η and η' Mixing and the Witten-Veneziano Formula

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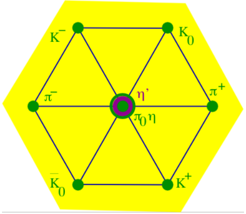
The $U_A(1)$ Problem

- nine lightest pseudo-scalar mesons show a peculiar spectrum:
  - 3 very light pions (140 MeV)
  - kaons and the η around 500 MeV
  - η' around 1 GeV



- this was called the $U_A(1)$-problem
- $M_\pi \ll M_\eta \ll M_{\eta'}$
- in contrast: $\rho - \omega - \phi$ appear ideally mixed as expected from OZI rule
- spontaneously broken chiral symmetry: nine Goldstone bosons?

[Glashow (1968)]

The $U_A(1)$ Problem

- the $U(1)$ axial current is anomalous at quantum level

  [Adler (1969), Jackiw and Bell (1969)]

$$\partial^\mu A^{0q}_\mu = \partial^\mu(\bar{q}\gamma_\mu\gamma_5 q) = 2m_q(\bar{q}i\gamma_5 q) + \frac{\alpha_s}{4\pi}F\tilde{F}$$

  and not spontaneously broken

- anomaly vanishes to all orders in perturbation theory at zero momentum

$\Rightarrow$ need additional mechanism

- instantons with non-trivial topology allow to explain η' mass

  [Kogut, Susskind (1975), Belavin et al. (1975), 't Hooft (1976)]

$\Rightarrow$ suggests consistency of QCD with nature

The Witten-Veneziano Formula

- π, η, η' masses can be computed using lattice QCD
- ... but how to establish the relation to the $U_A(1)$ anomaly?
- in the 't Hooft limit the Witten-Veneziano formula can be derived ($m_q = 0, N_c \to \infty$, $g^2 N_c$ and N_f fixed)

$$M_{\eta'}^2 = \frac{4N_f}{f_\pi^2}\chi_\infty$$

[Witten (1979), Veneziano (1979)]

- where χ_∞ is the topological susceptibility in pure Yang-Mills theory
- including $m_s > 0$ effects

$$M_\eta^2 + M_{\eta'}^2 - 2M_K^2 = \frac{4N_f}{f_\pi^2}\chi_\infty$$

- for exact SU(3) flavour symmetry ($m_u = m_d = m_s$) define (considering only $\bar{q}q$ states ...)

- the flavour octet state η_8

$$\eta_8 \quad \equiv \quad \frac{1}{\sqrt{6}}(\bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d - 2\bar{s}i\gamma_5 s)$$

$\Rightarrow$ a (pseudo) Goldstone boson

- the flavour singlet state η_0

$$\eta_0 \quad \equiv \quad \frac{1}{\sqrt{3}}(\bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d + \bar{s}i\gamma_5 s)$$

$\Rightarrow$ related to $U_A(1)$ anomaly

- SU(3) flavour symmetry broken by larger m_s: $m_s \gg m_u = m_d = m_\ell$

$\Rightarrow$ physical states will be a mixture, e.g.

$$\begin{pmatrix} |\eta\rangle \\ |\eta'\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \cdot \begin{pmatrix} |\eta_\ell\rangle \\ |\eta_s\rangle \end{pmatrix}$$

in the quark flavour basis

$$\eta_\ell \equiv \frac{1}{\sqrt{2}}(\bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d)\,, \qquad \eta_s = \bar{s}i\gamma_5 s$$

- in nature $m_u \neq m_d \qquad \Rightarrow \quad$ also π_0 mixes
- how did nature arrange the mixing pattern?
- can one determine the mixing angle(s)?

- mixing angle definition via Fock states theoretically difficult

$\Rightarrow$ use matrix elements instead

$$\begin{pmatrix} \mathbf{P}_\eta^\ell & \mathbf{P}_\eta^s \\ \mathbf{P}_{\eta'}^\ell & \mathbf{P}_{\eta'}^s \end{pmatrix} = \begin{pmatrix} c_\ell \cos\phi_\ell & -c_s \sin\phi_s \\ c_\ell \sin\phi_\ell & c_s \cos\phi_s \end{pmatrix}$$

defined via pseudo-scalar matrix elements

$$\mathbf{P}_P^q = \langle 0|\bar{q} i\gamma_5 q|P\rangle\,, \qquad P = \eta, \eta'\,, \qquad q = \ell, s$$

- note: usually defined via decay constants obtained from

$$\langle 0|A_\mu^q|P(p) = i f_P^q p_\mu$$

with

$$A_\mu^\ell = \frac{1}{\sqrt{2}}(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d)\,, \qquad A_\mu^s = \bar{s}\gamma_\mu\gamma_5 s$$

- angles are then obtained from

$$\frac{\mathbf{P}^{\ell}_{\eta'}}{\mathbf{P}^{\ell}_{\eta}} = \tan\phi_\ell\,, \qquad \frac{\mathbf{P}^{s}_{\eta}}{\mathbf{P}^{s}_{\eta'}} = -\tan\phi_s$$

- from χPT and $1/N_c$ arguments one expects

$$|\phi_\ell - \phi_s| \ll 1$$

- if correct, define a single angle ϕ obtained from

$$\tan^2\phi \;=\; -\frac{\mathbf{P}^{\ell}_{\eta'}\mathbf{P}^{s}_{\eta}}{\mathbf{P}^{\ell}_{\eta}\mathbf{P}^{s}_{\eta'}}$$

- note: all renormalisation constants cancel in the ratios

Motivation

fundamental questions:

- does QCD reproduce the experimental mass pattern?
- how does nature arrange for the mixing?
- can we relate the η' mass to the $U_A(1)$ anomaly

Challenges

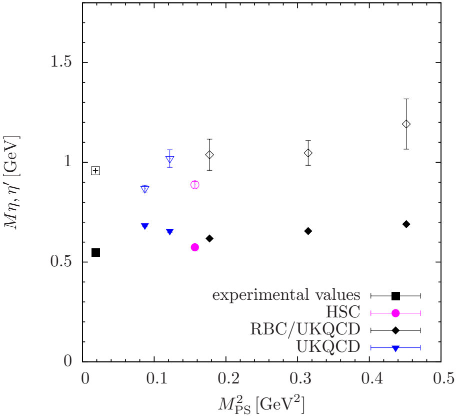
- η, η' involve OZI-rule violating diagrams

$\Rightarrow$ so-called fermionic disconnected diagrams

- due to noise computationally challenging

$\Rightarrow$ only few lattice results available

- lattice definition of χ_t
  - must be cleanly defined
  - must be properly renormalised



filled symbols: η open: η'

[HSC, J. J. Dudek et al., Phys. Rev. D83 (2011)]
[RBC/UKQCD, N. Christ et al., Phys. Rev. Lett. 105 (2010)]
[UKQCD, E. B. Gregory et al., Phys.Rev. D86 (2012)]

- let $O(x,t)$ be an operator with quantum numbers of a given state
- for instance for the pion it is given by

$$O(x,t) \;=\; \bar{u}i\gamma_5 d(x,t)\,, \qquad O(t) \;=\; \sum_x O(x,t)$$

  projected to zero momentum
- create pion at time 0 and annihilate at t

$$C_\pi(t) \;=\; \langle O(0)\; O^\dagger(t)\rangle \;\propto\; \sum_n \langle 0|O(0)|n\rangle\langle n|e^{-Ht}O(0)e^{Ht}|0\rangle$$

  inserting a complete set of states $|n\rangle$ and using the time evolution operator $\exp -Ht$

- which leads to

$$C_\pi(t) \ \propto\ \sum_n |\langle 0|\ O\ |n\rangle|^2\ e^{-(E_n - E_0)t}$$

- and for large times the lowest state dominates

$$\lim_{t\to\infty} C_\pi(t) \ \propto\ e^{-(E_1 - E_0)t}$$

- $E_1 - E_0$ corresponds at zero momentum to the mass M_π

- and matrix elements of the pion for $n = 1$

$$\langle 0|\ O\ |\pi\rangle$$

- also define effective masses

$$M = -\frac{d}{dt}\log C_\pi(t)$$

- $2 + 1 + 1$ quark flavour ensembles from ETM Collaboration
  $m_u = m_d < m_s < m_c$
  [ETMC, R. Baron et. al., JHEP 06 111 (2010)]

- three lattice spacings (A, B and D ensembles):
  $a_A = 0.086\ \mathrm{fm}$, $a_B = 0.078\ \mathrm{fm}$ and $a_D = 0.061\ \mathrm{fm}$

- charged pion masses range from $\approx 230\ \mathrm{MeV}$ to $\approx 500\ \mathrm{MeV}$

- $L \geq 3\ \mathrm{fm}$ and $M_\pi \cdot L \geq 3.5$ for most ensembles

- bare m_s and m_c fixed for each lattice spacing

- use $r_0 = 0.45(2)\ \mathrm{fm}$ (from f_π) for η, η'
  r_0 Sommer scale determined from static $\bar{q}q$ potential

$\Rightarrow$ $r_0 M$ is dimensionless (with M a mass)

Flavour Singlet Pseudo-Scalar Mesons

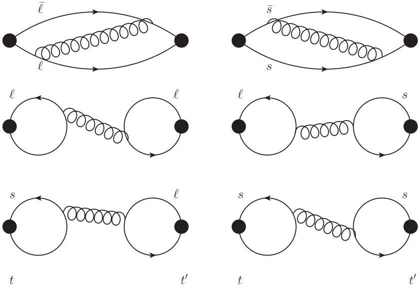
- need to estimate correlator matrix

$$\mathcal{C} = \begin{pmatrix} \eta_{\ell\ell} & \eta_{\ell s} \\ \eta_{s\ell} & \eta_{ss} \end{pmatrix}$$

- η_{XY} correlator of appropriate operators, e.g.

$$\eta_{ss}(t) \quad \equiv \quad \langle \bar{s} i\gamma_5 s(t)\ \bar{s} i\gamma_5 s(0) \rangle$$

- connected and disconnected contributions



- diagonalise matrix $\Rightarrow$ masses M_P and matrix elements $\mathbf{P}_P^q$

- $P = \eta$: lowest state, $P = \eta'$: second state, $P = \eta_c$ ...

η and η' Masses Overview

- η' mainly the flavour singlet
- disconnected contributions significant

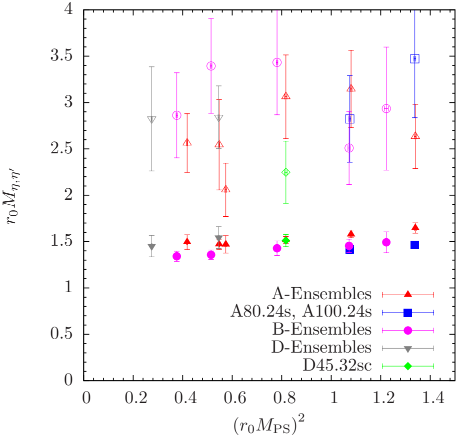
$\Rightarrow$ very noisy

- chiral extrapolation uncertain

$\Rightarrow$ no clear picture

- need for improvement

filled symbols: η open: η'

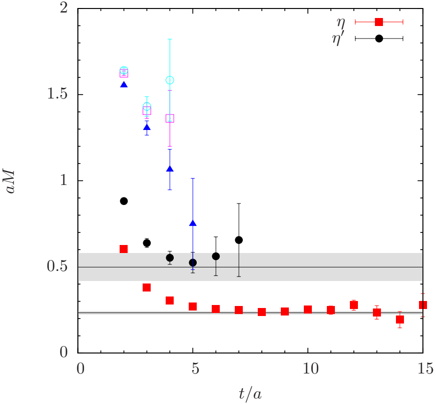


[C. Michael, K. Ottnad, S. Reker, C.U., JHEP 1211 (2012) 048]

Effective Mass Plots

- effective masses for $\eta, \eta', \eta_c, \ldots$
- η mass well determined
- η':
  - at small t/a excited states dominant
  - disconnected contributions noisy

    $\Rightarrow$ signal lost in noise before plateau reached



- large systematics in η'

$\Rightarrow$ need to reduce noise/increase statistics tremendously

or remove excited states to fit at smaller t/a

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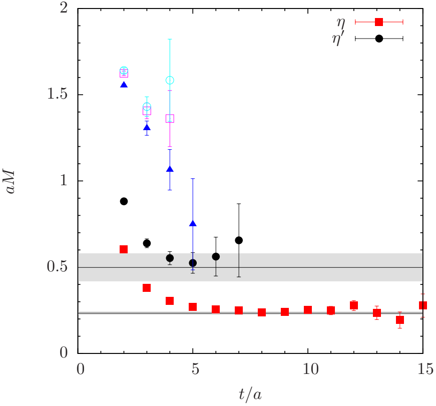


$\Rightarrow$ need to reduce noise/increase statistics tremendously

or remove excited states to fit at smaller t/a

Improved η' Extraction

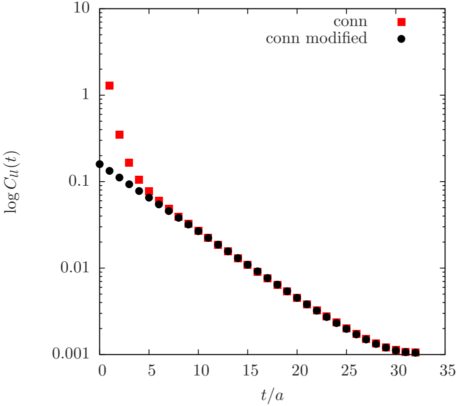
- recall: disconnected contributions noisy

- lets make an assumption: disconnected contributions couple only to η and η' states, not to higher states

  [Neff et al. (2001), K.Jansen, C.Michael, C.U. (2008)]

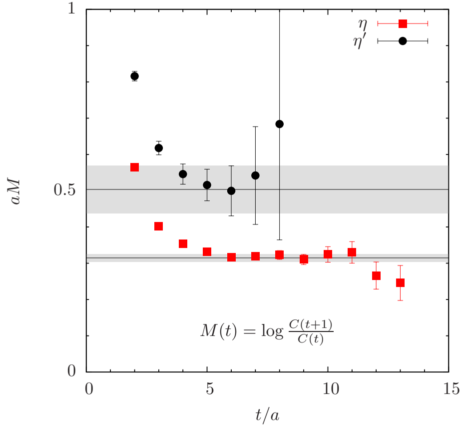
- replace connected contributions by only the ground states

- if assumption justified: there should be a plateau in the effective masses from very low times on!



w/o removal

- we see a plateau from $t/a = 2$ on
- for both η and η'
- η: good agreement with previous results
- η': possibly much better determination
- assumption justified?
- systematic uncertainties?



Excited State Removal

- we see a plateau from $t/a = 2$ on
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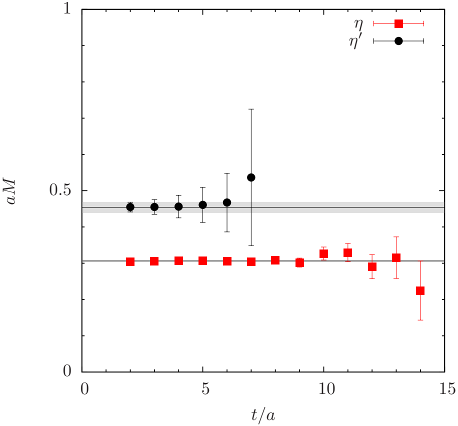
- systematic uncertainties?

w/ removal



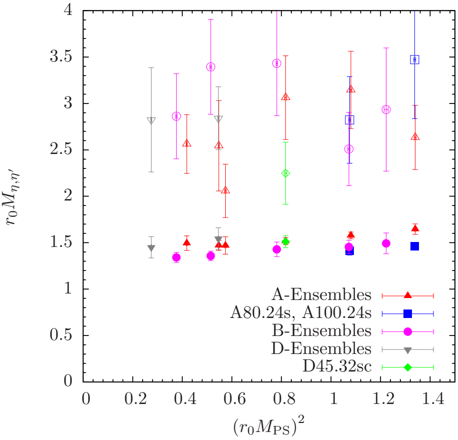
η:

- masses agree well
- improved precision

η':

- masses determined much better
- always agreement within 2σ
- systematics hard to quantify

w/o removal



Masses w/ and w/o Excited State Removal

η:

- masses agree well
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η':

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- always agreement within 2σ
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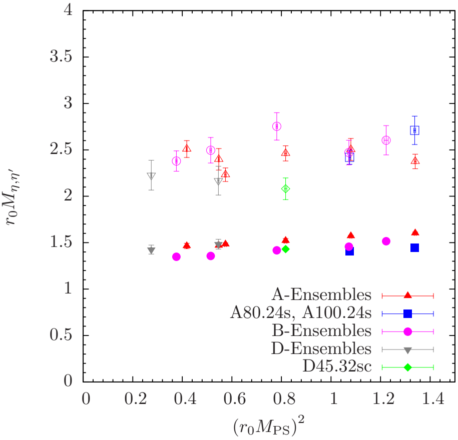
w/ removal



- can take the difference w/ and w/o to estimate systematics

Strange Quark Mass Dependence

- m_s not perfectly tuned to its physical value
- two re-tuned ensembles for a_A

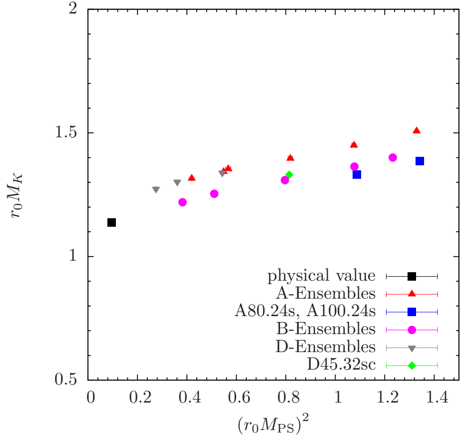
$\Rightarrow$ can estimate m_s dependence

- estimate

$$D_\eta \equiv \frac{d(aM_\eta)^2}{d(aM_\mathrm{K})^2} = 1.47(11)$$

- now assume: D_η independent of a, m_ℓ, m_s, m_c

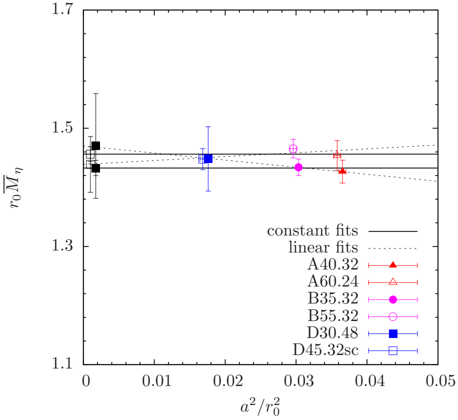
- ...correct η masses



- use two ensembles sets
  ($A60$, $B55$, $D45$)
  ($A40$, $B35$, $D30$)
  with $r_0 M_{\mathrm{PS}} \approx \mathrm{const}$
- correct M_η using D_η linearly in M_{K}^2

$\Rightarrow$ $r_0 M_{\mathrm{K}} = 1.34$ fixed

- compatible with both, constant and linear continuum extrapolation



$\Rightarrow$ lattice artifacts seem under control

[PRL **111** 181602 (2013)]

- now correct all $(r_0 M_\eta)^2$ using D_η

$\Rightarrow$ $(r_0 \bar{M}_\eta)^2 \propto const + (r_0 M_{\mathrm{PS}})^2$

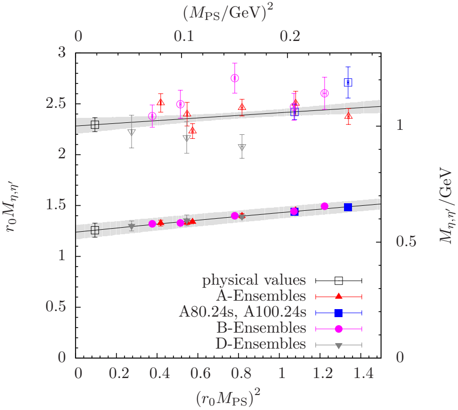
- all a-values fall on the same curve!

- extrapolate $(r_0 \bar{M}_\eta)^2$ linearly in $(r_0 M_{\mathrm{PS}})^2$ to $M_{\mathrm{PS}} = M_\pi$

$\Rightarrow$ result

$$M_\eta = 552(10)_{\mathrm{stat}}\ \mathrm{MeV}$$

- similarly with $(\bar{M}_\eta / \bar{M}_{\mathrm{K}})^2$ or GMO relation



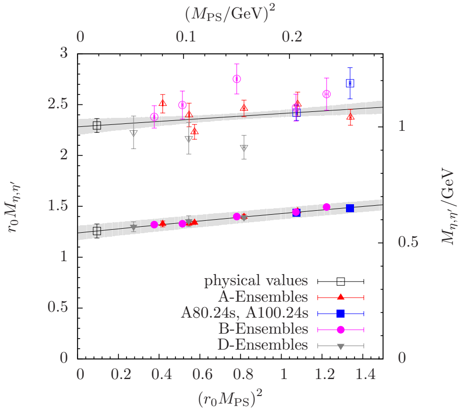
[PRL **111** 181602 (2013)]

- no clear dependence on
  - lattice spacing
  - strange quark mass
- errors still significant
- include all data in extrapolation
- $(r_0 M_{\eta'})^2 \propto const + (r_0 M_{\rm PS})^2$

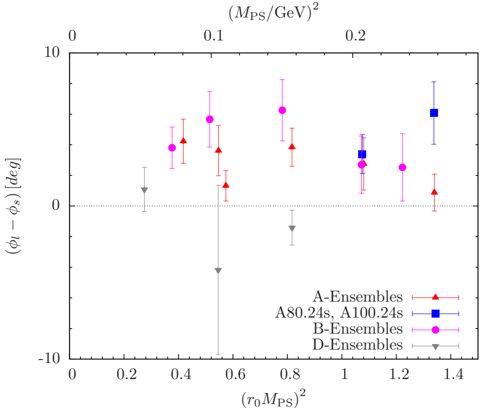
$\Rightarrow$ result

$$M_{\eta'} \;=\; 1006(54)_{\rm stat}\ \mathrm{MeV}$$

- fitting *A*, *B* and *D* separately gives compatible results



- $\Delta\phi = 3(1)_{\rm stat}(3)_{\rm sys}\,^\circ$ confirms expectation, smallness of OZI corrections
- data well described by a single angle
- but cannot exclude a finite difference



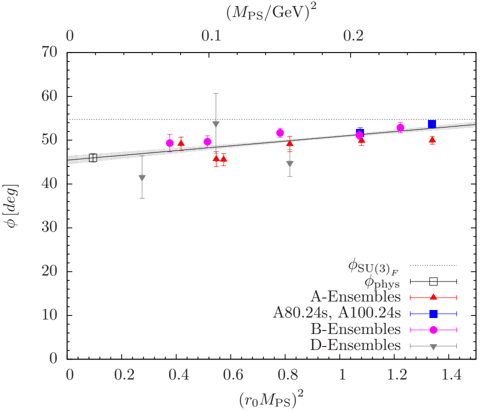
- systematic uncertainty from fits to data from different lattice spacings separately

[PRL **111** 181602 (2013)]

- extrapolate ϕ linearly in $(r_0 M_{\mathrm{PS}})^2$

$\Rightarrow$ $\phi = 46(1)_{\mathrm{stat}}(3)_{\mathrm{sys}}\,^\circ$

- systematic error from fitting different lattice spacings separately
- with $m_\ell = m_s$ we reproduce the SU(3) symmetric value 54.7°



- interpretation: η' meson mainly the flavour singlet state

$\Rightarrow$ far from ideally mixed

K. Cichy, E. Garcia-Ramos, K. Jansen

- spectral projectors R_M allow for a clean definition of χ_t

[Giusti, Lüscher, (2009)]

$$\chi_t = \frac{Z_S^2}{Z_P^2} \frac{\langle \mathrm{Tr}[\gamma_5 R_M^2]\, \mathrm{Tr}[\gamma_5 R_M^2] \rangle}{V}$$

- spectral projectors estimated stochastically
- dedicated quenched ensembles with Iwasaki gauge action

see Cichy, Garcia-Ramos, Jansen (2013) for $N_f = 2$ and $2+1+1$ results

- four values of the lattice spacing from 0.07 fm to 0.14 fm
- Wilson twisted mass valence quarks

$\Rightarrow$ automatic $\mathcal{O}(a)$ improvement

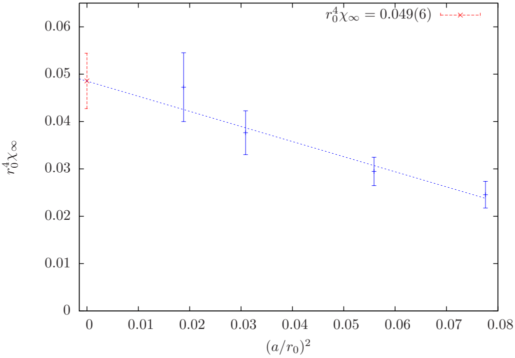
K. Cichy, E. Garcia-Ramos, K. Jansen

- box length fixed to

  $$L \sim 2.8\ \mathrm{fm}$$

- Z_S/Z_P computed with spectral projectors
- linear scaling in a^2 as expected

$\Rightarrow$ continuum limit: $r_0^4\chi_\infty = 0.049(6)$



- using $r_0 = 0.5\ \mathrm{fm}$ $\quad\Rightarrow\quad$ $\chi_\infty = (185(6)\ \mathrm{MeV})^4$

$\Rightarrow$ WV formula well fulfilled with physical mass values

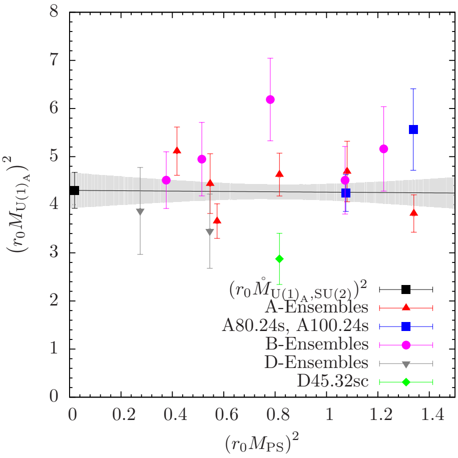
K. Cichy, E. Garcia-Ramos, K. Jansen, K. Ottnad, F. Zimmermann, C.U.

- define

$$M^2_{U_A(1)} = M^2_\eta + M^2_{\eta'} - 2M^2_K$$

- extrapolates almost constant in M^2_{PS}
- systematics very similar to $M_{\eta'}$
- dominant statistical noise
- the extrapolated value agrees with separat mass extrapolations

$\Rightarrow$ to be investigated further



- study $U_A(1)$ problem in LQCD
- η, η' mixing angle

$$\phi = 46(1)_{\mathrm{stat}}(3)_{\mathrm{sys}}\ ^\circ$$

- η, η' masses

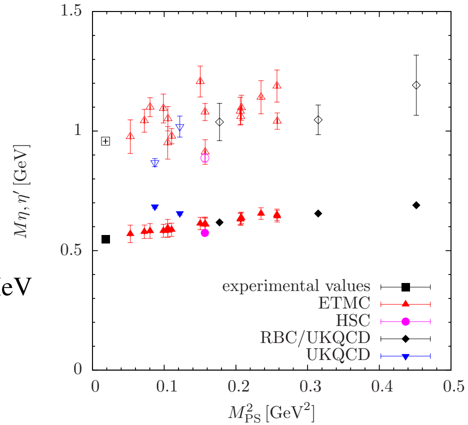
$$M_\eta = 551(8)_{\mathrm{stat}}(6)_{\mathrm{sys}}\ \mathrm{MeV}$$
$$M_{\eta'} = 1006(54)_{\mathrm{stat}}(38)_{\mathrm{sys}}(+61)_{\mathrm{ex}}\ \mathrm{MeV}$$

- Yang-Mills χ_∞

$$\chi_\infty = (185(6)\ \mathrm{MeV})^4$$

from spectral projectors

- test of Witten-Veneziano formula



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