```
> restart;
```

HyperInt

Copyright (C) 2014 Erik Panzer, Email: panzer@mathematik.hu-berlin.de, updates and resources might be posted on http://www.math.hu-berlin.de/~panzer/

```
> read "HyperInt.mpl";
HyperInt, version 1.0, Copyright (C) 2014 Erik Panzer
Please report any errors or suggestions to panzer@mathematik.hu-berlin.de
Loading periods from periodLookups.m
```

Helper to reset all options at the beginning of a manual section

```
> initialSettings:=proc() global _hyper_splitting_field, _hyper_abort_on_divergence, _hyper_check_divergences, _hyper_restrict_singularities, _hyper_ignore_nonlinear_polynomials: _hyper_splitting_field:={}: _hyper_abort_on_divergence:=true: _hyper_check_divergences:=true: _hyper_restrict_singularities:=false: _hyper_ignore_nonlinear_polynomials:=false: forgetAll(): end proc:
```

Massless propagators: T1 topology

Two-loop master integral

```
> initialSettings( ) :
```

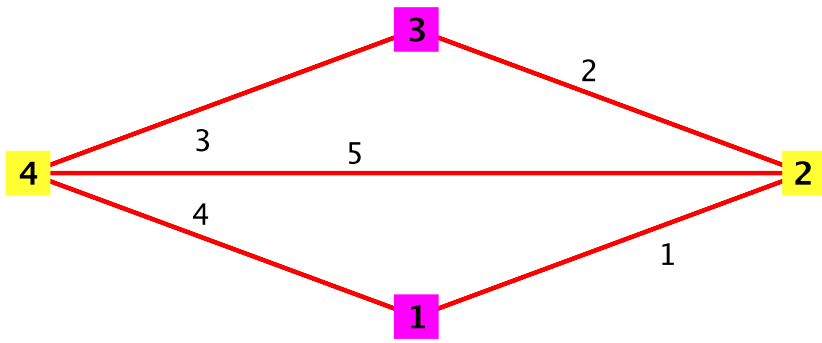
The well-known two-loop master function is just:

```
> _hyper_restrict_singularities := false: forgetAll( ) :
> E := [[1, 2], [2, 3], [3, 4], [4, 1], [2, 4]]: Momenta := [[1, 1], [3, 1]]: G := E, Momenta;
  psi := graphPolynomial(E); phi := secondPolynomial(G);
  drawGraph(G);
```

$$G := [[1, 2], [2, 3], [3, 4], [4, 1], [2, 4]], [[1, 1], [3, 1]]$$

$$\psi := x_1 x_2 + x_1 x_3 + x_1 x_5 + x_2 x_4 + x_2 x_5 + x_3 x_4 + x_3 x_5 + x_4 x_5$$

$$\phi := x_2 (x_4 x_5 + x_3 (x_5 + x_4)) + x_1 (x_4 x_5 + x_3 (x_5 + x_4) + x_2 (x_3 + x_4))$$



First four terms in the epsilon-expansion for unit exponents of the propagators:

> $Dim := 4 - 2\cdot\text{epsilon} : loops := 2 : sdd := nops(E) - \frac{Dim}{2}\cdot loops : f := \psi^{-\frac{Dim}{2}} \cdot \left(\frac{\text{phi}}{\text{psi}}\right)^{-sdd};$

$$f := \left(x_1 x_2 + x_1 x_3 + x_1 x_5 + x_2 x_4 + x_2 x_5 + x_3 x_4 + x_3 x_5 + x_4 x_5\right)^{-2} + \varepsilon \left(\frac{x_2 \left(x_4 x_5 + x_3 \left(x_5 + x_4\right)\right) + x_1 \left(x_4 x_5 + x_3 \left(x_5 + x_4\right) + x_2 \left(x_3 + x_4\right)\right)}{x_1 x_2 + x_1 x_3 + x_1 x_5 + x_2 x_4 + x_2 x_5 + x_3 x_4 + x_3 x_5 + x_4 x_5}\right)^{-1-2\varepsilon} \quad \textbf{(1.1.1)}$$

> $findDivergences(subs(\text{nu}[1] = 1, f));$

Analytic regularization around [epsilon = 0]

$$table([\,]) \quad \textbf{(1.1.2)}$$

> $convert(series(\text{GAMMA}(sdd)\cdot f, \text{epsilon} = 0, 2), polynom, \text{epsilon}) :$

```
> hyperInt(eval(%, x[1] = 1), [x[2], x[3], x[5], x[4]]) : fibrationBasis(%, [ ]) :
      subs(gamma = 0, π^2 = 6·zeta[2], π^4 = (6·zeta[2])^2, Zeta(3) = zeta[3],
      collect(%, epsilon));
Integrating variable x[2] from 0 to infinity, integrand has
3 terms
checking divergences after integration of x[2]
finished integration of variable x[2] in .59e-1 seconds,
produced 10 terms
Integrating variable x[3] from 0 to infinity, integrand has
10 terms
checking divergences after integration of x[3]
finished integration of variable x[3] in .314 seconds,
produced 32 terms
Integrating variable x[5] from 0 to infinity, integrand has
32 terms
checking divergences after integration of x[5]
finished integration of variable x[5] in .181 seconds,
produced 34 terms
Integrating variable x[4] from 0 to infinity, integrand has
34 terms
checking divergences after integration of x[4]
finished integration of variable x[4] in .103 seconds,
produced 10 terms
```

$$\left(\frac{18}{5}\,\zeta_2^2 + 12\,\zeta_3\right)\varepsilon + 6\,\zeta_3 \qquad \textbf{(1.1.3)}$$

With arbitrary epsilon-dependent deviations from unity powers:

```
> sdd := nops(E) + add(nu[n]·epsilon, n = 1..5) - Dim/2 · loops : f := ψ^(-Dim/2)
      · (phi/psi)^(-sdd) · mul(x[n]^(nu[n]·epsilon), n = 1..5);
```

$$f := \left(x_1 x_2 + x_1 x_3 + x_1 x_5 + x_2 x_4 + x_2 x_5 + x_3 x_4 + x_3 x_5 + x_4 x_5\right)^{-2} \qquad \textbf{(1.1.4)}$$

$$+\varepsilon \left(\frac{x_2\left(x_4 x_5 + x_3\left(x_5 + x_4\right)\right) + x_1\left(x_4 x_5 + x_3\left(x_5 + x_4\right) + x_2\left(x_3 + x_4\right)\right)}{x_1 x_2 + x_1 x_3 + x_1 x_5 + x_2 x_4 + x_2 x_5 + x_3 x_4 + x_3 x_5 + x_4 x_5}\right)^{-\varepsilon\nu_1 - \varepsilon\nu_2 - \varepsilon\nu_3 - \varepsilon\nu_4 - \varepsilon\nu_5 - 2\varepsilon - 1} x_1^{(\nu_1\varepsilon)}\, x_2^{(\nu_2\varepsilon)}\, x_3^{(\nu_3\varepsilon)}\, x_4^{(\nu_4\varepsilon)}\, x_5^{(\nu_5\varepsilon)}$$

```
> convert(series(f, epsilon = 0, 3), polynom, epsilon) :
> hyperInt(eval(%, x[3] = 1), [x[1], x[4], x[5], x[2]]) : fibrationBasis(%, [ ]) :
      collect(%, epsilon);
Integrating variable x[1] from 0 to infinity, integrand has
28 terms
checking divergences after integration of x[1]
finished integration of variable x[1] in .216 seconds,
```

```
produced 92 terms
Integrating variable x[4] from 0 to infinity, integrand has
92 terms
checking divergences after integration of x[4]
finished integration of variable x[4] in 2.686 seconds,
produced 311 terms
Integrating variable x[5] from 0 to infinity, integrand has
311 terms
checking divergences after integration of x[5]
finished integration of variable x[5] in 7.451 seconds,
produced 231 terms
Integrating variable x[2] from 0 to infinity, integrand has
231 terms
checking divergences after integration of x[2]
finished integration of variable x[2] in 2.264 seconds,
produced 26 terms
```

$$\left(-6\,\nu_2\,\nu_5\,\zeta_2\,\zeta_3 - 6\,\nu_3\,\nu_4\,\zeta_2\,\zeta_3 - 6\,\nu_3\,\nu_5\,\zeta_2\,\zeta_3 - 6\,\nu_4\,\nu_5\,\zeta_2\,\zeta_3 - 6\,\nu_1\,\nu_2\,\zeta_2\,\zeta_3 - 6\,\nu_1\,\nu_3\,\zeta_2\,\zeta_3 - 6\,\nu_1\,\nu_4\,\zeta_2\,\zeta_3 - 6\,\nu_1\,\nu_5\,\zeta_2\,\zeta_3 - 6\,\nu_2\,\nu_3\,\zeta_2\,\zeta_3 - 6\,\nu_2\,\nu_4\,\zeta_2\,\zeta_3 + 10\,\nu_2^2\,\zeta_5 + 10\,\nu_3^2\,\zeta_5 + 10\,\nu_4^2\,\zeta_5 + 15\,\nu_5^2\,\zeta_5 + 30\,\nu_1\,\zeta_5 + 30\,\nu_2\,\zeta_5 + 30\,\nu_3\,\zeta_5 + 30\,\nu_4\,\zeta_5 + 45\,\nu_5\,\zeta_5 + 10\,\nu_1^2\,\zeta_5 - 12\,\nu_1\,\zeta_2\,\zeta_3 - 12\,\nu_2\,\zeta_2\,\zeta_3 - 12\,\nu_3\,\zeta_2\,\zeta_3 - 12\,\nu_4\,\zeta_2\,\zeta_3 - 12\,\nu_5\,\zeta_2\,\zeta_3 + 10\,\nu_1\,\nu_2\,\zeta_5 + 5\,\nu_1\,\nu_3\,\zeta_5 + 10\,\nu_1\,\nu_4\,\zeta_5 + 15\,\nu_1\,\nu_5\,\zeta_5 + 10\,\nu_2\,\nu_3\,\zeta_5 + 5\,\nu_2\,\nu_4\,\zeta_5 + 15\,\nu_2\,\nu_5\,\zeta_5 + 10\,\nu_3\,\nu_4\,\zeta_5 + 15\,\nu_3\,\nu_5\,\zeta_5 + 15\,\nu_4\,\nu_5\,\zeta_5 - 18\,\zeta_2\,\zeta_3 + 24\,\zeta_3 + 42\,\zeta_5 + \frac{36}{5}\,\zeta_2^2\right)\varepsilon^2 + \left(\frac{18}{5}\,\zeta_2^2 + 12\,\zeta_3\right)\varepsilon + 6\,\zeta_3 \tag{1.1.5}$$

Note that for the full Feynman rule one has to include also the prefactor

> $\dfrac{\mathrm{GAMMA}(sdd)}{mul(\mathrm{GAMMA}(1+\mathrm{nu}[n]\cdot\mathrm{epsilon}),\ n=1..5)}$;

$$\frac{\Gamma(\varepsilon\,\nu_1+\varepsilon\,\nu_2+\varepsilon\,\nu_3+\varepsilon\,\nu_4+\varepsilon\,\nu_5+2\,\varepsilon+1)}{\Gamma(\varepsilon\,\nu_1+1)\,\Gamma(\varepsilon\,\nu_2+1)\,\Gamma(\varepsilon\,\nu_3+1)\,\Gamma(\varepsilon\,\nu_4+1)\,\Gamma(\varepsilon\,\nu_5+1)} \tag{1.1.6}$$

Pulling out a pre-factor to remove dependence on the Euler-Mascheroni constant and a factor (1+2*epsilon) to get a homogeneous weight result gives:

> $series\left(\dfrac{\exp(2\cdot\mathrm{gamma}\cdot\mathrm{epsilon})}{1+2\cdot\mathrm{epsilon}}\cdot((\mathbf{1.1.5})\cdot((\mathbf{1.1.6}),\ \mathrm{epsilon}=0,\ 3\right)$:
> $simplify(eval(\%,\ \pi^2=6\cdot\mathrm{zeta}[2]))$;

(1.1.7)

$$6\,\zeta_3 + \frac{18}{5}\,\zeta_2^2\,\varepsilon + \Big(10\,\nu_1^2\,\zeta_5 + 10\,\nu_1\,\nu_2\,\zeta_5 + 5\,\nu_1\,\nu_3\,\zeta_5 + 10\,\nu_1\,\nu_4\,\zeta_5 + 15\,\nu_1\,\nu_5\,\zeta_5 + 10\,\nu_2^2\,\zeta_5 + 10\,\nu_2\,\nu_3\,\zeta_5 + 5\,\nu_2\,\nu_4\,\zeta_5 + 15\,\nu_2\,\nu_5\,\zeta_5 + 10\,\nu_3^2\,\zeta_5 + 10\,\nu_3\,\nu_4\,\zeta_5 + 15\,\nu_3\,\nu_5\,\zeta_5 + 10\,\nu_4^2\,\zeta_5 + 15\,\nu_4\,\nu_5\,\zeta_5 + 15\,\nu_5^2\,\zeta_5 + 30\,\nu_1\,\zeta_5 + 30\,\nu_2\,\zeta_5 + 30\,\nu_3\,\zeta_5 + 30\,\nu_4\,\zeta_5 + 45\,\nu_5\,\zeta_5 - 6\,\zeta_2\,\zeta_3 + 24\,\zeta_3 + 42\,\zeta_5\Big)\,\varepsilon^2 + O\left(\varepsilon^3\right) \tag{1.1.7}$$

>