \omega + iB \; , \qquad\qquad dB = 0 \; , \qquad B \in H^{(1,1)} \; . \tag{3.6}$$

The closed B is the B-field, and it appears in the NLSM as a topological term

$$\mathcal{L} \supset i\varepsilon^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^{\bar{\jmath}}B_{i\bar{\jmath}} = i\phi^*(B) \; . \tag{3.7}$$

However what we are going to say is more generally valid for $\mathcal{N} = (2,2)$ SCFTs.

We can try (following [22]) to study the conformal manifold around $\lambda = 0$ by using

$$\langle \mathcal{O}(y) \rangle_\lambda = \big\langle \mathcal{O}(y) \, \exp(\delta S) \big\rangle_{\lambda=0} \;, \tag{3.8}$$

however this introduces divergences when $x \to y$, that have to be cured by local counterterms. So, let us consider the SCFT on S^2 and insist that the counterterms preserve $\mathfrak{su}(2|1)_A$ (the massive algebra that we studied so far).[26]

Chiral (complex structure) deformations come from F-terms of chiral multiplets of R-charge 2:

$$\int d\mathrm{vol}_2 \, F = \int d\mathrm{vol}_2 \, \delta(\dots) \tag{3.9}$$

is supersymmetric ($\delta F = D_\mu(\dots)^\mu$) but it also $\mathcal{Q}$-exact, therefore Z_{S^2} does not depend on chiral moduli (we already knew that there is no dependence on the superpotential). Twisted chiral (Kähler) deformations come from G-terms of twisted chiral multiplets of R_A-charge 2:

$$\int d\mathrm{vol}_2 \, G = 4\pi R \, \omega(\mathrm{NP}) + \int d\mathrm{vol}_2 \, \delta(\dots) \;. \tag{3.10}$$

The LHS is supersymmetric ($\delta G = D_\mu(\dots)^\mu$), not $\mathcal{Q}$-exact but almost: the integrated top operator is $\mathcal{Q}$-equivalent to a local insertion of the bottom operator, at a point that I called North Pole.[27] Notice that this relation is indeed satisfied by our localization formulæ, even in the non-conformal case. Similarly, insertions of integrated $\widetilde{G}$ are $\mathcal{Q}$-equivalent to local insertions of $-4\pi R \, \widetilde{\omega}$ at the opposite South Pole.

We can then conclude:

$$\begin{aligned} \partial_i \partial_{\bar{\jmath}} \log Z_{S^2} &= \frac{1}{\pi^2} \Big\langle \int d\mathrm{vol}_2 \, G_i(x) \int d\mathrm{vol}_2 \, G_{\bar{\jmath}}(y) \Big\rangle \\ &= -4R^2 \big\langle \omega_i(\mathrm{NP}) \; \omega_{\bar{\jmath}}(\mathrm{SP}) \big\rangle \\ &= -16 R^4 \Big\langle G_i(\mathrm{NP}) \; G_{\bar{\jmath}}(\mathrm{SP}) \Big\rangle = -g_{i\bar{\jmath}} = -\partial_i \partial_{\bar{\jmath}} K_{tc} \;. \end{aligned} \tag{3.11}$$

To go to the third line we used a supersymmetry Ward identity. This proves that

$$Z_{S^2} = e^{-K_{tc}} \qquad \text{up to} \qquad K_{tc} \to K_{tc} + f + \overline{f} \;. \tag{3.12}$$

The holomorphic ambiguity, which is the standard Kähler transformations, can be understood in the CFT as the possibility to add a local counterterm in supergravity which modifies the partition function.[28]

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[26] One might hope to be able to preserve the full superconformal group, but this is not possible [22]. Of course if we stay at one point $p \in \mathcal{S}$ we do preserve the superconformal group, the problem is in exploring $\mathcal{S}$ from p.

[27] The argument shows that the partition function is the same with the insertion of the LHS or RHS. However the RHS preserves all supersymmetries, while the RHS only preserves two.

[28] This coupling looks like

$$\int d^2x \, d\theta^+ d\widetilde{\theta}^- \, \widehat{\varepsilon}^{-1} F \, \mathcal{F}(\Omega^i) + c.c. \;,$$

where Ω^i are the twisted chiral multiplets whose lowest components are the couplings λ^i.

Now, we can compute the S^2 partition function of an $\mathcal{N} = (2, 2)$ GLSM that flows to a conformal NLSM, or more generally to a fixed point, in the IR. In this way, we can compute the Zamolodchikov metric on the conformal manifold. If we consider the case that the fixed point is a Calabi-Yau three-fold NLSM, then Z_{S^2} can be used to extract the so-called Gromov-Witten invariants of the manifold [23].

3.2 Higgs branch localization

Let us perform the localization in a different way. We will obtain the same result, because it is still the path-integral on S^2, but expressed in a different way. The new expression will manifestly look as a sum over non-perturbative contributions (instantons in two dimensions are *vortices*), and the computation of the vortex partition function will be a baby-version of Nekrasov's instanton partition function.

Let us add another localization term to the action, which starts with the D-term equation:

$$\mathcal{L}_H = \text{Tr}\Big[-i\Big(D - \frac{\sigma_1}{R}\Big)\big(\phi\phi^\dagger - \chi\big) + \dots \Big] = \mathcal{Q}(\dots) \ . \tag{3.13}$$

This action is not non-negative,[29] however it becomes non-negative if we integrate out D, which can be done exactly because it is an auxiliary field. This enforces the constraint:

$$D - \frac{\sigma_1}{R} = i\big(\phi\phi^\dagger - \chi\big) \ . \tag{3.14}$$

This is essentially the D-term equation, whose solutions are on the Higgs branch.

Due to the constraint, the BPS equations essentially reduce to the following. The Coulomb branch parameters are fixed to the roots of the Higgs branches:

$$(\sigma_1 - m)\phi = \sigma_2\phi = 0 \ . \tag{3.15}$$

Outside the poles of S^2 we are on the Higgs branch:

$$F_{12} = 0 \ , \qquad\qquad \phi\phi^\dagger = \chi \ . \tag{3.16}$$

In a neighborhood of the poles, though, we find the *vortex equations*:

$$\begin{aligned} F_{12} &= \pm\big(\chi - \phi\phi^\dagger\big) \\ D_\mp\phi &= 0 \ . \end{aligned} \tag{3.17}$$

Indeed, vortices are the instanton configurations in two dimensions, and we see that they appear in this "Higgs branch localization".

[29] The full term can be found in [16]. It has a real bosonic piece which is not non-negative.

In fact, there are other contributions called "deformed Coulomb branch", which however can be suppressed in the limit $\chi \to \infty$ (and one has to check that this is the case). In that limit the solutions to the vortex equations become pointlike, since their size scales as $\chi^{-1/2}$, making the system consistent.

Therefore, we find the following expression for the path-integral. We have to sum over the Higgs branches, and in each Higgs branch integrate over the moduli space of vortices. These spaces are the disjoint union of sectors with fixed vortex number, for instance for $U(N)$

$$k = \frac{1}{2\pi} \operatorname{Tr} \int_{S^2} F \; . \tag{3.18}$$

Then we should evaluate the classical action and the small quadratic fluctuations around them. The one-loop determinants on a vortex background can be computed with the index theorem [16, 17]. Pulling out the factors that do not depend on the vortex background we find:

$$Z_{S^2} = \sum_{\substack{\text{Higgs} \\ \text{branches}}} e^{S_{\text{cl}}} \, Z'_{\text{1-loop}} \, Z_{\text{vortex}}(q) \, Z_{\text{antivortex}}(\bar{q}) \; . \tag{3.19}$$

Here $Z'_{\text{1-loop}}$ is the one-loop determinant of all fields not taking VEV on the Higgs branch.

Z_{vortex} is the vortex partition function. Since, close to the poles, the action on S^2 at first order is equal to the action on $\mathbb{R}^2$ but in the so-called Ω-background, Z_{vortex} can be computed on $\mathbb{R}^2$ with Ω-background. This is a background that, essentially, contains a quadratic potential that traps non-trivial field configurations around the origin.

The vortex contribution is a sum over the disjoint topological sectors parametrized by the vortex number:

$$Z_{\text{vortex}} = \sum_k q^k \int_{\mathcal{M}_{k,\text{vortex}}} 1 \; , \qquad\qquad q = e^{-4\pi\xi - i\theta} \; . \tag{3.20}$$

The integral is an integral over the vortex moduli space, essentially its equivariant volume. We have

$$\int_{\mathcal{M}_{k,\text{vortex}}} 1 = \text{0d path-integral of a theory that has } \mathcal{M}_{k,\text{vortex}} \text{ as its moduli space} \tag{3.21}$$

and this is the *ADHM construction*.

3.3 Vortex moduli space and its partition function

Let us focus on $U(N)$ SQCD with N_f fundamentals and $\widetilde{N}_f$ antifundamentals (and $N_f \geq N, \widetilde{N}_f$). We want to compute its vortex partition function in Ω-background.

From a brane construction, Hanany and Tong [24] found that the k-vortex sector can be algebraically described as the Higgs branch of a 0d gauge theory (which is the dimensional reduction of a 2d $\mathcal{N} = (0,2)$) given by

$$\begin{array}{ll} U(k) \text{ vector} & \varphi, \lambda, \overline{\lambda}, D \\ \text{one adjoint} & X, \chi \\ N \text{ fundamental chirals} & I, \mu \\ N_f - N \text{ antifundamental chirals} & J, \nu \\ \widetilde{N}_f \text{ fundamental Fermi} & \xi, G \; . \end{array} \tag{3.22}$$

The 2d gauge coupling is mapped to the 0d FI term r, thus the moduli space is the Higgs branch. The equivariant action $U(1)_\varepsilon$ is mapped to a symmetry acting on the adjoint (and ε is identified with $1/R$).

The vacuum equations (D-terms and F-terms) are

$$\begin{aligned} &[X, X^\dagger] + II^\dagger - J^\dagger J = r\mathbb{1}_k \\ &\qquad \varphi I - I\hat{M} = 0 \qquad\qquad [\varphi, \widetilde{\varphi}] = 0 \\ &\qquad J\varphi - \check{M}J = 0 \qquad\qquad [\varphi, X] = \varepsilon X \; . \end{aligned} \tag{3.23}$$

Here $\hat{M}$ are the masses of the N chirals taking VEV, $\check{M}$ are the masses of the $N_f - N$ chirals not taking VEV, while $\widetilde{M}$ are the masses of the $\widetilde{N}_f$ antifundamentals.

We want to compute the partition function of this 0d gauge theory, which is the same as the integral over its moduli space. This is, once again, done with localization! The one-loop determinants are straightforward, because they are given by standard Gaussian integrals (path-integrals are standard integrals). For instance for a chiral multiplet:

$$\int dX\, dX^\dagger\, d\chi\, d\chi^\dagger\; e^{-X^\dagger \varphi^2 X - \chi^\dagger \varphi \chi} \;\sim\; \frac{1}{\varphi} \; .$$

This is in fact the equivariant volume of $\mathbb{C}$. The vector multiplet can be treated similarly to what we did before fixing the gauge, the final result is [25]:

$$Z_k = \oint_{\mathcal{C}} \prod_{I=1}^{k} \frac{d\varphi_I}{2\pi i}\; \mathcal{Z}_{\text{vec}}(\varphi)\; \mathcal{Z}_{\text{fund}}(M, \varphi)\; \mathcal{Z}_{\text{antifund}}(\widetilde{M}, \varphi) \; , \tag{3.24}$$

where the one-loop determinants are

$$\begin{aligned} \mathcal{Z}_{\text{vec}} &= \frac{1}{k!\, \varepsilon^k} \prod_{I \neq J} \frac{\varphi_I - \varphi_J}{\varphi_I - \varphi_J - \varepsilon} \\ \mathcal{Z}_{\text{fund}} &= \prod_{I=1}^{k} \prod_{f=1}^{N_f} \frac{1}{\varphi_I - M_f} \\ \mathcal{Z}_{\text{antifund}} &= \prod_{I=1}^{k} \prod_{a=1}^{\widetilde{N}_f} (\varphi_I + \widetilde{M}_a) \; . \end{aligned} \tag{3.25}$$

The integral is along a contour that encircles certain poles of the integrand, associated to the fixed points of the equivariant action on $\mathcal{M}_{k,\text{vortex}}$.[30] These poles are located at

$$\varphi_I = M_{p_i} + (l-1)\varepsilon \ , \qquad\qquad l = 1, \ldots, k_i \ , \qquad\qquad \sum k_i = k \tag{3.26}$$

and are parametrized by N one-dimensional Young diagrams (with total number of boxes equal to k). One can explicitly compute the residues, and collecting together all values of k one obtains

$$Z^{\text{SQCD}}_{\text{vortex}} = \sum_{\vec{k}} \frac{q^{|\vec{k}|}}{\vec{k}!} \, \frac{\prod_{i=1}^N \prod_{a=1}^{\widetilde{N}_f} \big(\frac{1}{\varepsilon}(M_{p_i} + \widetilde{M}_s)\big)_{k_i}}{\prod_{i\neq j}^N \big(\frac{1}{\varepsilon}(M_{p_i} - M_{p_j}) - k_j\big)_{k_j} \prod_{i=1}^N \prod_{f\notin\{p_j\}}^{N_f} \big(\frac{1}{\varepsilon}(M_{p_i} - M_s)\big)_{k_j}} \ . \tag{3.27}$$

This, plugged into the Higgs branch localization formula, gives the same result as the Coulomb branch integral formula.

A 2d supersymmetry transformations

Conventions. The epsilon tensor in vielbein coordinates is $\varepsilon_{12} = \varepsilon^{12} = 1$. This is the epsilon tensor with *vector* indices $\varepsilon^{\mu\nu}$. We decompose Weyl spinors as

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{A.1}$$

The contraction of spinors is upper-left to lower-right, and we declare $\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$:

$$\psi\,\chi = \psi^\alpha \chi_\alpha = \psi_+\chi_- - \psi_-\chi_+ \ , \qquad \psi^+ = -\psi_- \ , \qquad \psi^- = \psi_+ \ . \tag{A.2}$$

Notice that for anticommuting spinors $\psi\chi = \chi\psi$. Complex coordinates are

$$z = x^1 + ix^2 \ , \qquad \bar{z} = x^1 - ix^2 \ , \qquad \partial_z = \frac{\partial_1 - i\partial_2}{2} \ , \qquad \partial_{\bar{z}} = \frac{\partial_1 + i\partial_2}{2} \ . \tag{A.3}$$

The gamma matrices are

$$(\gamma^1)_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad (\gamma^2)_\alpha{}^\beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \qquad (\gamma^3)_\alpha{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.4}$$

and satisfy $\gamma^\mu\gamma^\nu = \delta^{\mu\nu} + i\epsilon^{\mu\nu}\gamma^3$. With complex indices

$$(\gamma_z)_\alpha{}^\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \qquad\qquad (\gamma_{\bar{z}})_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \tag{A.5}$$

[30] This can be rigorously obtained as the Jeffrey-Kirwan prescription. A physical derivation is in [10, 11], and its application to the instanton partition function of Nekrasov is in [26].

and in particular

$$\psi\gamma^3\chi = -\psi_+\chi_- - \psi_-\chi_+ \;, \qquad \psi\gamma_z\chi = \psi_+\chi_+ \;, \qquad \psi\gamma_{\bar z}\chi = -\psi_-\chi_- \;. \tag{A.6}$$

We see that $\gamma_z, \gamma_{\bar z}$ have spin $1, -1$ respectively. Under exchange:

$$\psi\gamma_{\mu_1}\ldots\gamma_{\mu_n}\chi = s\,(-1)^{n+1}\,\chi\gamma_{\mu_n}\ldots\gamma_{\mu_1}\psi \tag{A.7}$$

where $n \geq 0$, $\mu_i = 1, 23$, and $s = 1$ for commuting spinors and $s = -1$ for anticommuting. Another useful trick is

$$\big((\gamma_\mu\psi)\chi\big) = -\psi\gamma_\mu\chi \tag{A.8}$$

for any statistics. Moreover $\gamma_z\gamma_3 = -\gamma_3\gamma_z = \gamma_z$, $\gamma_{\bar z}\gamma_3 = -\gamma_3\gamma_{\bar z} = -\gamma_{\bar z}$. The Fierz identity, for spinors of any statistics, is

$$\zeta \cdot \chi = -\frac{1}{2}(\zeta\chi)\mathbb{1} + \frac{1}{2}\sum_{\mu=1,2,3}(\zeta\gamma_\mu\chi)\gamma^\mu \tag{A.9}$$

where $(\zeta\cdot\chi)_\alpha{}^\beta = \zeta_\alpha\chi^\beta$. The covariant derivative and the field strength are

$$D_\mu = \nabla_\mu - iA_\mu \;, \qquad\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{A.10}$$

where in the first one A_μ acts in the correct representation. We define the $U(1)$ spin connection $\omega_\mu = -\omega_\mu^{12}$, then the covariant derivative on fields of definite spin s is

$$\nabla_\mu\varphi^{(s)} = \big(\partial_\mu - is\omega_\mu\big)\,\varphi^{(s)} \;. \tag{A.11}$$

SUSY. The supersymmetry parameters are $\zeta, \widetilde{\zeta}$ with R-charge $1, -1$ respectively, and are commuting. The supercharges Q_α, $\widetilde{Q}_\beta$ are anticommuting spinorial operators. We construct the anticommuting variation

$$\delta = \tfrac{1}{\sqrt{2}}\big(\zeta Q + \widetilde{\zeta}\widetilde{Q}\big) \;. \tag{A.12}$$

The flat-space supersymmetry algebra is

$$\{Q_\alpha, \widetilde{Q}_\beta\} = \big[2\gamma^\mu P_\mu + 2i\mathbb{P}_+ Z + 2i\mathbb{P}_-\widetilde{Z}\big]_{\alpha\beta} \;, \tag{A.13}$$

where $P_\mu = -i\partial_\mu$. In components this reads:

$$\{Q_+, \widetilde{Q}_+\} = 4P_z \;, \quad \{Q_-, \widetilde{Q}_-\} = -4P_{\bar z} \;, \quad \{Q_+, \widetilde{Q}_-\} = -2iZ \;, \quad \{Q_-, \widetilde{Q}_+\} = 2i\widetilde{Z} \;. \tag{A.14}$$

In terms of variations this is

$$\{\delta_\zeta, \delta_{\widetilde{\zeta}}\} = i\mathcal{L}_{\zeta\gamma^\mu\widetilde{\zeta}} - i\zeta\begin{pmatrix} Z & 0 \\ 0 & \widetilde{Z}\end{pmatrix}\widetilde{\zeta} \;. \tag{A.15}$$

For a scalar, a spinor, a gauge field and a 2d field of generic spin s, the Lie derivative is:

$$\begin{aligned}
\mathcal{L}_K\phi &= K^\mu\partial_\mu\phi \\
\mathcal{L}_K\psi &= K^\mu\nabla_\mu\psi + \frac{1}{4}(\nabla_\mu K_\nu)\gamma^{\mu\nu}\psi \\
\mathcal{L}'_K A &= \mathcal{L}_K A - d^A(\imath_K A) = K^\rho F_{\rho\mu}dx^\mu \\
\mathcal{L}_K\varphi^{(s)} &= K^\mu(\partial_\mu - is\omega_\mu)\varphi^{(s)} + \frac{is}{2}\varepsilon^{\mu\nu}(\nabla_\mu K_\nu)\varphi^{(s)} \;.
\end{aligned} \tag{A.16}$$

The Killing spinor equations are (see *e.g.* [15]):

$$\begin{aligned}
D_\mu\zeta &= (\nabla_\mu - iV_\mu)\zeta = \frac{1}{2}H\gamma_\mu\zeta - \frac{i}{2}G\gamma_\mu\gamma_3\zeta = \frac{1}{2}\begin{pmatrix}\mathcal{H} & 0 \\ 0 & \widetilde{\mathcal{H}}\end{pmatrix}\gamma_\mu\zeta \\
D_\mu\widetilde{\zeta} &= (\nabla_\mu + iV_\mu)\widetilde{\zeta} = \frac{1}{2}H\gamma_\mu\widetilde{\zeta} + \frac{i}{2}G\gamma_\mu\gamma_3\widetilde{\zeta} = \frac{1}{2}\begin{pmatrix}\widetilde{\mathcal{H}} & 0 \\ 0 & \mathcal{H}\end{pmatrix}\gamma_\mu\widetilde{\zeta} \;.
\end{aligned} \tag{A.17}$$

We reserve ∇_μ for the pure metric-covariant derivative, and D_μ for the full covariant derivative:

$$D_\mu = \nabla_\mu - irV_\mu + \frac{1}{2}z\widetilde{C}_\mu - \frac{1}{2}\widetilde{z}C_\mu - iA_\mu \;. \tag{A.18}$$

Of course $\zeta, \widetilde{\zeta}$ are only charged under the R-symmetry. If we decompose $C_\mu = C^H_\mu + iC^G_\mu$, $\widetilde{C}_\mu = C^H_\mu - iC^G_\mu$, $z = z_G - iz_H$, $\widetilde{z} = z_G + iz_H$, we find $\frac{1}{2}(z\widetilde{C}_\mu - \widetilde{z}C_\mu) = -iz_H C^H_\mu - iz_G C^G_\mu$. If we have both $\zeta, \widetilde{\zeta}$ we can form

$$K_\mu = \zeta\gamma_\mu\widetilde{\zeta} \;. \tag{A.19}$$

From the Killing spinor equations follows

$$\nabla_\mu K_\nu = -\zeta\begin{pmatrix}\mathcal{H} & 0 \\ 0 & \widetilde{\mathcal{H}}\end{pmatrix}\gamma_{\mu\nu}\widetilde{\zeta} \;, \qquad \frac{1}{4}\nabla_\mu K_\nu\gamma^{\mu\nu} = \frac{1}{2}\zeta\begin{pmatrix}\mathcal{H} & 0 \\ 0 & \widetilde{\mathcal{H}}\end{pmatrix}\gamma_3\widetilde{\zeta}\cdot\gamma_3 \;, \tag{A.20}$$

in particular $\nabla_{(\mu}K_{\nu)} = 0$ and so K_μ is a Killing vector.

SUSY transformations. The transformations of a vector multiplet are:

$$\begin{aligned}
\delta A_\mu &= \frac{i}{\sqrt{2}}\big(\zeta\gamma_\mu\widetilde{\lambda} + \widetilde{\zeta}\gamma_\mu\lambda\big) & \delta F_{12} &= -\frac{1}{\sqrt{2}}D_\mu\big(\zeta\gamma^\mu\gamma_3\widetilde{\lambda} + \widetilde{\zeta}\gamma^\mu\gamma_3\lambda\big) \\
\delta(\sigma+\widetilde{\sigma}) &= -\sqrt{2}\,\zeta\widetilde{\lambda} + \sqrt{2}\,\widetilde{\zeta}\lambda & \delta\sigma &= -\sqrt{2}\,\zeta\mathbb{P}_-\widetilde{\lambda} + \sqrt{2}\,\widetilde{\zeta}\mathbb{P}_+\lambda \\
\delta(\sigma-\widetilde{\sigma}) &= \sqrt{2}\,\zeta\gamma_3\widetilde{\lambda} + \sqrt{2}\,\widetilde{\zeta}\gamma_3\lambda & \delta\widetilde{\sigma} &= -\sqrt{2}\,\zeta\mathbb{P}_+\widetilde{\lambda} + \sqrt{2}\,\widetilde{\zeta}\mathbb{P}_-\lambda
\end{aligned}$$

$$\begin{aligned}
\delta\lambda &= \frac{i}{\sqrt{2}}\Big[-iD + \Big(-F_{12} + \frac{1}{2}[\sigma,\widetilde{\sigma}]\Big)\gamma_3 + \begin{pmatrix}\widetilde{\mathcal{H}}\sigma & 0 \\ 0 & \mathcal{H}\widetilde{\sigma}\end{pmatrix} + D_\mu\begin{pmatrix}\sigma & 0 \\ 0 & \widetilde{\sigma}\end{pmatrix}\gamma^\mu\Big]\zeta \\
\delta\widetilde{\lambda} &= -\frac{i}{\sqrt{2}}\Big[-iD + \Big(F_{12} + \frac{1}{2}[\sigma,\widetilde{\sigma}]\Big)\gamma_3 + \begin{pmatrix}\mathcal{H}\widetilde{\sigma} & 0 \\ 0 & \widetilde{\mathcal{H}}\sigma\end{pmatrix} + D_\mu\begin{pmatrix}\widetilde{\sigma} & 0 \\ 0 & \sigma\end{pmatrix}\gamma^\mu\Big]\widetilde{\zeta} \\
\delta D &= -\frac{i}{\sqrt{2}}D_\mu\big(\zeta\gamma^\mu\widetilde{\lambda} - \widetilde{\zeta}\gamma^\mu\lambda\big) - \frac{i}{\sqrt{2}}\zeta\begin{pmatrix}[\sigma,\cdot] & 0 \\ 0 & [\widetilde{\sigma},\cdot]\end{pmatrix}\widetilde{\lambda} - \frac{i}{\sqrt{2}}\widetilde{\zeta}\begin{pmatrix}[\widetilde{\sigma},\cdot] & 0 \\ 0 & [\sigma,\cdot]\end{pmatrix}\lambda \;.
\end{aligned} \tag{A.21}$$

To write the transformations of charged chiral multiplets in gauge representation $\mathfrak{R}$, we introduce the short-hand notation

$$\mathbf{Q} = \begin{pmatrix} z - \sigma - \frac{r}{2}\mathcal{H} & 0 \\ 0 & \widetilde{z} - \widetilde{\sigma} - \frac{r}{2}\widetilde{\mathcal{H}} \end{pmatrix} \tag{A.22}$$

and $\widetilde{\mathbf{Q}}$ which has inverted $\widetilde{\ }$, *i.e.* $\widetilde{\mathbf{Q}} = \gamma^\mu \mathbf{Q} \gamma^\mu$ for fixed $\mu = 1, 2$. Here $\sigma, \widetilde{\sigma}$ are valued in $\mathfrak{R}$. Then

$$\begin{aligned} \delta\phi &= \zeta\psi & \delta\widetilde{\phi} &= -\widetilde{\zeta}\widetilde{\psi} \\ \delta\psi &= i\gamma^\mu\widetilde{\zeta}\, D_\mu\phi - i\mathbf{Q}\widetilde{\zeta}\,\phi + i\zeta F & \delta\widetilde{\psi} &= -i\gamma^\mu\zeta\, D_\mu\widetilde{\phi} + i\widetilde{\mathbf{Q}}\zeta\,\widetilde{\phi} + i\widetilde{\zeta}\widetilde{F} \\ \delta F &= \widetilde{\zeta}\gamma^\mu D_\mu\psi + \widetilde{\zeta}\widetilde{\mathbf{Q}}\psi + \sqrt{2}\,\widetilde{\zeta}\widetilde{\lambda}\phi & \delta\widetilde{F} &= \zeta\gamma^\mu D_\mu\widetilde{\psi} + \zeta\mathbf{Q}\widetilde{\psi} + \sqrt{2}\,\widetilde{\phi}\zeta\lambda\ . \end{aligned} \tag{A.23}$$

Recall that $(\widetilde{\phi}, \widetilde{\psi}, \widetilde{F})$ transform in gauge representation $\overline{\mathfrak{R}}$. For all fields but A_μ,[31] the algebra can be written as

$$\{\delta_\zeta, \delta_{\widetilde{\zeta}}\}\varphi_{(r,z,\tilde{z})} = i\big(\mathcal{L}'_K - \zeta\mathbf{Q}\widetilde{\zeta}\big)\,\varphi\ , \qquad \{\delta_\zeta, \delta_\eta\} = \{\delta_{\widetilde{\zeta}}, \delta_{\widetilde{\eta}}\} = 0\ , \tag{A.24}$$

where $\mathcal{L}'_K$ is the gauge-covariant Lie derivative.

Actions. The action for chiral multiplets is

$$\begin{aligned} \mathcal{L}_\Phi = D_\mu\widetilde{\phi}D^\mu\phi - i\widetilde{\phi}D\phi + \frac{1}{2}\big(\tfrac{r}{2}R_s + \mathcal{H}\widetilde{z} + \widetilde{\mathcal{H}}z\big)\widetilde{\phi}\phi + \frac{1}{2}\widetilde{\phi}\{\mathbf{Q}, \widetilde{\mathbf{Q}}\}\phi \\ + \widetilde{F}F + i\widetilde{\psi}\gamma^\mu D_\mu\psi + i\widetilde{\psi}\widetilde{\mathbf{Q}}\psi + i\sqrt{2}\,\widetilde{\psi}\widetilde{\lambda}\phi + i\sqrt{2}\,\widetilde{\phi}\lambda\psi\ . \end{aligned} \tag{A.25}$$

This is the D-term of $-\frac{1}{2}\widetilde{\Phi}e^{-2\mathcal{V}}\Phi$.

The YM action is

$$\begin{aligned} \mathcal{L}_\mathcal{V} = \frac{1}{2}\Big(F_{12} - \frac{1}{2}\widetilde{\mathcal{H}}\sigma + \frac{1}{2}\mathcal{H}\widetilde{\sigma}\Big)^2 + \frac{1}{2}D_\mu\widetilde{\sigma}D^\mu\sigma + \frac{1}{8}[\sigma, \widetilde{\sigma}]^2 \\ + i\widetilde{\lambda}\gamma^\mu D_\mu\lambda - i\widetilde{\lambda}\begin{pmatrix} [\widetilde{\sigma}, \cdot] & 0 \\ 0 & [\sigma, \cdot] \end{pmatrix}\lambda - \frac{1}{2}\Big(-iD + \frac{1}{2}\widetilde{\mathcal{H}}\sigma + \frac{1}{2}\mathcal{H}\widetilde{\sigma}\Big)^2 \end{aligned} \tag{A.26}$$

with trace implicit. This is the D-term of a gauge-invariant multiplet whose lowest component is $\frac{1}{2}\operatorname{Tr}\widetilde{\sigma}\sigma$.

B Wilson loop operators as defect theories

Wilson loop operators are defined by

$$W_\mathcal{R}[\gamma] = \operatorname{Tr}_\mathcal{R}\mathcal{P}\exp\oint_\gamma A\ . \tag{B.1}$$

[31]For all fields including A_μ, the algebra can be written as the gauge-invariant part of (A.24) plus a gauge transformation, which comes from imposing Wess-Zumino gauge.

We would like to find a defect theory description of these operators.

First, consider a 1d theory along γ, given by a free complex spinor ψ in representation $\mathcal{R}$ of the bulk gauge group G, minimally coupled to the bulk. Its Lagrangian is

$$\mathcal{L}_D = \overline{\psi}\not{D}\psi = \overline{\psi}(\partial_\tau - iA_\tau)\psi = \sum_{\rho\in\mathcal{R}} \overline{\psi}_\rho\big(\partial_\tau - i\rho(A_\tau)\big)\psi_\rho \ . \tag{B.2}$$

Here τ is a coordinate along γ, A is the bulk connection pulled back to γ, ρ are the weights of $\mathcal{R}$, and the 1d gamma matrix $\gamma_\tau = 1$ in einbein basis. For simplicity, we will take τ such that the pulled back metric is 1. Let γ be a circle of length β and let us choose antiperiodic (thermal) boundary conditions for the fermions. Then the path-integral is easily evaluated, since ψ is free. Let us choose a gauge where A_τ is constant. Then

$$Z_D = \int \mathcal{D}\overline{\psi}\,\mathcal{D}\psi\; e^{-\int d\tau\,\overline{\psi}(\partial_\tau - iA_\tau)\psi} = \prod_{\rho\in\mathcal{R}}\prod_{k\in\mathbb{Z}}\Big(\tfrac{2\pi i}{\beta}\big(k+\tfrac{1}{2}\big) - i\rho(A_\tau)\Big) \ . \tag{B.3}$$

That is because the modes of ψ are $e^{2\pi i\left(k+\frac{1}{2}\right)\tau/\beta}$. The regularization has some ambiguity, as the function should have zeros at $\rho(A_\tau) = 2\pi\big(k+\frac{1}{2}\big)$, but we can choose

$$Z_D = \prod_{\rho\in\mathcal{R}}\big(1+e^{i\beta\rho(A_\tau)}\big) \;\equiv\; \prod_{\rho\in\mathcal{R}}(1+x_\rho) \ . \tag{B.4}$$

This is just the partition function of the fermionic Fock space, where the excited levels have energies $-i\rho(A_\tau)$. Notice that x_ρ are the eigenvalues of the holonomy $\mathcal{P}\exp\oint_\gamma A$ in representation $\mathcal{R}$, therefore the gauge-invariant expression for Z_D is

$$Z_D = \det\nolimits_{\mathcal{R}}\Big(1 + \mathcal{P}\exp\oint_\gamma A\Big) \ . \tag{B.5}$$

This is not yet the Wilson line operator in representation $\mathcal{R}$. However notice that if we decompose $\prod_\rho(1+x_\rho)$ into characters, we find all antisymmetric products of $\mathcal{R}$, which can be further decomposed into irreducible representations:

$$\prod_\rho(1+x_\rho) \;\sim\; \sum_{\ell=0}^{\dim\mathcal{R}} \mathcal{R}^{\otimes_A \ell} \ .$$

Each level ℓ is the partition function restricted to fermion number ℓ. To select a specific fermion number, we gauge it – which corresponds to imposing Gauss law – and include a Chern-Simons coupling which includes $-\ell$ units of electric charge so that gauge-invariant states have fermion number ℓ. Thus, we consider the action

$$\widetilde{\mathcal{L}}_D = \overline{\psi}\big(\partial_\tau - iA_\tau - i\widetilde{A}_\tau\big)\psi + i\ell\widetilde{A}_\tau \ , \tag{B.6}$$

where $\widetilde{A}$ is a 1d gauge field. The path-integral over $\tilde{A}$ gives a delta function on $\psi\overline{\psi} = \ell$, which projects the partition function to the sector with fermion number ℓ. Alternatively, we perform the path-integral over ψ first and introduce a fugacity $y = e^{i\beta\widetilde{A}_\tau}$ for the 1d $U(1)$ symmetry; then the CS term gives a classical contribution $y^{-\ell}$ and finally the path-integral over $\widetilde{A}$ – imposing Gauss law – reduces to a contour integral along $|y| = 1$:

$$\widetilde{Z}_D = \oint_{|y|=1} \frac{dy}{2\pi i\, y}\, y^{-\ell} \prod_{\rho\in\mathcal{R}} \big(1 + x_\rho y\big) = \sum_{\rho_1<\ldots<\rho_\ell} x_\rho \;, \tag{B.7}$$

where, with some abuse of notation, we have assumed an ordering of the weights.

If now we consider the special case $\ell = 1$, we precisely produce the trace of the holonomy in representation $\mathcal{R}$:

$$\widetilde{Z}_D(\ell = 1) = \sum_\rho x_\rho = \mathrm{Tr}_\mathcal{R}\, \mathcal{P} \exp \oint_\gamma A \;. \tag{B.8}$$

Representations $\mathcal{R}$ which are the antisymmetric product of some representation $\mathcal{R}'$ can be obtained either by choosing higher ℓ, or by choosing $\mathcal{R}$ directly.

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