MASSIVE TADPOLES:

Techniques & Applications

I. RHO-PARAMETER

II. QUARK MASSES

III. MISCELLANEOUS

J. Kühn / Project A1

Karlsruhe Institute of Technology

Computational Theoretical Particle Physics
SFB TR9

I. RHO-PARAMETER

1. Definition, work before 2003

2. Three-loop electroweak results with full m_H-dependence
   (Faisst, JK, Seidensticker, Veretin)

3. Four-loop QCD contributions, $\mathcal{O}\left(G_F \alpha_s^3\right)$
   (Chetyrkin, Faisst, JK, Maierhöfer, Meier, Sturm)

I.1 Definition, work before 2003

central prediction of SM:

$$M_W = f(\underbrace{G_F,\ M_Z,\ \alpha}_{\text{Born}};\quad \underbrace{M_t,\ M_H,\ \ldots}_{\text{radiative corrections}})$$

[similarly for couplings of fermions: $\sin^2\theta_{\text{eff}}$]

$$M_W^2\left(1-\frac{M_W^2}{M_Z^2}\right)=\frac{\pi\alpha}{\sqrt{2}G_F}(1+\Delta r)$$

Δr dominated by: $\Delta r = -\frac{c^2}{s^2}\Delta\rho + \Delta\alpha$

$\Delta\rho \sim \left(G_F\, M_t^2\right) + \ldots$ (Veltman)

aim: compete with experiment

δM_W [MeV]	δM_t [GeV]	
33	5	status 2003 (LEP, TEVATRON)
15	0.76	now (TEVATRON, LHC)
$8 \to 5$	0.6	aim (LHC), theory limited
3, < 1.2	0.1 - 0.2	ILC, TLEP

theory: correlation (for $\alpha(M_Z), M_H$ fixed): $\delta M_W \approx 6 \cdot 10^{-3}\, \delta M_t$

$\Rightarrow$ shifts in $M_W \lesssim \left\{ \begin{array}{l} 5 \text{ MeV} \\ 1 \text{ MeV} \end{array} \right\}$ are relevant for $\left\{ \begin{array}{l} \text{LHC} \\ e^+e^- \text{ collider} \end{array} \right.$

Theory: Status 2002

status: two-loop $\left\{ \begin{array}{l} \text{Barbieri, Beccaria, Ciafaloni, Curci, Vicere} \\ \text{Fleischer, Jegerlehner, Tarasov} \end{array} \right.$

approximation: $M_t^2 \gg M_W^2$ $\Rightarrow$ scalar bosons only, gaugeless limit

$$\Delta\rho = X_t^2 \, f(M_t/M_H) \quad \text{with} \quad X_t \equiv \frac{G_F \, M_t^2}{8\sqrt{2}\pi^2} = \frac{g^2_{\text{Yukawa}}}{16\pi^2}$$

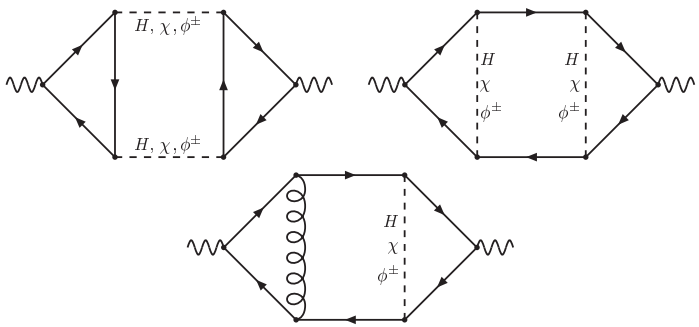
status: three-loop $M_H = 0$

(van der Bij, Chetyrkin, Faisst, Jikia, Seidensticker)

poor approximation, leads to tiny corrections for terms of order X_t^3 and $\alpha_s X_t^2$

$M_H \neq 0$ (Faisst, JK, Seidensticker, Veretin, 2003)

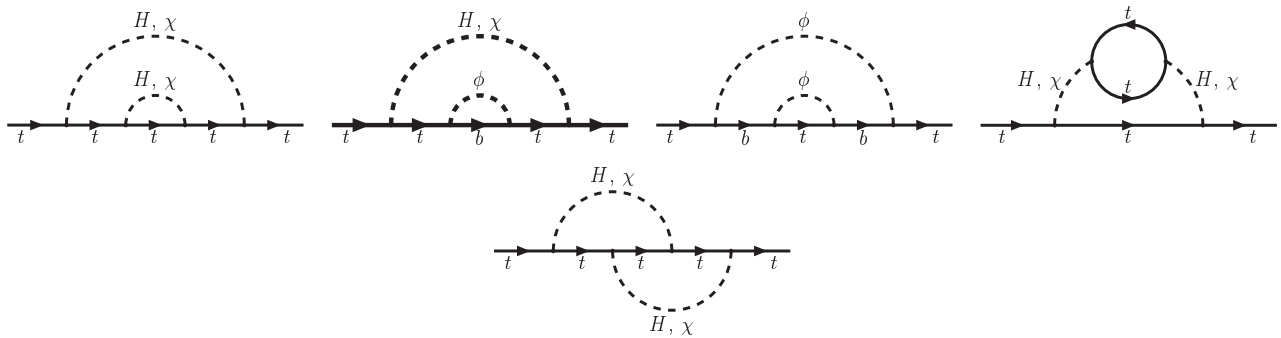
requires: 3-loop tadpoles $\left(\Pi_{WW}(0),\ \Pi_{ZZ}(0) \right)$



$X_t^3\, f(m_t/M_H)$

$\alpha_s X_t^2\, f(m_t/M_H)$

and two-loop on-shell diagrams: $M_t \iff m_t(\overline{\text{MS}})$



in general two (!) mass scales, three loops

special cases:

$M_H = 0$: $\surd$
$M_H \gg M_t$: hard mass expansion in $(M_t^2/M_H^2)^n$ mod. log
$M_H = M_t$: one scale

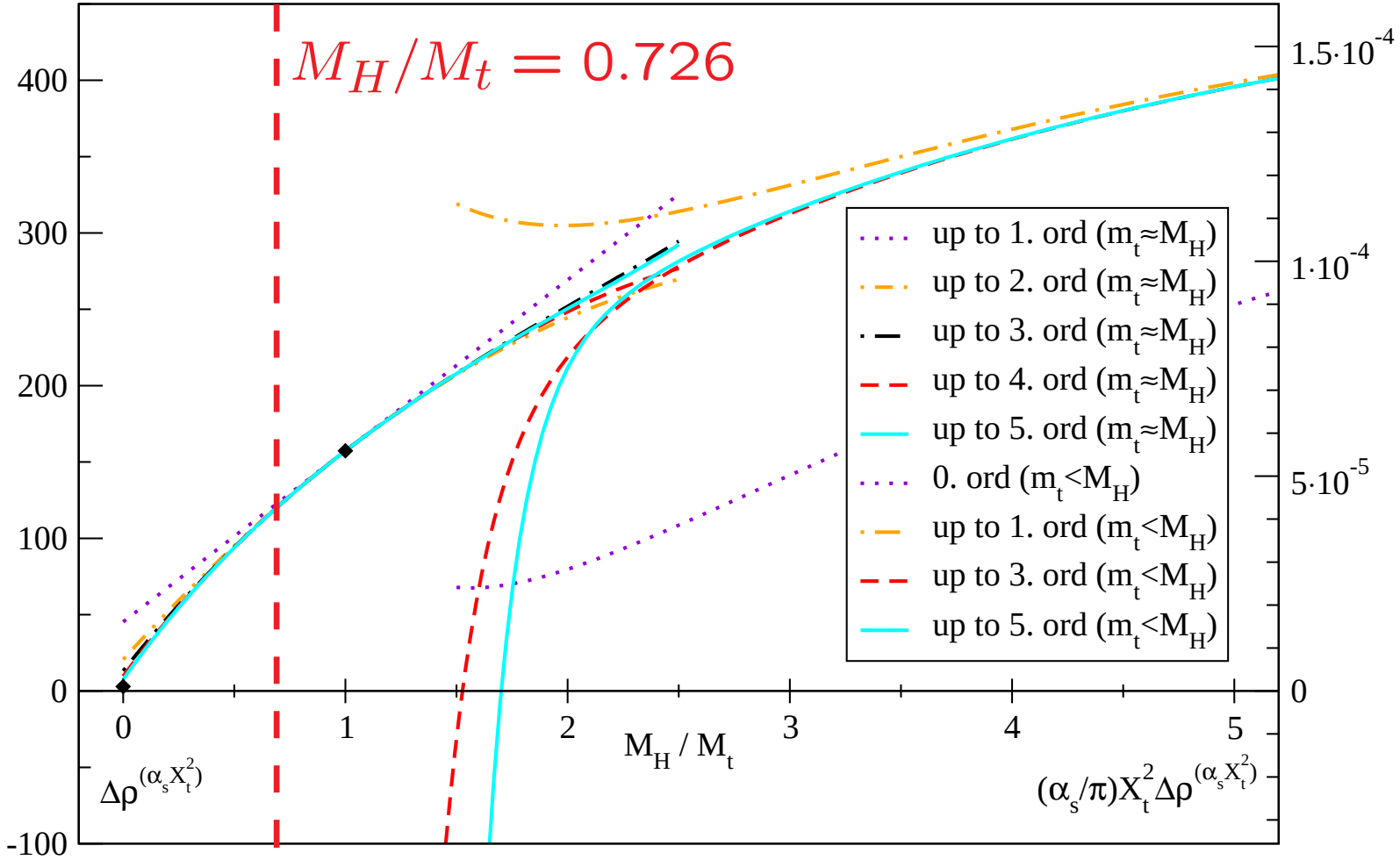
M_H in neighbourhood of M_t:

Taylor expansion: $\delta = (M_H - M_t)/M_t$

excellent approximation

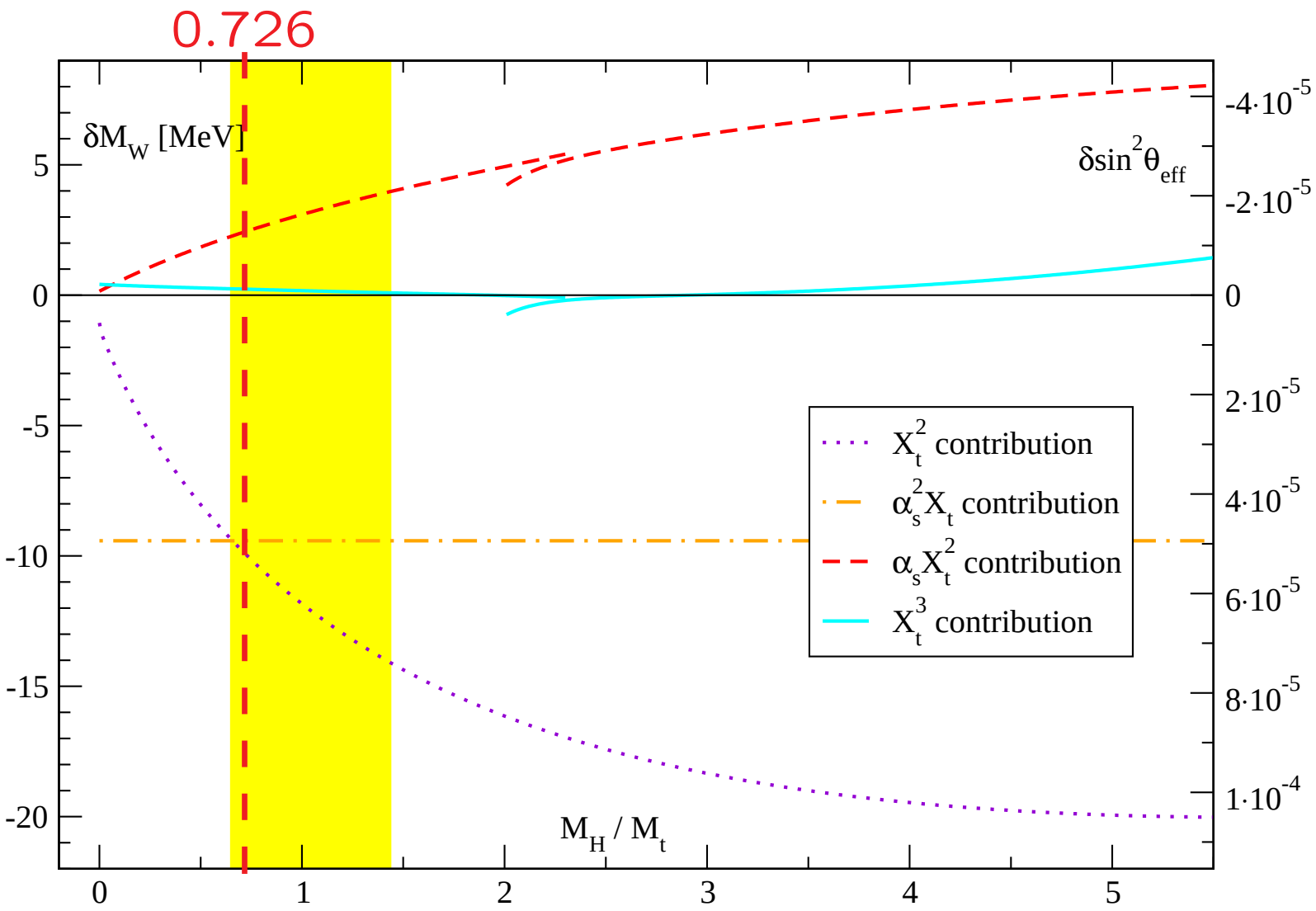
$\Rightarrow$ reduction to one-scale two- or three-loop integrals

on-shell result $\mathcal{O}(\alpha_s X_t^2)$



Contributions of order $\alpha_s X_t^2$ to $\Delta\rho$ in the on-shell definition of the top quark mass. The black squares indicate the points where the exact result is known.

$M_H = (0|126) \Rightarrow \Delta\rho(\alpha_s X_t^2) = (2.9|120)$

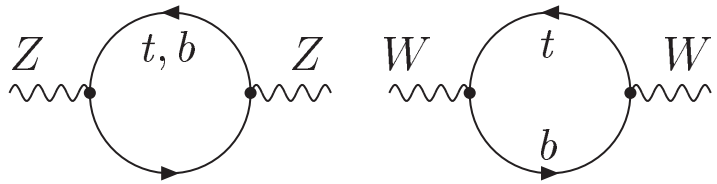


$\delta M_W \approx 2.3$ MeV

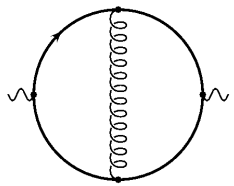
$\delta \sin^2\theta_{\text{eff}} \approx 1.5 \cdot 10^{-5}$

I.3 Four-loop QCD contributions

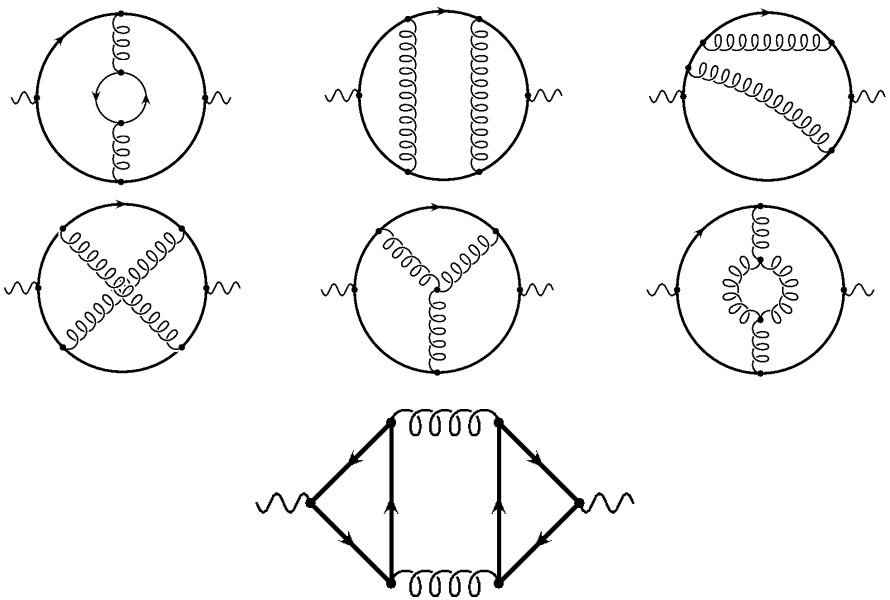
QCD: Status 2002



one-loop (1977)
(Veltman)



two-loop (1987)
(Djouadi; Kniehl, JK, Stuart)



tadpoles; zero scale
three-loop (1995)
(Chetyrkin, JK, Steinhauser)
(Avdeev,. . .)

δM_W in MeV	α_s^0	α_s^1	α_s^2
M_t^2	611.9	-61.3	-10.9
const.	136.6	-6.0	-2.6
$1/M_t^2$	-9.0	-1.0	-0.2

four loop?

Techniques:

3-loop $M_{\text{pole}} \Leftrightarrow \bar{m}$ relation (1999/2000)

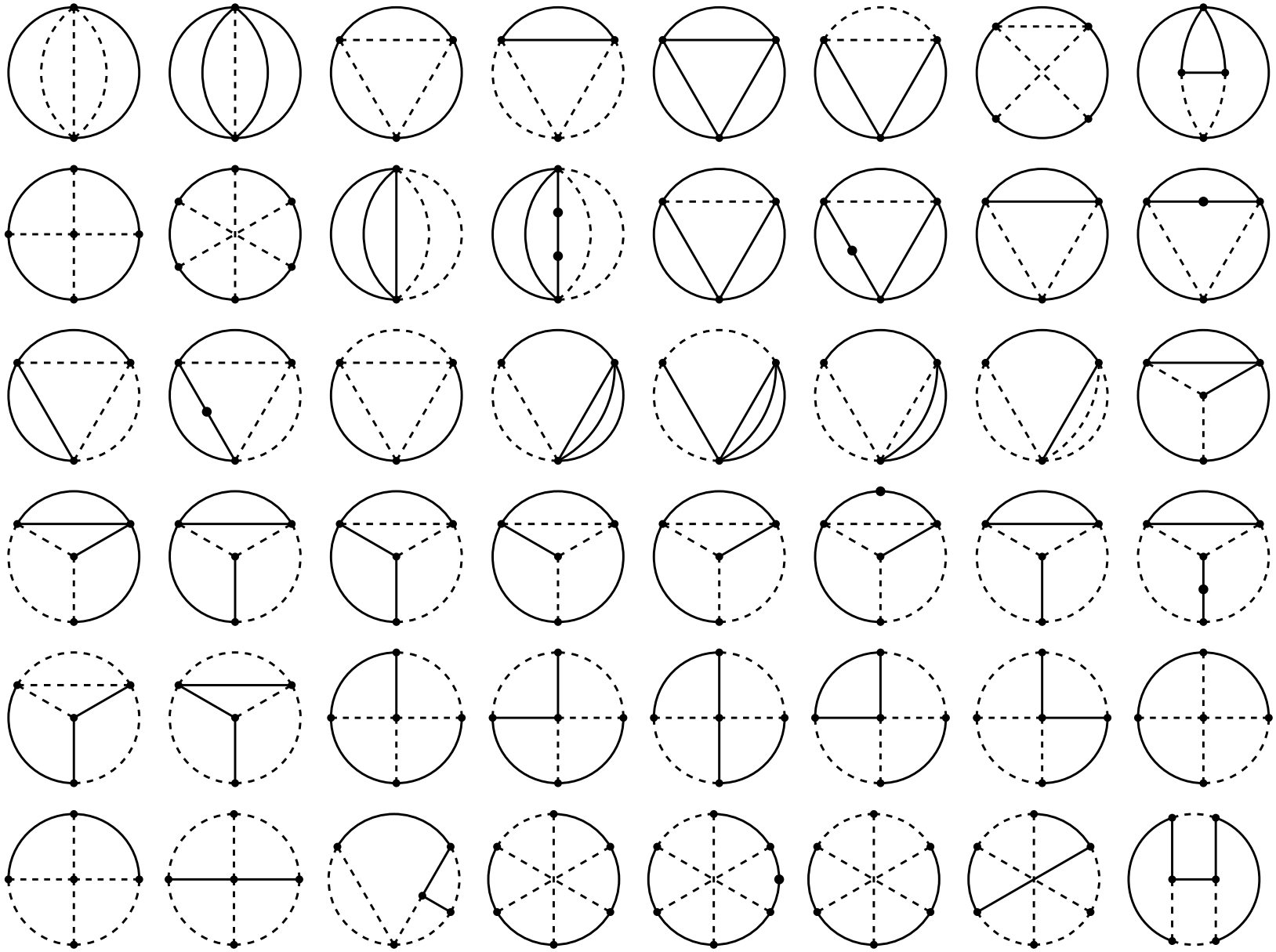
(Chetyrkin+Steinhauser, Melnikov+van Ritbergen)

and 4-loop tadpoles: Laporta algorithm

[previously: 3 loop tadpoles $\Rightarrow$ recursive algorithm
(Broadhurst, Steinhauser: MATAD)]

analytical and numerical evaluation of ~ 50 four-loop master integrals:

difference equations, semi-numerical integration



result (2006) (Chetyrkin, Faisst, JK, Maierhöfer, Sturm)

$$\delta\rho_t^{(4\text{ loop})} = 3\,\frac{G_F m_t^2}{8\sqrt{2}\pi^2}\alpha_s^3\,(\,-\,3.2866 \qquad +1.6067 \qquad = -1.6799)$$

$\uparrow$ singlet piece (Schröder+ Steinhauser, 2005)

$\uparrow$ this result

result immediately independently confirmed (Boughezal+Czakon)

conversion to pole mass:

$$\delta\rho_t^{(4\text{ loop})} = 3\,\frac{G_F M_t^2}{8\sqrt{2}\pi^2}\alpha_s^3\,(-93.1501) \quad !$$

corresponds to a shift $\delta M_W \sim 2$ MeV (similar to $\mathcal{O}(x_t^2\alpha_s)$)

II. QUARK MASSES

from relativistic 4 loop moments

1. Why

2. Theory

3. Results, from experiment and from lattice

in collaboration with
K. Chetyrkin, Y. Kiyo, A. Maier, P. Maierhöfer, P. Marquard, A. Smirnov, M. Steinhauser, C. Sturm and the HPQCD Collaboration

II.1 WHY precise masses?

B-decays:

$\Gamma(B \to X_{\mathsf{u}} l \bar{\nu}) \sim G_{\mathsf{F}}^2 \, m_{\mathsf{b}}^5 \, |V_{\mathsf{ub}}|^2$

$\Gamma(B \to X_{\mathsf{c}} l \bar{\nu}) \sim G_{\mathsf{F}}^2 \, m_{\mathsf{b}}^5 \, f(m_{\mathsf{c}}^2/m_{\mathsf{b}}^2) \, |V_{\mathsf{cb}}|^2$

$B \to X_{\mathsf{s}} \gamma$

comparison with Υ-spectroscopy:

$M\,(\Upsilon(1\mathsf{s})) = 2M_{\mathsf{b}} - \left(\frac{4}{3}\alpha_{\mathsf{s}}\right)^2 \frac{M_{\mathsf{b}}}{4} + \ldots +$ excitations

(Penin & Zerf,... $\delta m_b \sim 9$ MeV)

H decay (ILC, TLEP)

$\boxed{H \to b\bar{b}}$ dominant decay mode, all branching ratios are affected!

status: $\Gamma_b = \frac{G_F M_H^2}{4\sqrt{2}\pi} m_b^2(M_H) R^S(M_H)$

$$\begin{aligned} R^S(M_H) &= 1 + 5.667\left(\frac{\alpha_s}{\pi}\right) + 29.147\left(\frac{\alpha_s}{\pi}\right)^2 + 41.758\left(\frac{\alpha_s}{\pi}\right)^3 - 825.7\left(\frac{\alpha_s}{\pi}\right)^4 \\ &= 1 + 0.19551 + 0.03469 + 0.00171 - 0.00117 \end{aligned}$$

(Chetyrkin, Baikov, JK, 2006)

Theory uncertainty ($M_H/3 < \mu < 3M_H$) : 5‰ (four loop)
reduced to 1.5‰ (five loop)

present uncertainties from m_b

$m_b(10\ \text{GeV}) = 3610 - \frac{\alpha_s - 0.1189}{0.002}\, 12 \pm 11$ MeV (Karlsruhe, arXiv:0907.2110)

running from 10 GeV to M_H depends on anomalous mass dimension, β-function and α_s

$m_b(M_H) = 2759 \pm 8|_{m_b} \pm 27|_{\alpha_s}$ MeV aim ± 4 MeV ($\widehat{=} 1.5 \times 10^{-3}$)

γ_4 (five loop): Baikov + Chetyrkin, 2013

β_4 under construction

$\frac{\delta m_b^2(M_H)}{m_b^2(M_H)} = -1.4 \times 10^{-4} (b_4 = 0) \mid -4.3 \times 10^{-4} (b_4 = 100) \mid -7.3 \times 10^{-4} (b_4 = 200)$

to be compared with $\delta\Gamma/\Gamma = 2.0 \times 10^{-3}$ (TLEP)

Yukawa Unification

$\lambda_\tau \sim \lambda_{\mathsf{b}}$ or $\lambda_\tau \sim \lambda_{\mathsf{b}} \sim \lambda_{\mathsf{t}}$ at GUT scale

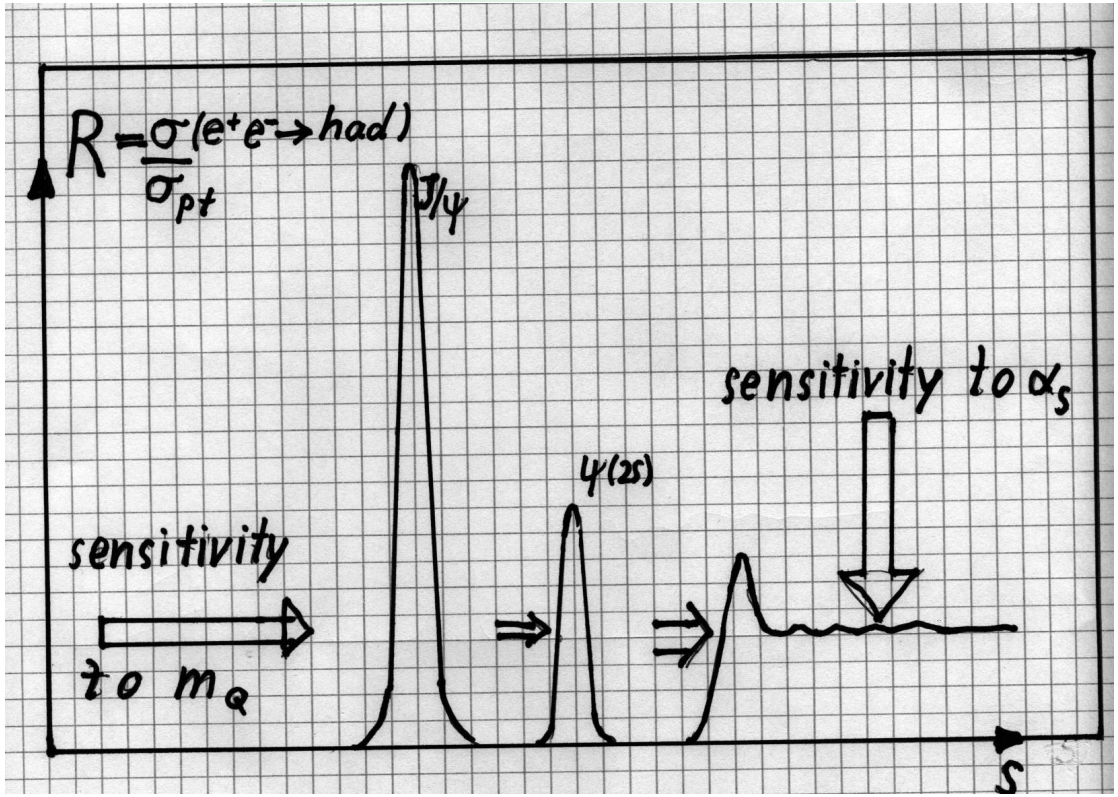
top-bottom $\rightarrow m_{\mathsf{t}} \big/ m_{\mathsf{b}} \sim$ ratio of vacuum expectation values

request $\frac{\delta m_{\mathsf{b}}}{m_{\mathsf{b}}} \sim \frac{\delta m_{\mathsf{t}}}{m_{\mathsf{t}}} \Rightarrow \delta m_{\mathsf{t}} \approx 0.5\,\mathrm{GeV} \Rightarrow \delta m_{\mathsf{b}} \approx 12\,\mathrm{MeV}$

II.2 Theory

m_Q from SVZ Sum Rules, Moments and Tadpoles

Main Idea (SVZ)



Some definitions:

$$\left(-q^2 g_{\mu\nu} + q_\mu\, q_\nu\right)\, \Pi(q^2) \;\equiv\; i \int \mathrm{d}x\, e^{iqx} \langle T j_\mu(x) j_\nu(0)\rangle$$

with the electromagnetic current j_μ.

$$R(s) \;=\; 12\pi\, \mathrm{Im}\left[\Pi(q^2 = s + i\epsilon)\right]$$

Taylor expansion: $\Pi_Q(q^2) \;=\; Q_Q^2\, \dfrac{3}{16\pi^2} \sum\limits_{n\geq 0} \bar{C}_n\, z^n$

with $z = q^2/(4m_Q^2)$ and $m_Q = m_Q(\mu)$ the $\overline{\mathrm{MS}}$ mass.

$$\bar{C}_n = \bar{C}_n^{(0)} + \frac{\alpha_s}{\pi}\bar{C}_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}_n^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \bar{C}_n^{(3)} + \ldots$$

generic form

$$
\begin{aligned}
\bar{C}_n = \; & \bar{C}_n^{(0)} \\
& + \frac{\alpha_s}{\pi}\left(\bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_{m_c} \right) \\
& + \left(\frac{\alpha_s}{\pi}\right)^2 \left(\bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_{m_c} + \bar{C}_n^{(22)} l_{m_c}^2 \right) \\
& + \left(\frac{\alpha_s}{\pi}\right)^3 \left(\bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_{m_c} + \bar{C}_n^{(32)} l_{m_c}^2 + \bar{C}_n^{(33)} l_{m_c}^3 \right) \\
& + \ldots
\end{aligned}
$$

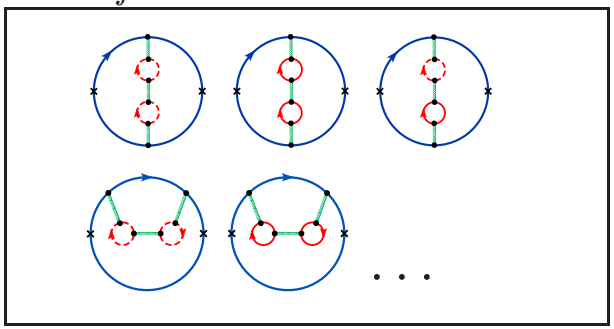
Analysis in NNLO

- FORM program MATAD
  Coefficients $\bar{C}_n$ up to $n = 8$
  (also for axial, scalar and pseudoscalar correlators)
  (Chetyrkin, JK, Steinhauser, 1996)

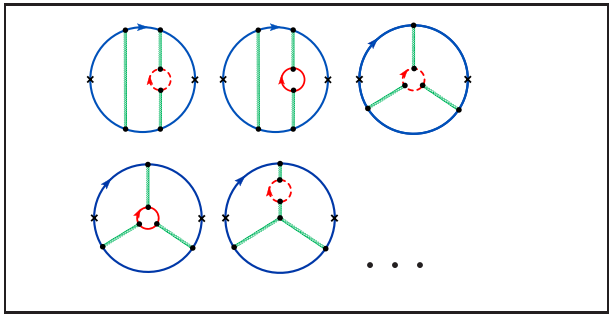
Analysis in N^{3}LO

Algebraic reduction to 13 master integrals (Laporta algorithm); numerical and analytical evaluation of master integrals

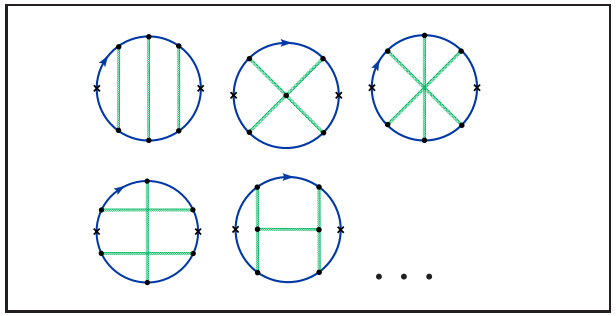
n_f^2-contributions



n_f^1-contributions



n_f^0-contributions



: heavy quarks, : light quarks,

n_f: number of active quarks

$\Rightarrow$ About 700 Feynman-diagrams

➪ Reduction to master integrals

$\bar{C}_0$ and $\bar{C}_1$ in order α_s^3 (four loops!) Program "Sturman" (Sturm) (2006)

(Chetyrkin, JK, Sturm; Boughezal, Czakon, Schutzmeier)

$\bar{C}_2$ and $\bar{C}_3$ (2008) Program "Crusher", Marquard & Seidel

(Maier, Maierhöfer, Marquard, A. Smirnov)

All master integrals known analytically and double checked.

(Schröder + Vuorinen, Chetyrkin et al., Schröder + Steinhauser, Laporta, Broadhurst, Kniehl et al.)

$\bar{C}_4 - \bar{C}_{10}$: extension to higher moments by Padé method, using analytic information from low energy ($q^2 = 0$), threshold ($q^2 = 4m^2$), high energy ($q^2 = -\infty$) (Kiyo, Maier, Maierhöfer, Marquard, 2009)

(Also: q^2-dependence of scalar, vector,... correlator)

Relation to measurements

$$\mathcal{M}_n^{\text{th}} \equiv \frac{12\pi^2}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n \Pi_c(q^2)\bigg|_{q^2=0} = \frac{9}{4}Q_c^2\left(\frac{1}{4m_c^2}\right)^n \bar{C}_n$$

Perturbation theory: $\bar{C}_n$ is function of α_s and $\ln\frac{m_c^2}{\mu^2}$

dispersion relation:

$$\Pi_c(q^2) = \frac{q^2}{12\pi^2}\int \mathrm{d}s\,\frac{R_c(s)}{s(s-q^2)} + \text{subtraction}$$

➪ $\mathcal{M}_n^{\text{exp}} = \int \frac{\mathrm{d}s}{s^{n+1}} R_c(s)$

constraint: $\mathcal{M}_n^{\text{exp}} = \mathcal{M}_n^{\text{th}}$

➪ m_c

II.3a Results from Experiment

Ingredients (charm)

experiment:

- $\Gamma_e(J/\psi, \psi')$ from BES & CLEO & BABAR (PDG)
- $\psi(3770)$ and $R(s)$ from BES
- $\alpha_s = 0.1187 \pm 0.0020$

theory:

- N^{3}LO for $n = 1, 2, 3, 4$
- include condensates

$$\delta\mathcal{M}_n^{\text{np}} = \frac{12\pi^2 Q_c^2}{(4m_c^2)^{(n+2)}} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle a_n \left(1 + \frac{\alpha_s}{\pi} \bar{b}_n\right)$$

- estimate of non-perturbative terms (oscillations, based on Shifman)
- careful extrapolation of R_{uds}
- careful definition of R_c

Results (m_c) (2009)

Error budget

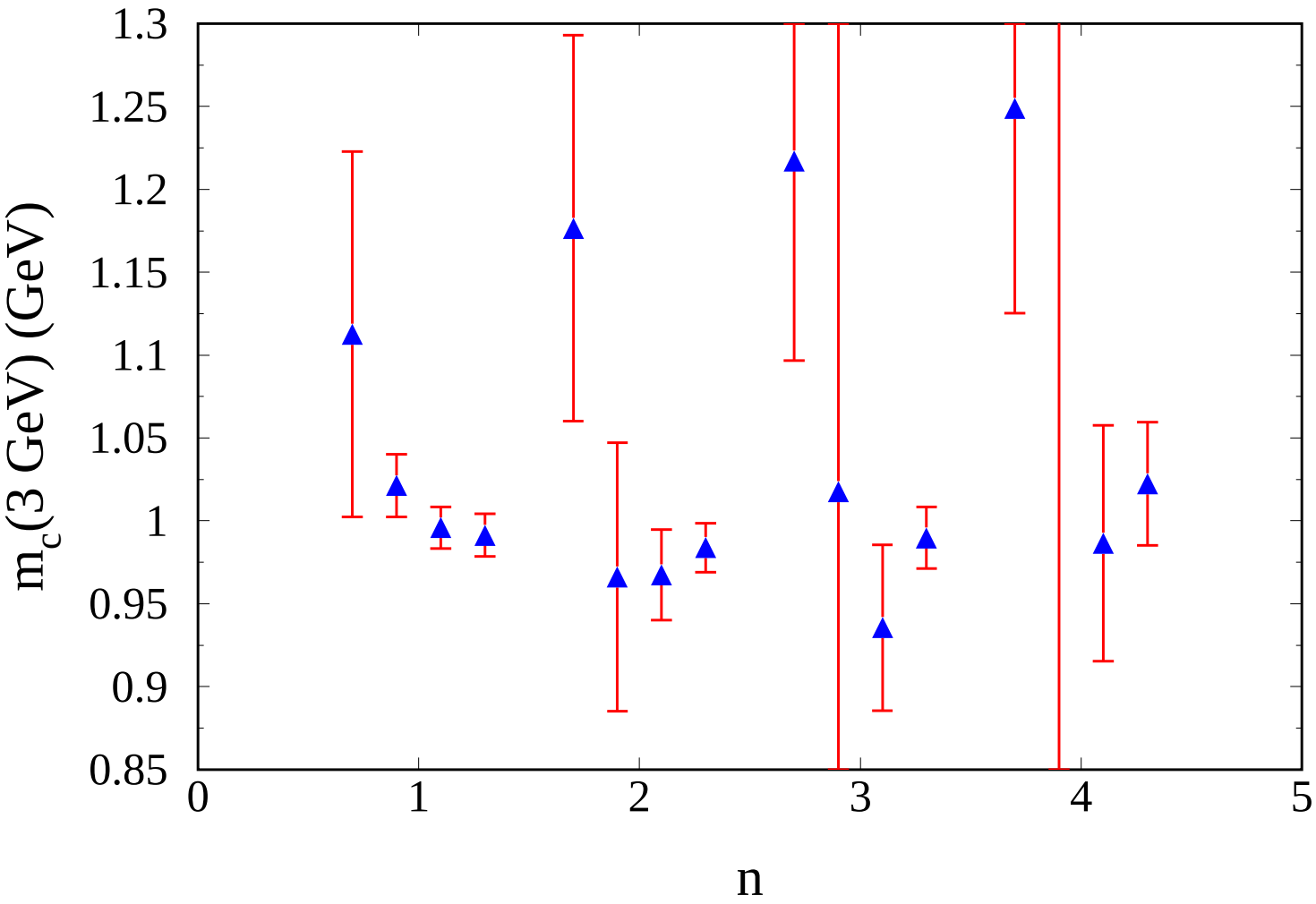
n	$m_c(3\,\text{GeV})$	exp	α_s	μ	np	total
1	986	9	9	2	1	13
2	976	6	14	5	0	16
3	978	5	15	7	2	17
4	1004	3	9	31	7	33

Remarkable consistency between $n = 1, 2, 3, 4$ and stability ($\mathcal{O}(\alpha_s^2)$ vs. $\mathcal{O}(\alpha_s^3)$);
prefered scale: $\mu = 3\,\text{GeV}$,

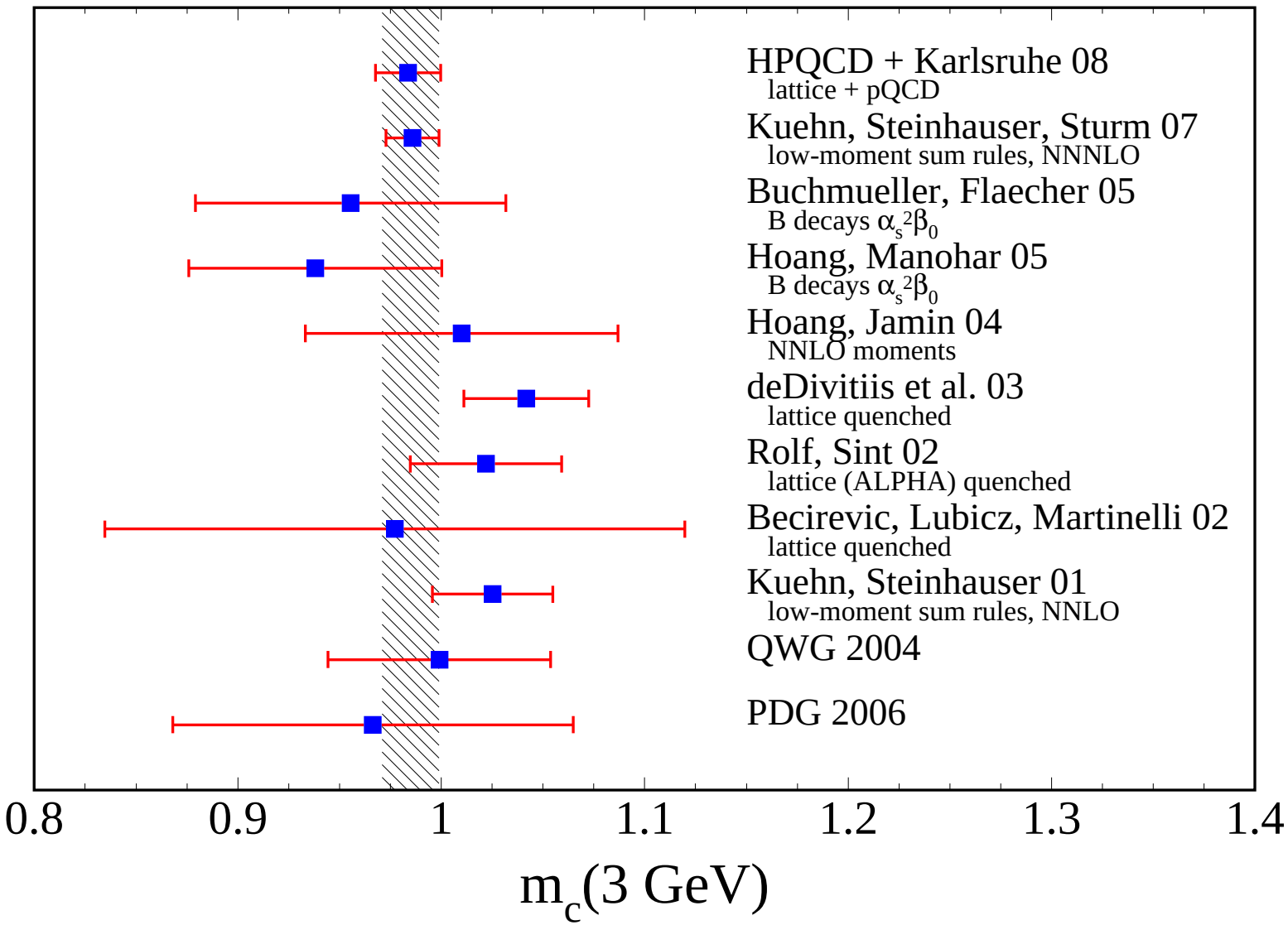
- $m_c(3\,\text{GeV}) = 986 \pm 13\,\text{MeV}$

conversion to $m_c(m_c)$:

- $m_c(m_c) = 1279 \pm 13\,\text{MeV}$



dependence of m_c on number of moment n and on $\mathcal{O}(\alpha_s^i)$ for $i = 0, \ldots, 3$

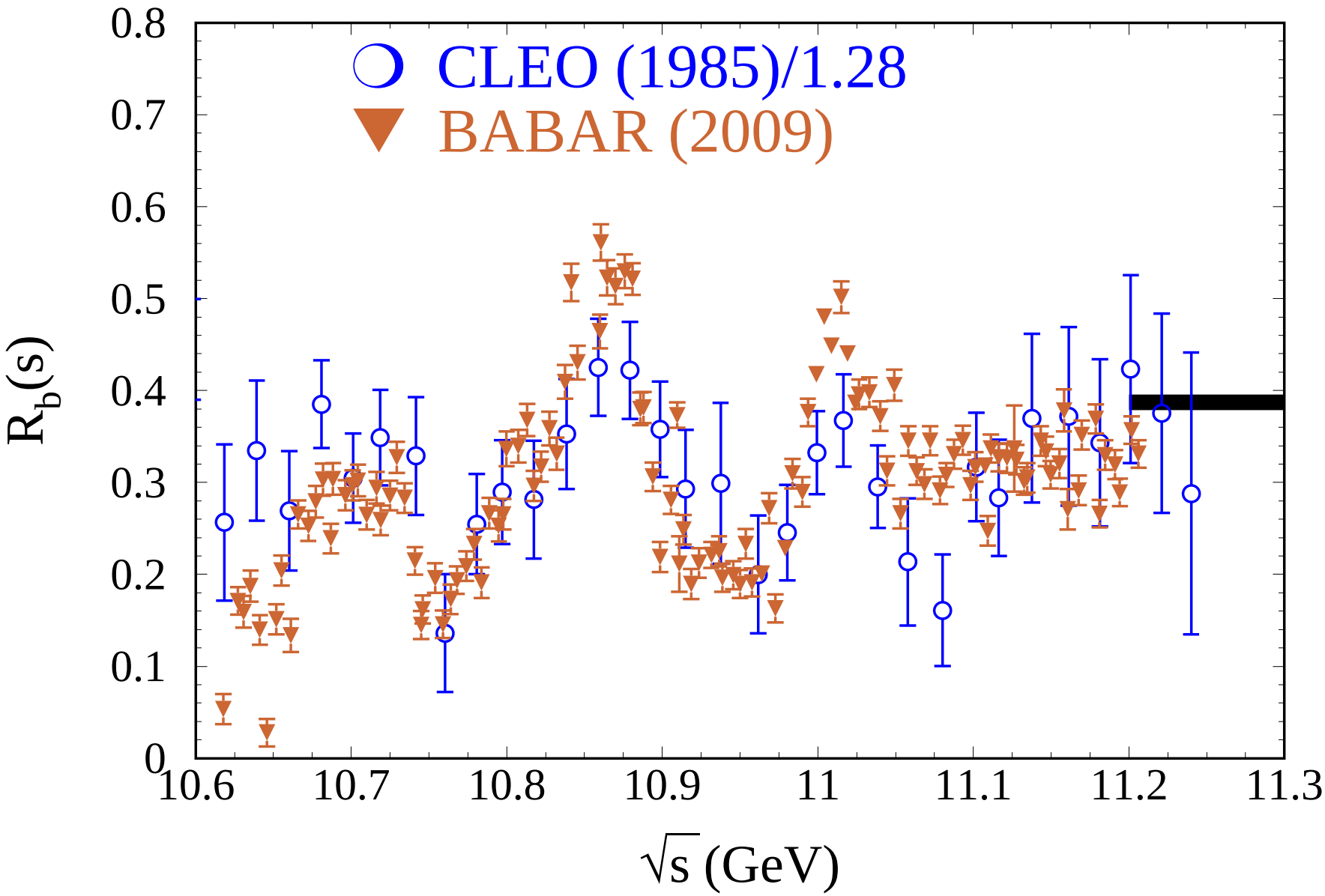

HPQCD + Karlsruhe 08
lattice + pQCD
Kuehn, Steinhauser, Sturm 07
low-moment sum rules, NNNLO
Buchmueller, Flaecher 05
B decays $\alpha_s^2\beta_0$
Hoang, Manohar 05
B decays $\alpha_s^2\beta_0$
Hoang, Jamin 04
NNLO moments
deDivitiis et al. 03
lattice quenched
Rolf, Sint 02
lattice (ALPHA) quenched
Becirevic, Lubicz, Martinelli 02
lattice quenched
Kuehn, Steinhauser 01
low-moment sum rules, NNLO
QWG 2004
PDG 2006
0.8
0.9
1
1.1
1.2
1.3
1.4
m_c(3 GeV)

Experimental Ingredients for m_b

Contributions from

- narrow resonances ($\Upsilon(1S)$ – $\Upsilon(4S)$) (PDG)
- threshold region (10.618 GeV – 11.2 GeV) (BABAR 2009)
- perturbative continuum ($E \geq 11.2$ GeV) (Theory)
- different relative importance of resonances vs. continuum for $n = 1, 2, 3, 4$

n	$\mathcal{M}_n^{\text{res},(1S-4S)}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{cont}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{exp}}$ $\times 10^{(2n+1)}$
1	1.394(23)	0.287(12)	2.911(18)	4.592(31)
2	1.459(23)	0.240(10)	1.173(11)	2.872(28)
3	1.538(24)	0.200(8)	0.624(7)	2.362(26)
4	1.630(25)	0.168(7)	0.372(5)	2.170(26)

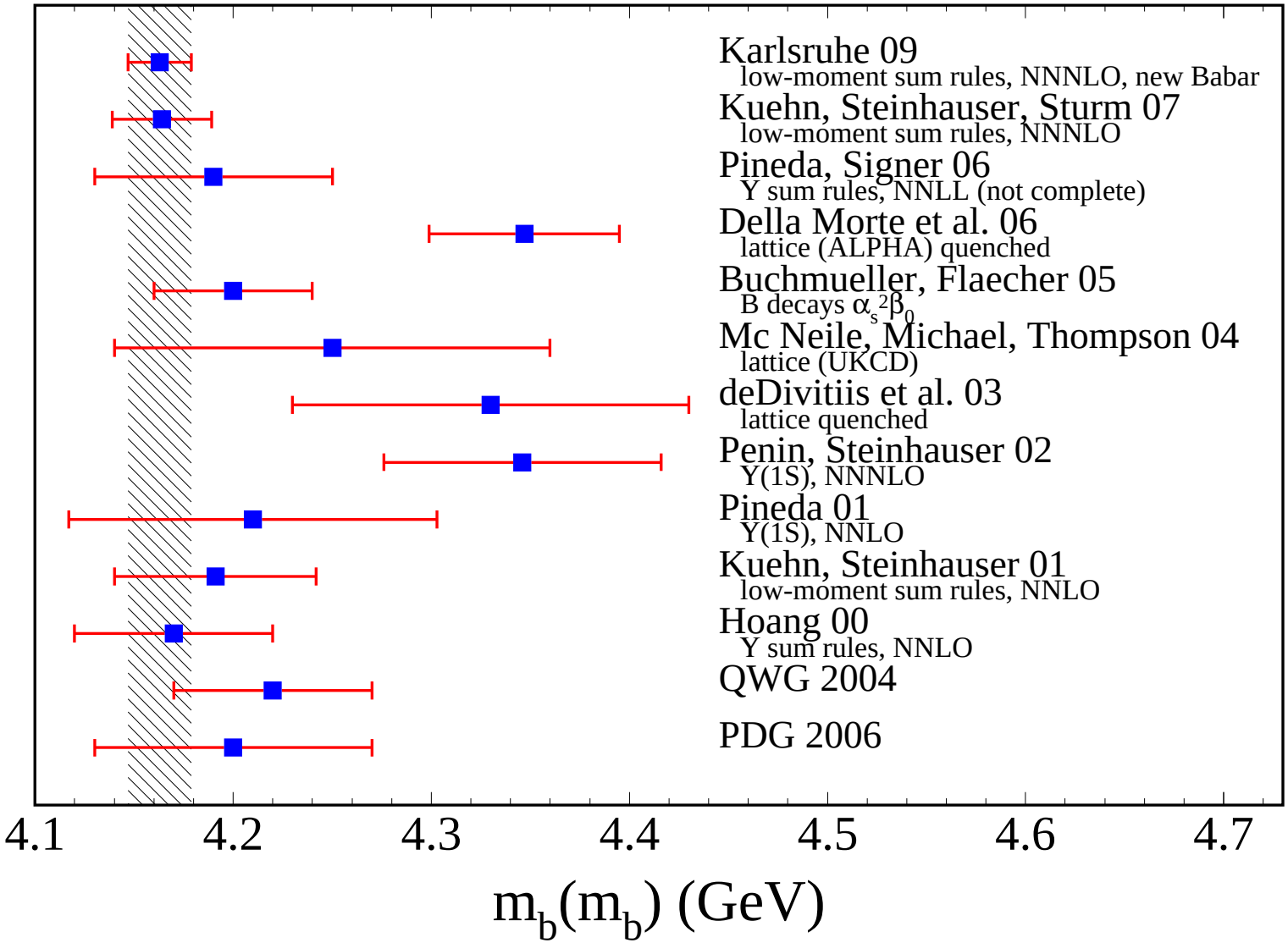

CLEO (1985)/1.28
BABAR (2009)
$R_b(s)$
$\sqrt{s}$ (GeV)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
10.6
10.7
10.8
10.9
11
11.1
11.2
11.3

n	$m_b(10\,\text{GeV})$	exp	α_s	μ	total	$m_b(m_b)$
1	3597	14	7	2	16	4151
2	3610	10	12	3	16	4163
3	3619	8	14	6	18	4172
4	3631	6	15	20	26	4183

Consistency ($n = 1, 2, 3, 4$) and stability ($\mathcal{O}(\alpha_s^2)$ vs. $\mathcal{O}(\alpha_s^3)$);

- $m_\text{b}(10\,\text{GeV}) = 3610 \pm 16\,\text{MeV}$
- $m_\text{b}(m_\text{b}) = 4163 \pm 16\,\text{MeV}$

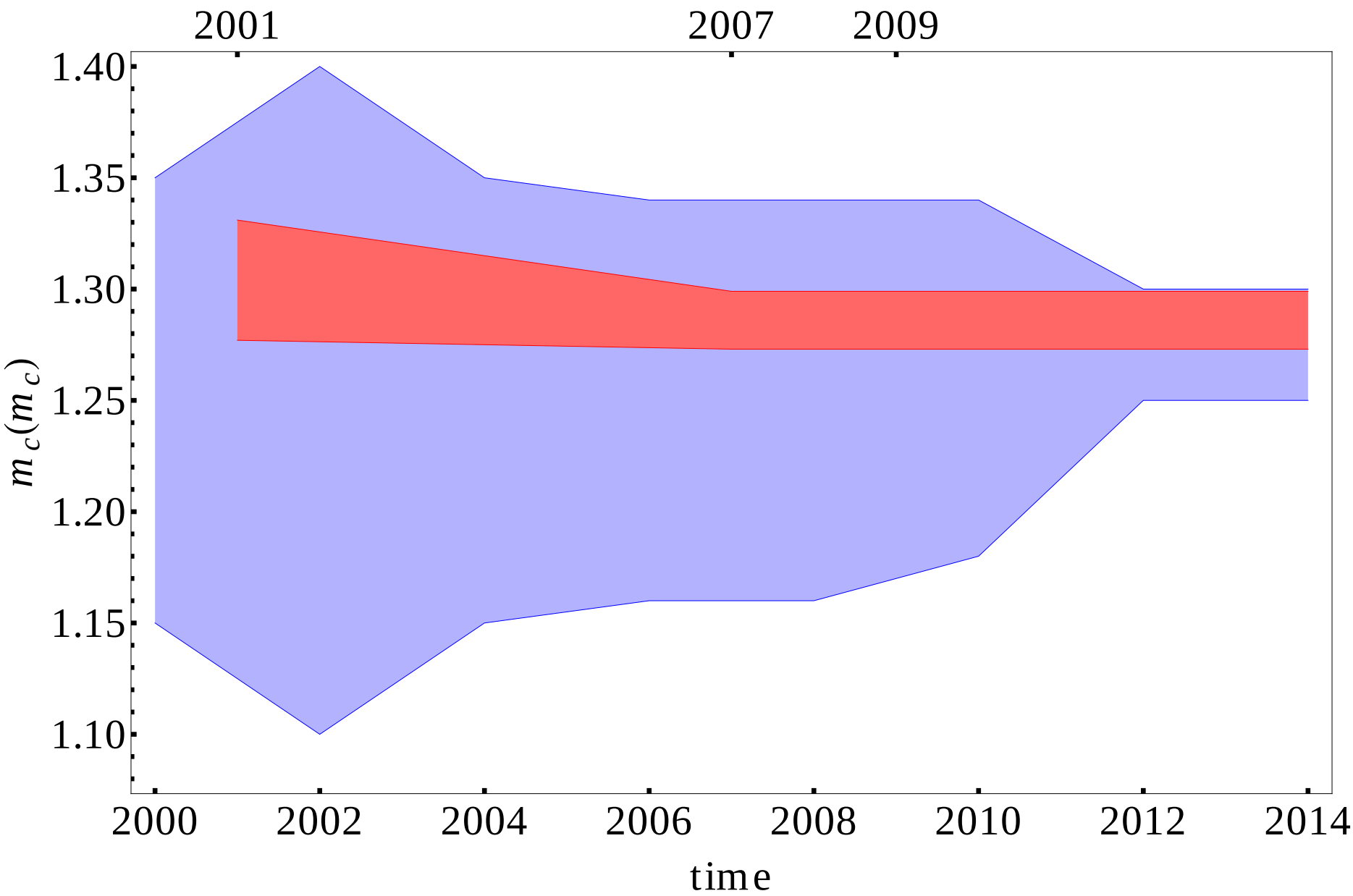
well consistent with KSS 2007


Karlsruhe 09
low-moment sum rules, NNNLO, new Babar
Kuehn, Steinhauser, Sturm 07
low-moment sum rules, NNNLO
Pineda, Signer 06
Y sum rules, NNLL (not complete)
Della Morte et al. 06
lattice (ALPHA) quenched
Buchmueller, Flaecher 05
B decays $\alpha_s^2\beta_0$
Mc Neile, Michael, Thompson 04
lattice (UKCD)
deDivitiis et al. 03
lattice quenched
Penin, Steinhauser 02
Y(1S), NNNLO
Pineda 01
Y(1S), NNLO
Kuehn, Steinhauser 01
low-moment sum rules, NNLO
Hoang 00
Y sum rules, NNLO
QWG 2004
PDG 2006
4.1
4.2
4.3
4.4
4.5
4.6
4.7
$m_b(m_b)$ (GeV)

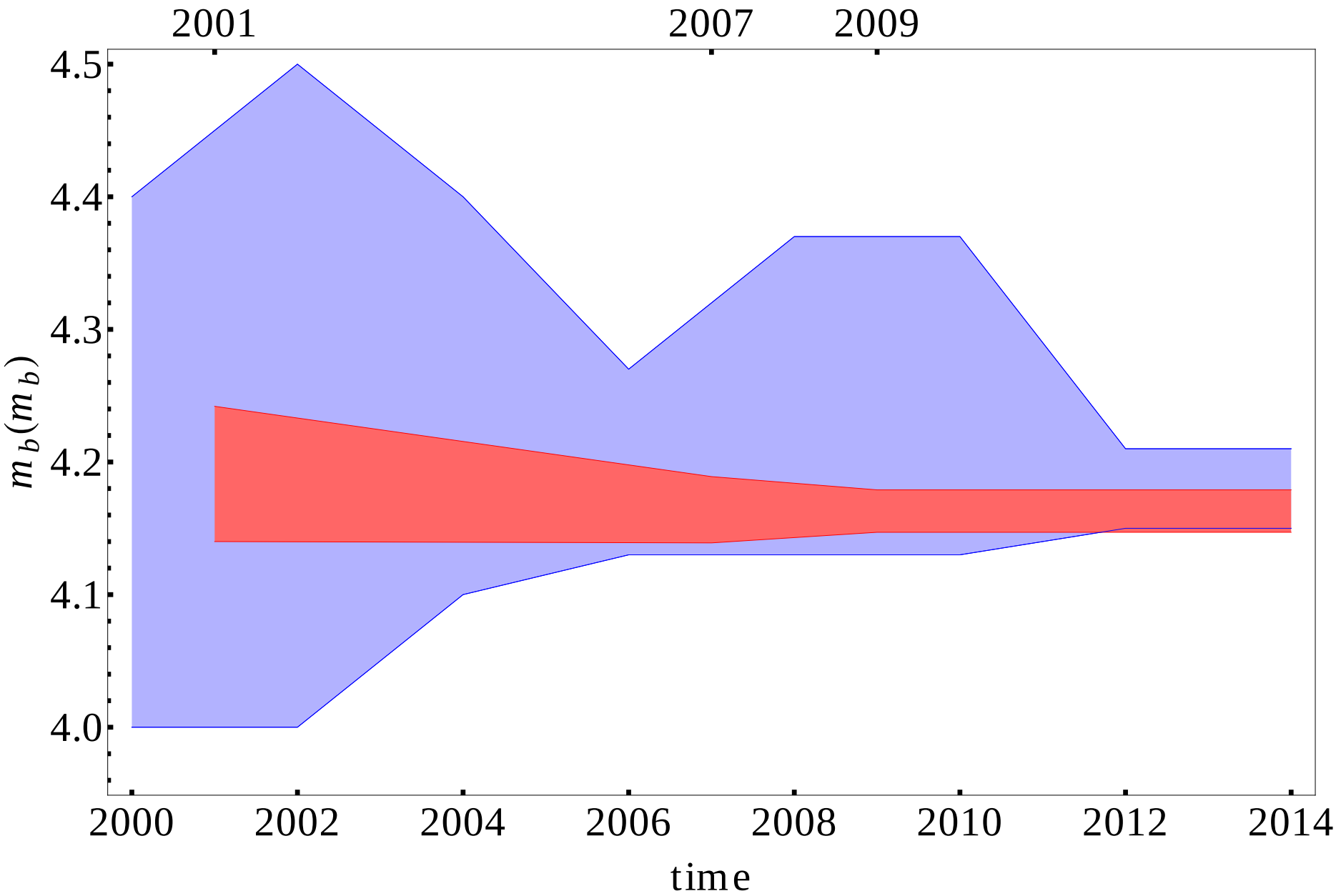
Time evolution:

results from 2001, 2007 and 2009 internally consistent, driving the PDG result

	$m_c(m_c)$	$m_b(m_b)$
PDG 2000	$1.15 - 1.35$ GeV	$4.0 - 4.4$ GeV
sum rules 2001	1304 ± 27 MeV	4191 ± 51 MeV
sum rules 2007	1286 ± 13 MeV	4164 ± 25 MeV
sum rules 2009	1286 ± 13 MeV	4163 ± 16 MeV
PDG 2014	1275 ± 25 MeV	4180 ± 30 MeV


2001
2007
2009
1.40
1.35
1.30
1.25
1.20
1.15
1.10
$m_c(m_c)$
2000
2002
2004
2006
2008
2010
2012
2014
time

oder


2001
2007
2009
4.5
4.4
4.3
4.2
4.1
4.0
$m_b(m_b)$
2000
2002
2004
2006
2008
2010
2012
2014
time

oder

II.3b Results from Lattice

lattice & pQCD (HPQCD + SFB/A1)

lattice evaluation of pseudoscalar correlator

$\Rightarrow$ replace experimental moments by lattice simulation

input: $M(\eta_c) \widehat{=} m_c\,, \quad M(\Upsilon(1S)) - M(\Upsilon(2S)) \widehat{=} \alpha_s$

pQCD for pseudoscalar correlator available:

"all" moments in $\mathcal{O}(\alpha_s^2)$

three lowest moments in $\mathcal{O}(\alpha_s^3)$.

the lowest moment is dimensionless

$\Rightarrow \quad \alpha_s(3\text{GeV}) \Rightarrow \alpha_s(M_Z) = 0.1174(12)$

higher moments: $\sim m_c^2 \times \left(1 + ...\frac{\alpha_s}{\pi}...\right)$,

$\Rightarrow \quad m_c(3\text{GeV}) = 986(10)$ MeV

to be compared with 986(13) MeV from e^+e^- !

update: HPQCD 2010

$\alpha_s(3\text{GeV}) \Rightarrow \alpha_s(M_Z) = 0.1183(7)$

$m_c(3\text{GeV}) = 986(6)$ MeV

$m_b(10\text{GeV}) = 3617(25)$ MeV

SUMMARY on m_Q

best determinations of m_c(3 GeV)

986(13) MeV	low moments; e^+e^- data	SFB/A1
986(6) MeV	low moments; lattice	SFB/A1+HPQCD
combined 986(5) MeV		

best determinations of m_b(10 GeV)

3610(16) MeV	low moments; e^+e^- data	
3617(25) MeV	HPQCD	
m_b(10 GeV) = 3621(9) MeV	non relativistic sum rules,	Penin, 2014
combined 3618(7) MeV		

further improvements needed and possible

III. MISCELLANEOUS

1. Matching and running

Decoupling in QCD at four loops
(Schröder, Steinhauser; Chetyrkin, JK, Sturm 2005)

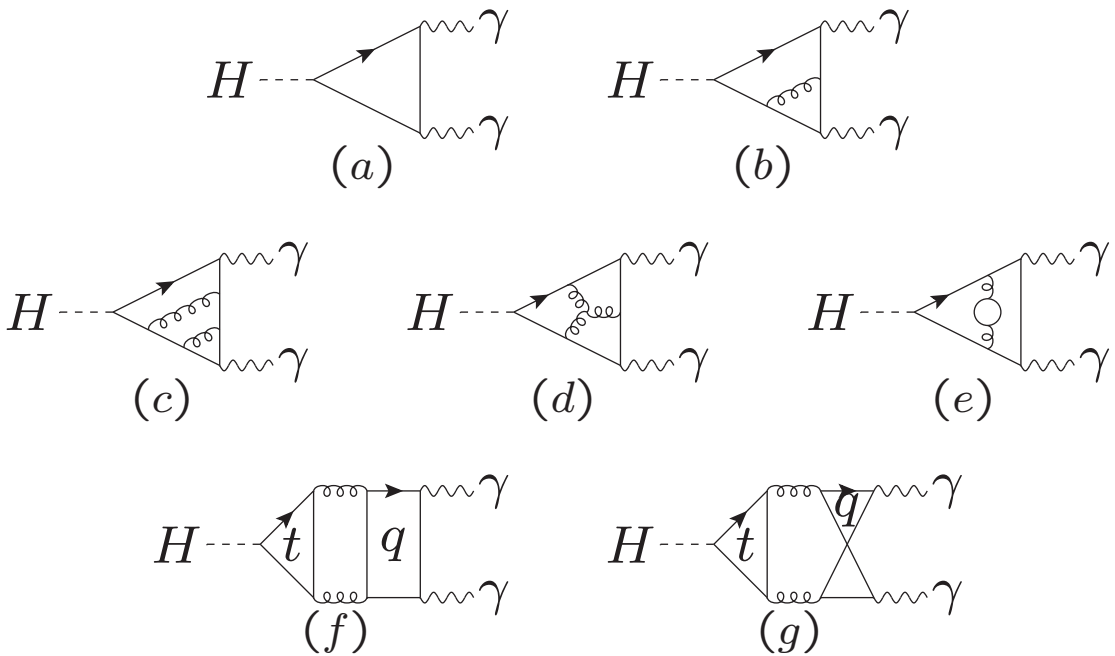
first evaluation of four-loop tadpoles

required for running of α_s

complementary to five-loop beta-function ($\rightarrow$ Chetyrkin)

2. Higgs decay

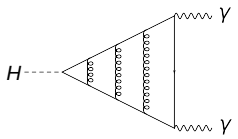
Three-loop QCD contributions to $H \to \gamma\gamma$
(Maierhöfer & Marquard 2012), full m_H/m_t dependence



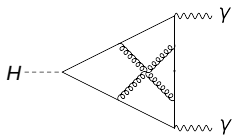
Sample diagrams of the 1-loop (a), 2-loop (b), 3-loop non-singlet (c)-(e), and 3-loop singlet (f)-(g) top quark induced contribution to $H \to \gamma\gamma$.

NNLO non-singlet: Steinhauser 1996
singlet: Maierhöfer & Marquard, 2012

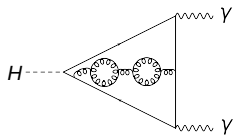
Four (and five) loop QCD contributions to $H \to \gamma\gamma$ in large m_t limit (Sturm 2014)



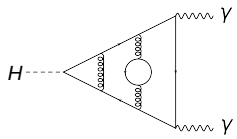
C_F^3



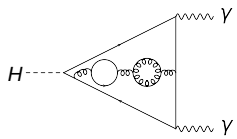
$C_F^2 C_A$



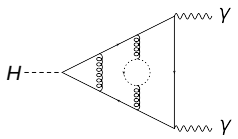
$C_F C_A^2$



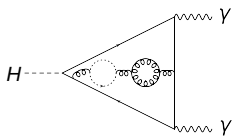
$n_h C_F^2 T_F$



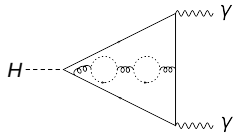
$n_h C_F C_A T_F$



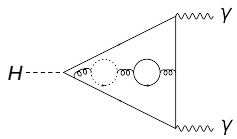
$n_l C_F^2 T_F$



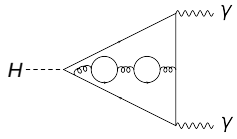
$n_l C_F C_A T_F$



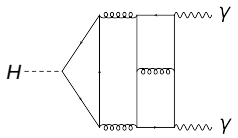
$n_l^2 C_F T_F^2$



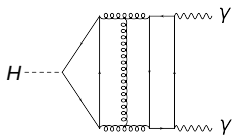
$n_h n_l C_F T_F^2$



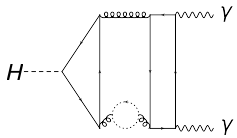
$n_h^2 C_F T_F^2$



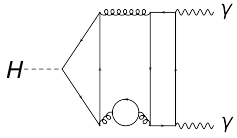
si $n_h C_F^2 T_F$



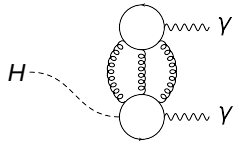
si $n_h C_F C_A T_F$



si $n_h n_l C_F T_F^2$



si $n_h^2 C_F T_F^2$



$n_h d^{abc} d^{abc}$

non-singlet and singlet pieces in four loop

Q_t^2 terms in five loop

Physics of top quarks

- electroweak corrections (Scharf, JK, Uwer)
- QCD asymmetry (Rodrigo, JK)
- threshold cross section (Kiyo, JK, Moch, Steinhauser, Uwer)

⋮

SUMMARY

12 PhD-Thesis in A1

- many postdocs $\nearrow$ research
  $\searrow$ industry

actively ongoing program