Introduction to Computer Algebra

lecture III

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– Computer Algebra and Particle Physics 2015, Hamburg, March 23, 2015 –

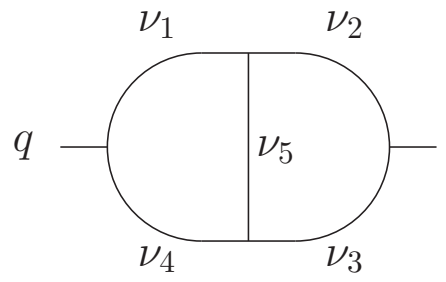
Plan

- Particle physics and perturbation theory at work
  Monday, March 23, 2015
- Symbolic sums and recurrence relations
  Tuesday, March 24, 2015
- *Polylogarithms and generating functions*
  Thursday, March 24, 2015

Loop integrals

- Example: massless two-loop two-point function

$$\mathrm{T1} = \int d^D p_1\, d^D p_2\, \frac{1}{(p_1^2)^{\nu_1}\,(p_2^2)^{\nu_2}\,(p_3^2)^{\nu_3}\,(p_4^2)^{\nu_4}\,(p_5^2)^{\nu_5}}$$



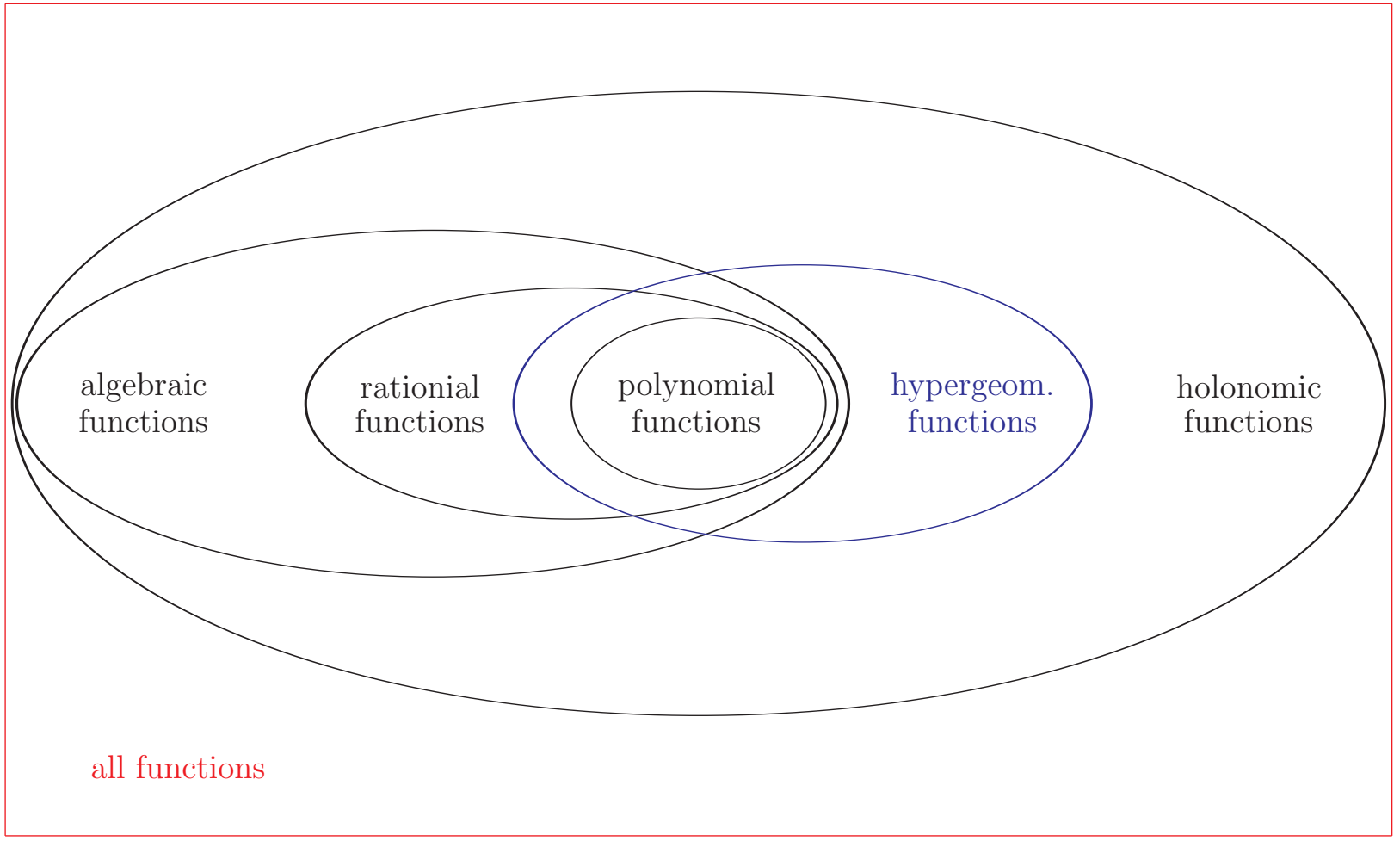
- $S_6 \times Z_2$ symmetry group of $\mathrm{T1}$ (1440 elements)

Solution up to four loops

- All massless propagators up to four loops are linearly reducible. Their ϵ-expansions only contain alternating Euler sums $\mathrm{Li}_{n_1,\ldots,n_r}(\pm 1,\ldots,\pm 1)$
  Brown '09; Panzer '13

Classes of functions

- Some common classes of functions



Holonomic functions

- Definition (continuous case of one variable): A function f is called holonomic if there exist polynomials $p_0, \ldots, p_r$ not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0$$

- Examples:
  - $\exp(x)$: $\quad f'(x) - f(x) = 0$
  - $\ln(1-x)$: $\quad (x-1)f''(x) - f'(x) = 0$
  - $\frac{1}{1+\sqrt{1-x^2}}$: $\quad (x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0$
  - Bessel functions, Hankel functions, Struve functions, Airy functions, Polylogarithms, Elliptic integrals, the Error function, Kelvin functions, Mathieu functions, ...
  - many functions which have no name and no closed form

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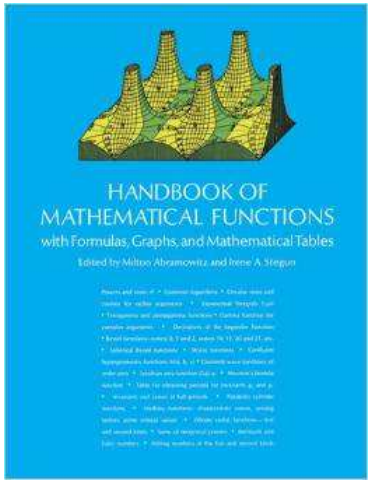
- Not holonomic:
  - $\exp(\exp(x) - 1)$
  - Riemann Zeta function
  - many functions which have no name and no closed form
- This means that these functions can (provably) not be viewed as solutions of a linear differential equation with polynomial coefficients.

Holonomic functions

- Approximately 60% of the functions in Abramowitz and Stegun's handbook fall into the category of holonomic functions in one variable.

  - *Handbook of Mathematical Functions*
    M. Abramowitz, I. Stegun



Differential equations

Theorem

- The solution set of a linear differential equation of order r is a vector space of dimension r.

Consequences

- A holonomic function f is uniquely determined by
  - the differential equation
  - a finite number of initial values $f(0)$, $f'(0)$, $f''(0)$, $\ldots$, $f^{(k)}(0)$ (usually, $k = r$ suffices.)
- A holonomic function can be represented exactly by a finite amount of data. (assuming that the constants appearing in equation and initial values belong to a suitable subfield of $\mathbb{C}$, e.g., to $\mathbb{Q}$.)

Examples

- $f(x) = \exp(x)$
  $\longleftrightarrow f'(x) - f(x) = 0$ with $f(0) = 1$

- $f(x) = \ln(1 - x)$
  $\longleftrightarrow (x - 1)f''(x) - f'(x) = 0$ with $f(0) = 0, f'(0) = -1$

- $f(x) = \frac{1}{1+\sqrt{1-x^2}}$
  $\longleftrightarrow (x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0$ with
  $f(0) = \frac{1}{2}, f'(0) = 0$

- $f(x) = I_5(x)$ (fifth modified Bessel function of the first kind)
  $\longleftrightarrow x^2 f''(x) + xf'(x) - (x^2 + 25)f(x) = 0$
  with $f(0) = f'(0) = \cdots = f^{(4)}(0) = 0, f^{(5)}(0) = \frac{1}{32}$

- ...

Holonomic sequences

- Definition (discrete case of one variable): A sequence $(a_n)_{n=0}^{\infty}$ is called holonomic if there exist polynomials $p_0, \ldots, p_r$ not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0$$

- Examples:
  - 2^n : $\quad a_{n+1} - 2a_n = 0$
  - $n!$: $\quad a_{n+1} - (n+1)a_n = 0$
  - $\sum_{i=0}^{n} \frac{(-1)^i}{i!}$: $\quad (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0$
  - Fibonacci numbers, Harmonic numbers, Perrin numbers, diagonal Delannoy numbers, Motzkin numbers, Catalan numbers, Apery numbers, Schröder numbers, ...
  - many sequences which have no name and no closed form

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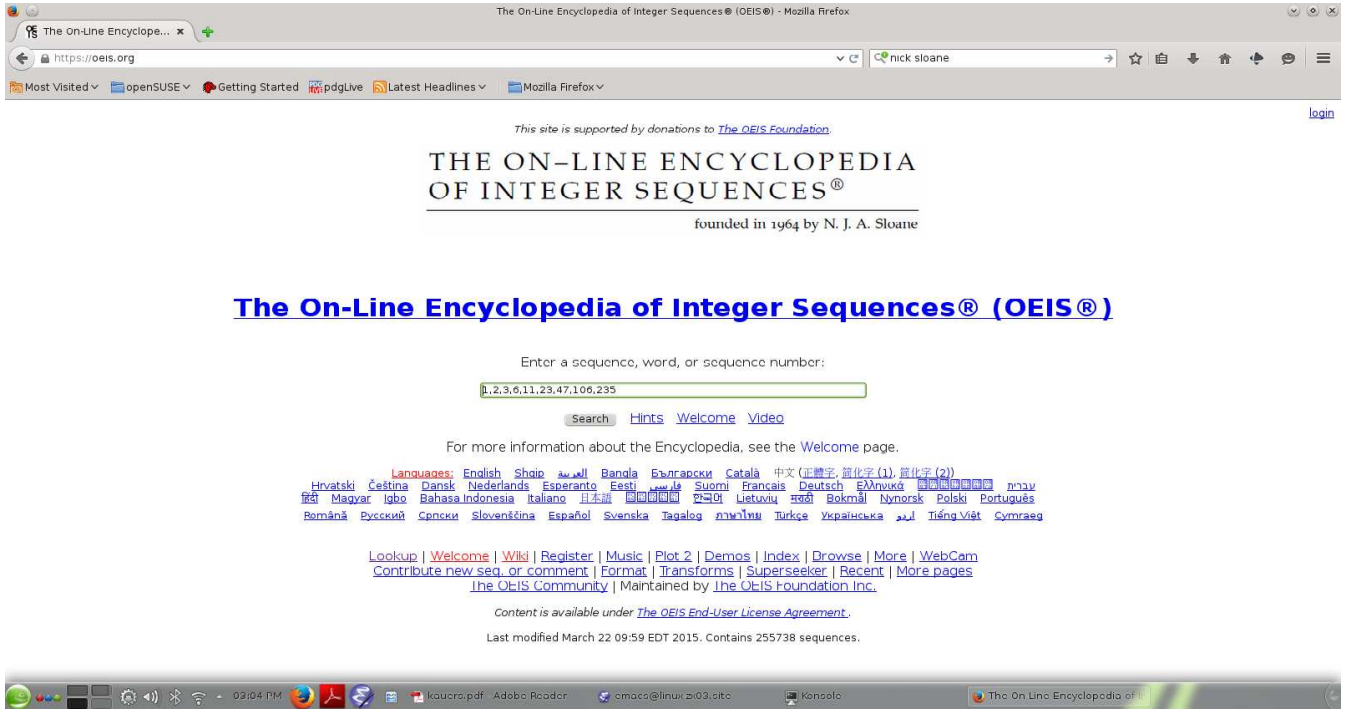
$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0$$

- Not holonomic:
  - 2^{2^n}
  - sequence of prime numbers
  - many sequences which have no name and no closed form
- This means that these sequences can (provably) not be viewed as solutions of a linear recurrence equation with polynomial coefficients.

Holonomic sequences

- Approximately 25% of the sequences in Sloane's Online Encyclopedia of Integer Sequences fall into the category of holonomic sequences in one variable.
  - Online Encyclopedia of Integer Sequences `https://oeis.org/`



Difference equations

Theorem

- The solution set of a linear recurrence equation of order r whose leading coefficient has s integer roots greater than r is a vector space of dimension $s + r$.

Consequences

- A holonomic sequence $(a_n)_{n=0}^{\infty}$ is uniquely determined by A holonomic function f is uniquely determined by
  - the recurrence equation
  - a finite number of initial values a_0, a_1, a_2, $\ldots$, a_k (usually, $k = r$ suffices.)
- A holonomic sequence can be represented exactly by a finite amount of data. (assuming that the constants appearing in equation and initial values belong to a suitable subfield of $\mathbb{C}$, e.g., to $\mathbb{Q}$.)

Examples

- $a_n = 2^n$
  $\longleftrightarrow a_{n+1} - 2a_n = 0$ with $a_0 = 1$

- $a_n = n!$
  $\longleftrightarrow a_{n+1} - (n+1)a_n = 0$ with $a_0 = 1$

- $a_n = \sum_{i=0}^{n} \frac{(-1)^i}{i!}$
  $\longleftrightarrow (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0$ with $a_0 = 1, a_1 = 0$

- $a_n = I(n)$ (number of involutions of n letters)
  $\longleftrightarrow a_{n+3} + na_{n+2} - (3n+6)a_{n+1} - (n+1)(n+2)a_n = 0$ with
  $a_0 = 1, a_1 = 1, a_2 = 2$

- ...

Conversion

Theorem

- Conversion of difference to differential equations:
  Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$, then $a(x)$ is holonomic as function $\longleftrightarrow$ $(a_n)_{n=0}^{\infty}$ is holonomic as sequence

Consequences

- Given a differential equation for $a(x)$, one can compute a recurrence for $(a_n)_{n=0}^{\infty}$
- Given a recurrence for $(a_n)_{n=0}^{\infty}$, we can compute a differential equation for $a(x)$

Polynomials

Polynomials as sequences

- Examples

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} 1\, x^i$$

$$D_x \frac{1}{1-x} = \frac{1}{(1-x)^2} = \sum_{i=0}^{\infty} i\, x^{i-1} = \sum_{i=0}^{\infty} (i+1)\, x^i$$

$$D_x^2 \frac{1}{1-x} = \frac{2}{(1-x)^3} = \sum_{i=0}^{\infty} i\,(i-1)\, x^{i-2} = \sum_{i=0}^{\infty} (i+1)\,(i+2)\, x^i$$

$$D_x^3 \frac{1}{1-x} = \frac{6}{(1-x)^4} = \sum_{i=0}^{\infty} i\,(i-1)\,(i-2)\, x^{i-3} = \sum_{i=0}^{\infty} (i+1)\,(i+2)\,(i+3)\, x^i$$

Harmonic polylogarithms

- Harmonic polylogarithms $H_{m_1,\ldots,m_k}(x)$
  Remiddi, Vermaseren '99
  - physical quantities in momentum (x)-space

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  - physical quantities in momentum (x)-space
    - basic functions of lowest weight
      $H_0(x) = \ln x\,, \quad H_1(x) = -\ln(1-x)\,, \quad H_{-1}(x) = \ln(1+x)$
    - higher functions defined by recursion
      $$H_{m_1,\ldots,m_w}(x) = \int_0^x dz\; f_{m_1}(z)\; H_{m_2,\ldots,m_w}(z)$$
      $$f_0(x) = \frac{1}{x}\,, \qquad f_1(x) = \frac{1}{1-x}\,, \qquad f_{-1}(x) = \frac{1}{1+x}$$

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      $$f_0(x) = \frac{1}{x}\,, \qquad f_1(x) = \frac{1}{1-x}\,, \qquad f_{-1}(x) = \frac{1}{1+x}$$
- Algebra under multiplication
  $H_{m_1,\ldots,m_r}(x) H_{n_1,\ldots,n_s}(x) \longrightarrow H_{m_1,\ldots,m_{r+s}}(x)$
- Integral transformation (Mellin transform to discrete N)
  $$\tilde{f}(N) \;=\; \int_0^1 dx\; x^N\, f(x)$$
  - unique mapping $\dfrac{H_{m_1,\ldots,m_w}(x)}{(1 \pm x)} \longleftrightarrow S_{n_1,\ldots,n_{w+1}}(N)$

Algebra of words

- Consider alphabet alphabet of length $l = 3$
  - harmonic polylogarithms arise from iterated integrals over letters $x, 1 \pm x$

Iterated integrals

- Generalization:
  - hyperlogarithms (mathematics definition Poincaré)
  - generalized polylogarithms $\mathrm{Li}_{m_k,\ldots,m_1}(x_k, \ldots, x_1)$ Goncharov '98; Borwein, Bradley, Broadhurst, Lisonek '99
- Words $w = m_{\sigma_1} \ldots m_{\sigma_n}$ from letters $w = m_{\sigma_i}$ associated to generalized polylogarithms

$$\mathrm{Li}_{m_k,\ldots,m_1}(x_k, \ldots, x_1) =$$

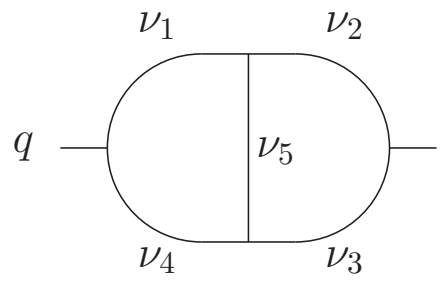
$$= \int_0^{x_1 x_2 \ldots x_k} \underbrace{\underbrace{\left(\frac{dt'}{t'}\circ\right)^{m_1-1}}_{\frac{dt'_{m_1-1}}{t'_{m_1-1}} \cdots \frac{dt'_1}{t'_1}}}_{(m_1-1)\text{ times}} \frac{dt_k}{x_2 x_3 \ldots x_k - t_k} \cdots \int_0^{t_2} \left(\frac{dt'}{t'}\circ\right)^{m_k-1} \frac{dt_1}{1-t_1}$$

Feynman integrals from iterated integrals

Graph polynomials

- Graph polynomial Ψ, associated polynomial Φ
  - $\Psi = \sum_{T \in T^1} \prod_{l \notin T} x_l$ (sum over 1-trees of graph; maximally connected subgraphs without loop)
  - $\Phi = \sum_{T \in T^2} \prod_{l \notin T} x_l (q_T)^2$ (sum over 2-trees of graph; two connectivity components with sum of momenta q_T)
- Graph polynomial for massless two-loop two-point function

$$\mathrm{T1} = \int d^D p_1 \, d^D p_2 \, \frac{1}{(p_1^2)^{\nu_1} \, (p_2^2)^{\nu_2} \, (p_3^2)^{\nu_3} \, (p_4^2)^{\nu_4} \, (p_5^2)^{\nu_5}}$$



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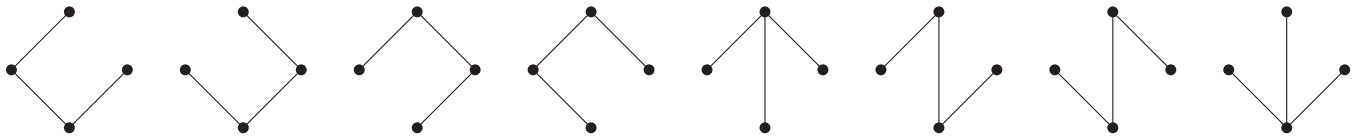
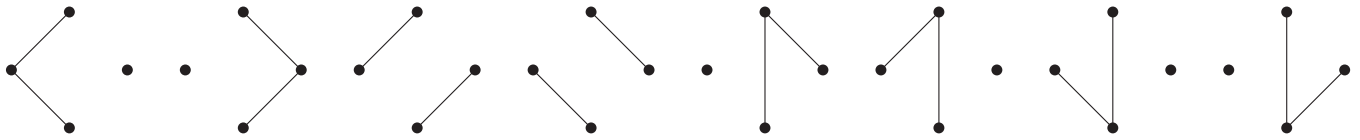


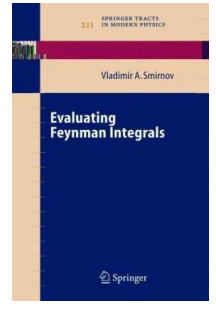
$$\Psi = x_1x_2 + x_1x_3 + x_1x_5 + x_2x_4 + x_2x_5 + x_3x_4 + x_3x_5 + x_4x_5$$



$$\Phi = x_2\big(x_4x_5 + x_3(x_5 + x_4)\big) + x_1\big(x_4x_5 + x_3(x_5 + x_4) + x_2(x_3 + x_4)\big)$$

Integrals over Schwinger parameters (I)

- *Evaluating Feynman integrals*
  V. Smirnov
  In spite of the fact, that the method of alpha paramters is not an optimal method for this class of integrals, let us, for the sake of illustration, evaluate this diagram . . .



- Function f for intergration over Schwinger parameters x_1

$$f = \Psi^{-D/2} \left(\frac{\Phi}{\Psi}\right)^{-E+lD/2}$$

  - number of edges $E = 5$
  - number of loops $l = 2$
  - dimension $D = 4 - 2\epsilon$

- Integral $\mathrm{T1}(D, 1, 1, 1, 1, 1)$ (all powers of propagators equal unity)

$$\begin{aligned} \mathrm{T1} &= \Gamma(E - lD/2) \int dx_1 \ldots dx_5 \Psi^{-D/2} \left(\frac{\Phi}{\Psi}\right)^{E-lD/2} \delta\left(1 - \sum x_i\right) \\ &= 6\zeta_3 + \mathcal{O}(\epsilon) \end{aligned}$$

Integrals over Schwinger parameters (II)

- MAPLE package `HyperInt` intergration over Schwinger parameters Panzer '14
  - use linearly reducibilty of integrals over Schwinger parameters to generalized polylogarithms
  - alternating Euler sums $\mathrm{Li}_{n_1,\ldots,n_r}(\pm 1,\ldots,\pm 1)$ for massless propagators up to four loops Brown '09; Panzer '13
- Modification of integrand for all powers of propagators shifted by ν_i
  - $\nu_{12345} = \sum_{i=1}^{5} \nu_i$

$$f = \Psi^{-D/2} \left(\frac{\Phi}{\Psi}\right)^{-E-\nu_{12345}+lD/2} x_1^{\nu_1} x_2^{\nu_2} x_3^{\nu_3} x_4^{\nu_4} x_5^{\nu_5}$$

- Integral

$$\mathrm{T1}(D,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) = \\ = \frac{\Gamma(E-lD/2)}{\Gamma(1-\nu_1)\ldots\Gamma(1-\nu_5)} \int dx_1 \ldots dx_5\, f\, \delta\left(1-\sum x_i\right)$$

Multiple scales

- Generalized sums $S(n; m_1, ..., m_k; x_1, ..., x_k)$
  - recursive definition

    $$S(n; m_1, ..., m_k; x_1, ..., x_k) = \sum_{i=1}^{n} \frac{x_1^i}{i^{m_1}} S(i; m_2, ..., m_k; x_2, ..., x_k)$$

  - multiple scales $x_1, ..., x_k$
  - depth k, weight $w = m_1 + ... + m_k$

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  - multiple scales $x_1, ..., x_k$
  - depth k, weight $w = m_1 + ... + m_k$
- Special cases

$S(\infty; m_1, ..., m_k; x_1, ..., x_k) \quad \rightarrow \quad \mathrm{Li}_{m_k,...,m_1}(x_k, ..., x_1)$

multiple polylogarithms

Goncharov '98; Borwein, Bradley, Broadhurst, Lisonek '99

$S(\infty; m_1, ..., m_k; x, 1, ..., 1) \quad \rightarrow \quad H_{m_1,...,m_k}(x)$

harmonic polylogarithms

Remiddi, Vermaseren '98

$S(n; m_1, ..., m_k; 1, ..., 1) \quad \rightarrow \quad S_{m_1,...,m_k}(n)$

Euler-Zagier sums

Algorithms for nested sums

- Same structures as for harmonic sums, in particular
  - multiplication
    $$S(n; m_1, ...; x_1, ...) \times S(n; m'_1, ...; x'_1, ...)$$
  - convolution
  - conjugation
  - binomial convolution
- Recursive algorithms analogous to harmonic sums solve multiple nested sums

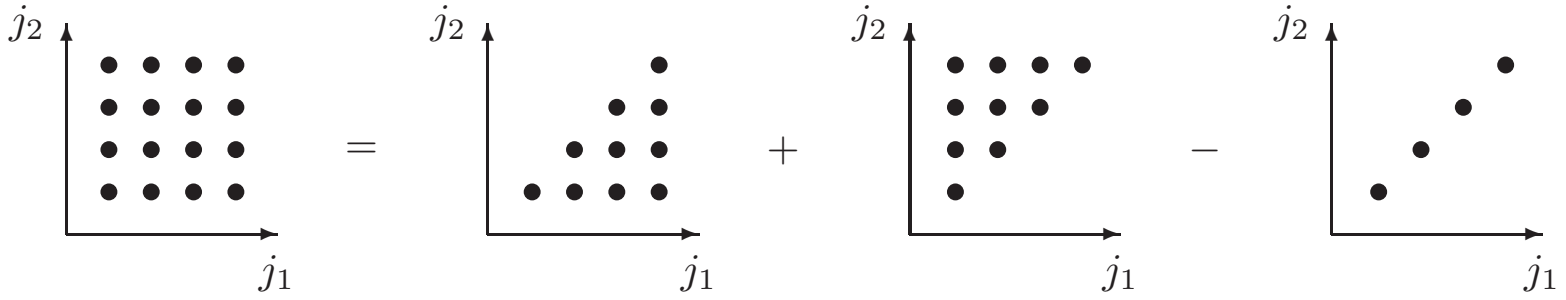
Algorithms for harmonic sums

- Multiplication (Hopf algebra)
  - basic formula (recursion)

$$
\begin{aligned}
S_{m_1,\ldots,m_k}(n) \times S_{m'_1,\ldots,m'_l}(n) \quad = \quad & \sum_{j_1=1}^{n} \frac{1}{j_1^{m_1}} S_{m_2,\ldots,m_k}(j_1) S_{m'_1,\ldots,m'_l}(j_1) \\
& + \sum_{j_2=1}^{n} \frac{1}{j_2^{m'_1}} S_{m_1,\ldots,m_k}(j_2) S_{m'_2,\ldots,m'_l}(j_2) \\
& - \sum_{j=1}^{n} \frac{1}{j^{m_1+m'_1}} S_{m_2,\ldots,m_k}(j) S_{m'_2,\ldots,m'_l}(j)
\end{aligned}
$$

- Proof uses decomposition

$$
\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \quad = \quad \sum_{i=1}^{n}\sum_{j=1}^{i} a_{ij} + \sum_{j=1}^{n}\sum_{i=1}^{j} a_{ij} - \sum_{i=1}^{n} a_{ii}
$$



Algorithms for harmonic sums (cont'd)

- Convolution (sum over $n-j$ and j)

$$\sum_{j=1}^{n-1} \frac{1}{j^{m_1}} S_{m_2,\ldots,m_k}(j) \frac{1}{(n-j)^{n_1}} S_{n_2,\ldots,n_l}(n-j)$$

- Conjugation

$$-\sum_{j=1}^{n} \binom{n}{j} (-1)^j \frac{1}{j^{m_1}} S_{m_2,\ldots,m_k}(j)$$

- Binomial convolution (sum over binomial, $n-j$ and j)

$$-\sum_{j=1}^{n-1} \binom{n}{j} (-1)^j \frac{1}{j^{m_1}} S_{m_2,\ldots,m_k}(j) \frac{1}{(n-j)^{n_1}} S_{n_2,\ldots,n_l}(n-j)$$

Higher transcendental functions

- Expansion of higher transcendental functions in small parameter
  - expansion parameter ϵ occurs in the rising factorials (Pochhammer symbols)
- Hypergeometric function

$$ {}_2F_1(a,b;c,x_0) = \sum_{i=0}^{\infty} \frac{a^{\overline{i}} b^{\overline{i}}}{c^{\overline{i}}} \frac{x_0^i}{i!} $$

- First Appell function

$$ F_1(a,b_1,b_2;c;x_1,x_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{a^{\overline{m_1+m_2}} b_1^{\overline{m_1}} b_2^{\overline{m_2}}}{c^{\overline{m_1+m_2}}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} $$

- Second Appell function

$$ F_2(a,b_1,b_2;c_1,c_2;x_1,x_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{a^{\overline{m_1+m_2}} b_1^{\overline{m_1}} b_2^{\overline{m_2}}}{c_1^{\overline{m_1}} c_2^{\overline{m_2}}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} $$

Massive loop integrals

- In real life, particles often have mass, e.g. $W^{\pm}, Z$-bosons, t-quark, ...
  - particle propagator $\dfrac{1}{p^2 - m^2}$
  - presence of mass term complicates analysis/algebra a lot

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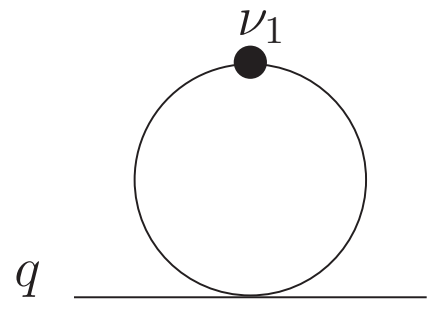
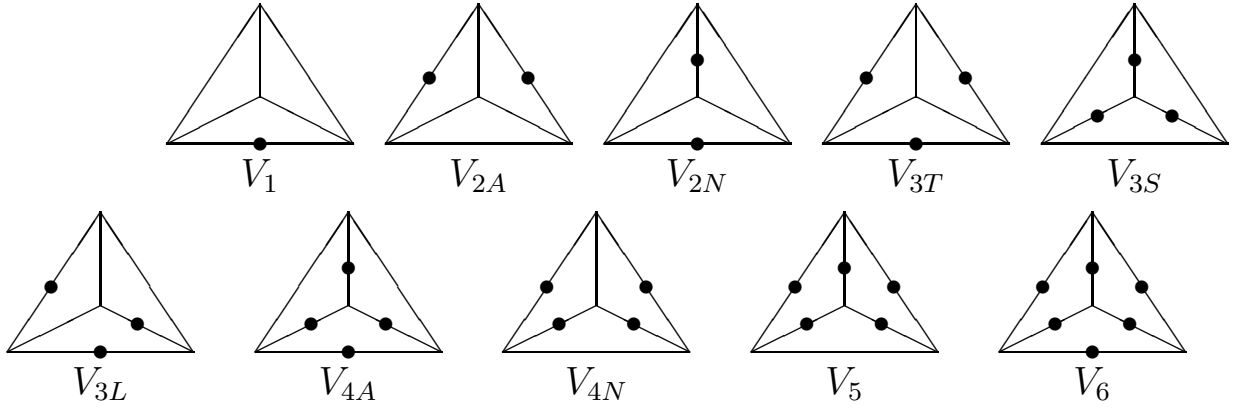
- In real life, particles often have mass, e.g. $W^{\pm}, Z$-bosons, t-quark, ...
  - particle propagator $\dfrac{1}{p^2 - m^2}$
  - presence of mass term complicates analysis/algebra a lot
- Tadpole with a mass (denoted by a blob)

$$0 \neq \int d^D p_1 \, \frac{1}{(p_1^2 - m^2)^{\nu_1}}$$



Tetrahedron

- Colourings of the tetrahedron by mass (denoted by a blob)



- Computation of finite part of three-loop scalar tetrahydral diagram
  Broadhurst '98
- Result in terms of double nested sums
  - constant related to sixth root of unity

$$\sum_{i>j>0} \frac{(-1)^{i+j}}{i^3 j}, \qquad \sum_{i>j>0} \frac{(-1)^i \cos(2\pi j/3)}{i^3 j}$$

Figure-eight knot complement

- Hyperbolic volume of figure-eight knot complement
  - double quotient space $U(2)\backslash SL(2,\mathbf{C})/\Gamma$
    (Γ fundamental group of figure-eight knot complement)
  - sculpture Helaman Ferguson `www.helasculpt.com`

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$$
\begin{aligned}
V &= 2\sqrt{3}\sum_{k=1}^{\infty}\frac{1}{k\binom{2k}{k}}\sum_{i=k}^{2k-1}\frac{1}{i} \\
&= \frac{\sqrt{3}}{9}\sum_{k=0}^{\infty}\frac{(-1)^k}{27^k}\left(\frac{18}{(6k+1)^2}-\frac{18}{(6k+2)^2}-\frac{24}{(6k+3)^2}-\frac{6}{(6k+4)^2}+\frac{2}{(6k+5)^2}\right)
\end{aligned}
$$

Roots of unity

- Roots of unity $r_q^p = \exp\left(\frac{2\pi \mathrm{i} p}{q}\right)$
  - q positive integer and p non-negative integer
  - powers of the q-th roots of unity periodic modulo q
    $\left(r_q^p\right)^{j+q} = \left(r_q^p\right)^j$
  - sums of powers of the q-th roots of unity

$$\sum_{p=0}^{q-1} \left(r_q^p\right)^m = \begin{cases} q, & m = 0 \text{ mod } q, \\ 0, & m \neq 0 \text{ mod } q. \end{cases}$$

- Conversion of $S(n; ...)$ to a combination of sums $S(q \cdot n; ...)$ (iterative algorithm)

$$\sum_{i=1}^{n} \frac{x^i}{i^m} S(q \cdot i; ...) = q^{m-1} \sum_{p=0}^{q-1} \sum_{i=1}^{q \cdot n} \frac{1}{i^m} \left(r_q^p x^{1/q}\right)^i S(i; ...).$$

- Gamma-function expanded around rational numbers ($n - p/q$) in $\epsilon = 2 - \frac{D}{2}$
  - sums over roots of unity r_q^l and sums of argument $S(q \cdot n, k; r_q^l)$ emerge naturally Weinzierl '04

$$\frac{\Gamma\left(n+1-\frac{p}{q}+\epsilon\right)}{\Gamma\left(1-\frac{p}{q}+\epsilon\right)} = \frac{\Gamma\left(n+1-\frac{p}{q}\right)}{\Gamma\left(1-\frac{p}{q}\right)} \times \exp\left(-\frac{1}{q}\sum_{l=0}^{q-1}\left(r_q^l\right)^p \sum_{k=1}^{\infty}\epsilon^k \frac{(-q)^k}{k} S(q \cdot n; k; r_q^l)\right)$$

Binomial and inverse binomial sums

- Feynman integrals with massive propagators lead to structures
  Davydychev, Kalmykov '02

$$\chi = \frac{1-\sqrt{1-4x}}{1+\sqrt{1-4x}}, \quad x = \frac{\chi}{(1+\chi)^2}, \quad \sqrt{1-4x} = \frac{1-\chi}{1+\chi}.$$

- Simple example of binomial sum (involve Catalan numbers)

$$\sum_{k=1}^{\infty} \binom{2k}{k} \frac{x^k}{k^a}$$

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Algorithms

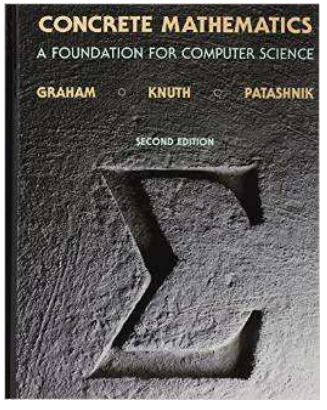
- Ideas for general recursive algorithms over basic hypergeometric sums
  Weinzierl '04

$$\frac{1}{\Gamma\left(1-\frac{p}{q}\right)} \sum_{k=1}^{\infty} \frac{\Gamma\left(k+1-\frac{p}{q}\right)}{\Gamma(k+1)} x^k = -1 + (1-x)^{-(1-p/q)}$$

- Yet, no closed algorithmic recursion derived

Literature (I)

- Text books
  - *Modern Computer Algebra*
    J. von zur Gathen, J. Gerhard
  - *Concrete Mathematics*
    R. L Graham, D. E. Knuth, O. Pataschnik
  - *Concrete Tetrahydron*
    M. Kauers, P. Paule
  - *A=B*
    M. Petkovsek, H. S. Wilf, D. Zeilberger `www.math.upenn.edu/~wilf/AeqB.html`



Literature (II)

- Selected research articles
  - *Harmonic sums, Mellin transforms and integrals*, J. Vermaseren; hep-ph/9806280
  - *Nested sums, expansion of transcendental functions and multi-scale multi-loop integrals*, S.M., P. Uwer, S. Weinzierl; hep-ph/0110083
  - *Gauss hypergeometric function: Reduction, epsilon-expansion for integer/half-integer parameters and Feynman diagrams*, M. Yu. Kalmykov; hep-th/0602028
  - *HypExp 2, Expanding Hypergeometric Functions about Half-Integer Parameters*, T. Huber, D. Maitre; 0708.2443
  - *On the analytic computation of massless propagators in dimensional regularization*, E. Panzer; 1305.2161
  - ...

Software (I)

Requirements in particle physics

- Symbolic calculations characterized by need for basic operations
  - sorting, gcd, factorization, multiplication
  - symbolic integration/summation
  - solution of systems of equations
  - ...
- Specialized code usually written by the user
  - largely dependent on the physics problem
  - add-on libraries
- No need for a system which knows more than the user

Software (II)

- Commercial programs: *Mathematica*, *Maple*, . . .
- Freeware/Add-on packages
  - *Mathematica*, *Maple*
    - several packages for hypergeometric summation
      [see for instance `www.math.upenn.edu/~wilf/AeqB.html`]
    - RISC software for symbolic summation and integration
      `www.risc.jku.at/research/combinat/software`
    - expansion of hypergeometric functions *HypExp*, T. Huber, D. Maitre
    - reduction of hypergeometric functions *HyperDire*, V. Bytev
    - hyperlogarithmic integration *HyperInt*, E. Panzer
  - GiNaC `www.ginac.de`
    - *nestedsums*, S. Weinzierl
  - Form `www.nikhef.nl/~form`
    - *Summer6*, J. Vermaseren
    - *XSummer*, S.M., P. Uwer

Exercises 1

Hypergeometric functions

- Use the FORM package XSUMMERfor the expansion of the hypergeometric function. Download the package from
  `http://www-zeuthen.desy.de/~moch/xsummer/`.

$${}_2F_1(1, -\epsilon; 1-\epsilon; x_1) \qquad \text{and} \qquad {}_3F_2(-2\epsilon, -2\epsilon, 1-\epsilon, 1-2\epsilon, 1-2\epsilon; x_1)$$

  - Check `ex-hyper.frm` for an implementation

- Alternatively, you can use the *Mathematica* package *HypExp* for this exercise

Exercises 2

Bonus material (I)

- Program the expansion of the Gamma-function around any integer value in FORM. To do so, use the result for expansions around positive integers ($n \geq 0$)

$$\frac{\Gamma(n+1+\epsilon)}{\Gamma(1+\epsilon)} = \Gamma(n+1)\,\exp\left(-\sum_{k=1}^{\infty}\epsilon^k\,\frac{(-1)^k)}{k}\,S_k(n)\right)$$

  For expansions around negative integers ($n \leq 0$) use the well-known relation

$$\frac{\Gamma(-n+1+\epsilon)}{\Gamma(1+\epsilon)} = (-1)^n\,\frac{\Gamma(-\epsilon)}{\Gamma(n-\epsilon)}$$

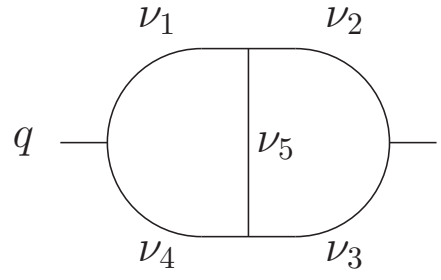
- Check your results with *Mathematica* or *Maple* or `ExpandGam.prc` in the XSUMMER package under FORM.

Exercises 3

Bonus material (II)

- Use the MAPLE package `HyperInt` to compute the massless two-loop two-point function for the parameter choices $\nu_i = 1$ and $\nu_i = 1 + \epsilon$

$$\mathrm{T1} = \int d^D p_1 \, d^D p_2 \, \frac{1}{(p_1^2)^{\nu_1} \, (p_2^2)^{\nu_2} \, (p_3^2)^{\nu_3} \, (p_4^2)^{\nu_4} \, (p_5^2)^{\nu_5}}$$



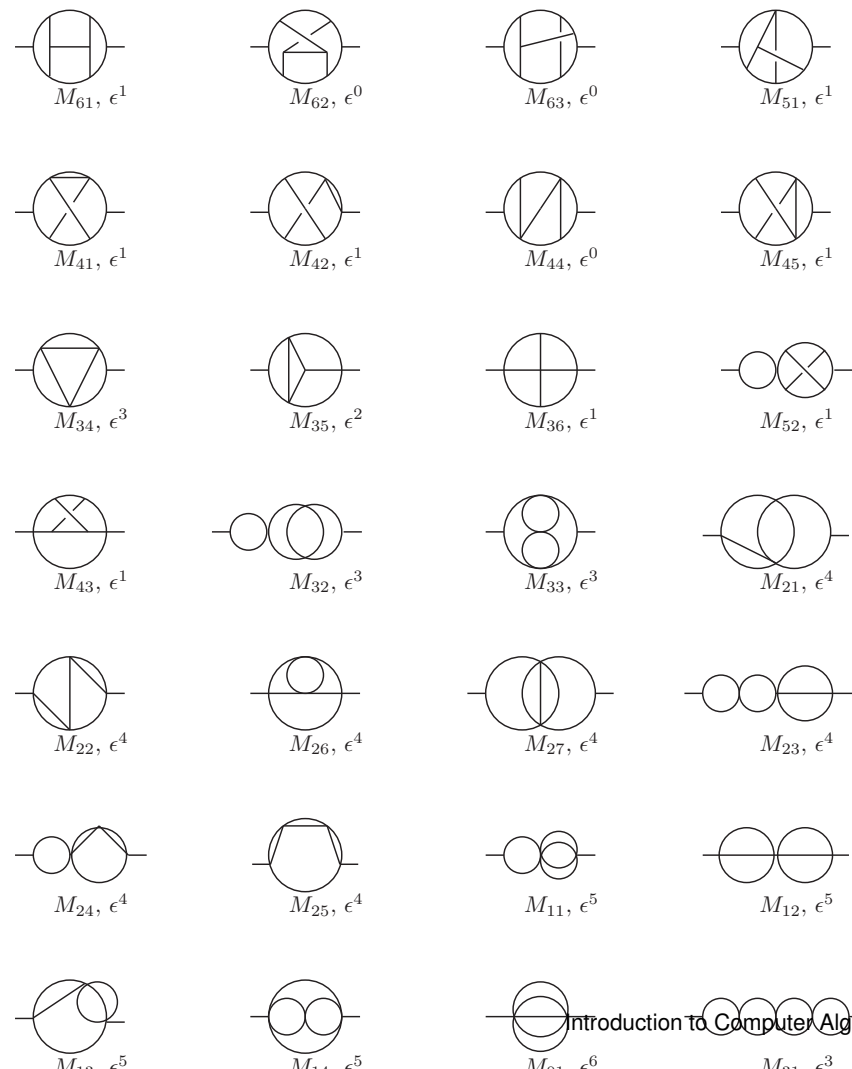
- Check `T1topo.mw` for an implementation

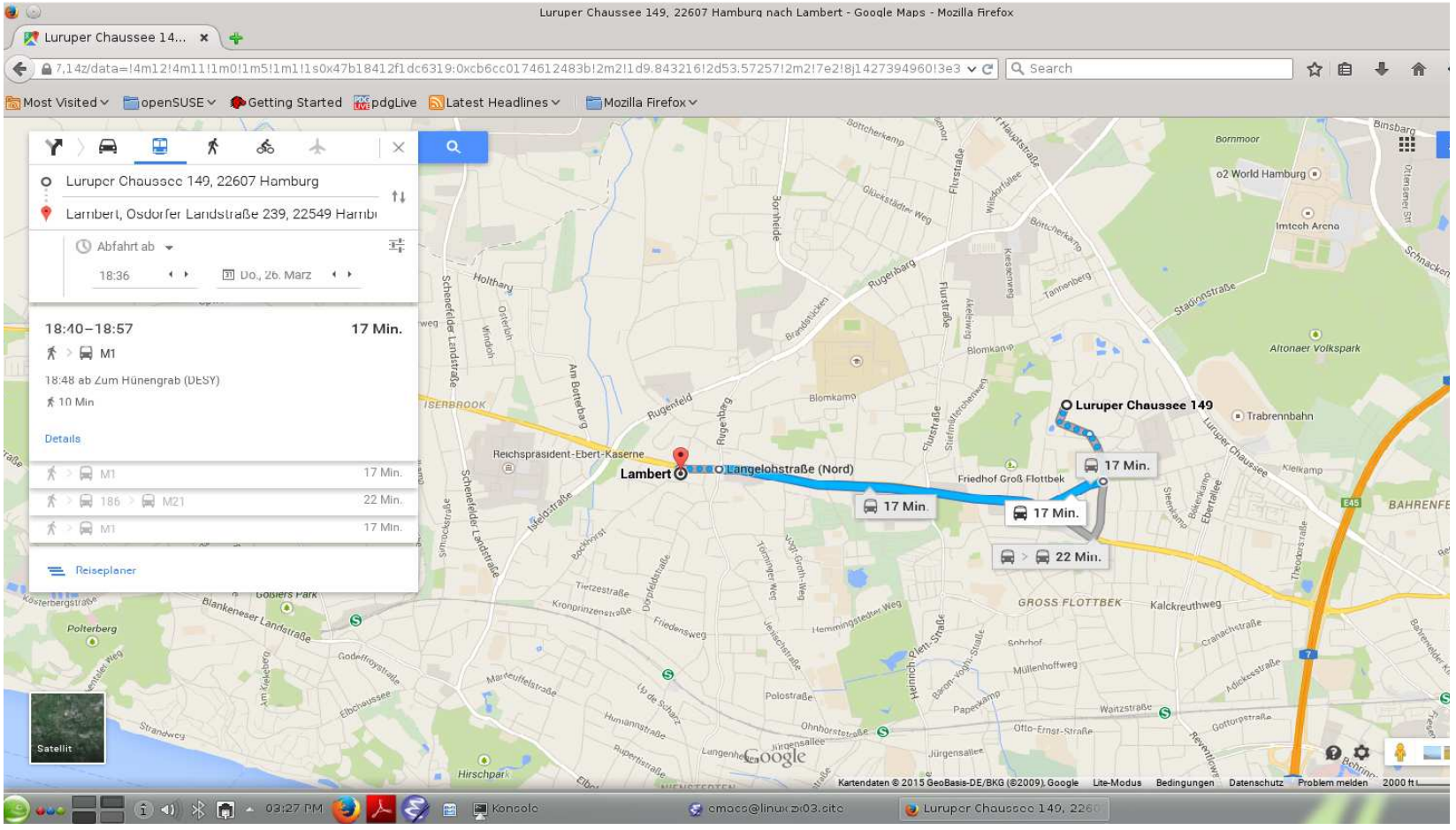
Exercises 4

Difficult bonus material (III)

- Use the MAPLE package `HyperInt` to compute the massless four-loop two-point function M_{44}

$$\mathrm{M}_{44} = \frac{441}{8}\zeta_7 + \mathcal{O}(\epsilon)$$



Social dinner

- Thursday evening 19:00 hrs at Restaurant Lambert
  http://www.lambert-hamburg.de