# Non-perturbative computation of the strong coupling constant on the lattice

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Many contributors to B2 over the years:

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# Opportunities and problems

- free parameters of QCD: $\alpha(\mu), m_f \quad f \in \{\text{flavors}\}$, some scheme
- extracted by matching PT to experiments at high energy [in principle...]

On the lattice:

- non-PT (NP) computations possible, but other (practical) limitations apply
  - for large size $L \lesssim 10$ fm cutoff $a^{-1} \lesssim 5\text{GeV}$ (2014+, generous!)
  - often (still) unphysical quark masses
  - systematic and statistical errors
- also on the lattice PT is needed to match a *lattice experiment* at large $\mu$
  - often the dominant error today

principle steps in a naive approach:

- compute a known hadronic scale $M \sim 1\text{GeV}$ [need $a \ll M^{-1} \ll L$]
- compute some observable $\mathcal{O}(\mu)$ at $\mu/M = \rho \gg 1$ [need $a \ll \mu^{-1}$]
  with $\mathcal{O}(\mu) = \alpha(\mu) + p_1\,\alpha^2(\mu) + p_2\,\alpha^3(\mu) + \ldots.$ $\Rightarrow$ extract $\alpha(\mu = \rho M)$
- need large $\rho$ to control PT truncation (and intrinsic) error

$\Rightarrow$

$$a \ll \mu^{-1} \ll M^{-1} \ll L \qquad \text{BUT} \qquad L/a \lesssim \mathrm{O}(100)$$

- forces compromises [on the control of systematic errors]

strategies on the market:

A) compromise as well as you can

B) $\mathcal{O}$, PT at scale $\mu \sim a^{-1}$ [difficult to control PT and cutoff effects]

C) finite-size step-scaling with $\mu = L^{-1}$ [ALPHA Collaboration project B2]

# Step scaling method

pioneered by ALPHA/B2, very popular for NP renormalization and recent investigations of nearly conformal 'walking'

- we need to invent $\bar{g}^2(L)$ [normalized $\mathcal{O}(\mu)$] such that
  - NP well-defined for $L^{-1} \in [M, \infty]$
  - good signal to noise ratio
  - well-behaved computable PT at small $L$ in terms of $\alpha(\mu = L^{-1})$

the method (logically, other order of steps in practice):

1. scale setting: $a \ll M^{-1} \ll L$ [expensive, large volume]
2. compress $L \searrow L_{\rm max}$ until $\rho_0 = M L_{\rm max} = \mathrm{O}(1)$
   obtain $\bar{g}^2(L_{\rm max}) = u_{\rm max}$ [intermediate cost, volume]
3. repeat with $a \searrow 0$, extrapolate $\Rightarrow$ universal continuum results $u_{{\rm max},\rho_0}$

recursive step scaling step $[\bar{g}^{\,2}(L) \leftrightarrow \bar{g}^{\,2}(2L)]$

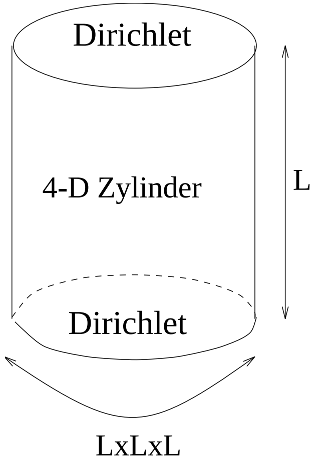
1. for given $u$ pick $L/a$, find bare parameters $[m_f \equiv 0^*]$ for $\bar{g}^{\,2}(L) = u$
2. change $L \to 2L$, find $\bar{g}^{\,2}(2L) = \Sigma(u, a/L)$
3. step scaling function: $\sigma(u) = \lim_{a/L \to 0} \Sigma(u, a/L)$ [like a $\beta -$ function]
4. obtain $\sigma(u)$ for range of $u$: universal, continuum, relatively cheap

*: massless scheme

Now couplings: $u_{\max} = u_0 \to u_1 \to \cdots \to u_n, \qquad u_i = \sigma(u_{i+1})$
scales: $\qquad \rho_0 \, M^{-1} = L_{\max} \to \frac{1}{2} L_{\max} \to \frac{1}{4} L_{\max} \cdots \frac{1}{2^n} \, L_{\max} = \rho_0 (2^n M)^{-1}$

- perturbative conversion $\bar{g}^{\,2} \to \alpha_{\overline{\rm MS}}$ at scale $\mathrm{O}(2^n M \sim 100\mathrm{GeV})$
- 'arbitrarily' high $\Rightarrow$ full PT-error control

# Schrödinger functional coupling




$$\bar{g}^{\,-2}(L) \propto \frac{\partial(\mathrm{free\ energy})}{\partial(\mathrm{boundary-value})}$$

$$\mathrm{e}^{-\Gamma(\eta)} = \langle C'(\eta)|\mathrm{e}^{-T\hat{H}}|C(\eta)\rangle = \int_{C,C'} DA\,\mathrm{e}^{-S_G[A]}\prod_{f=1}^{n_f}\det\,[\not{D}+m_f]$$

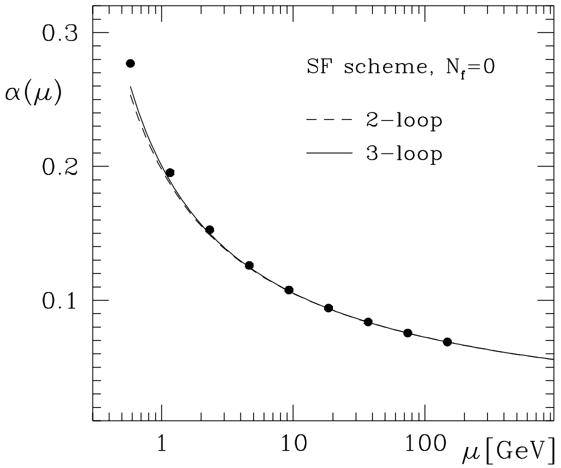
- $\eta$: dimensionless parameter in $C, C'$

$$\frac{1}{\bar{g}^{\,2}(L)} = \mathcal{N}\frac{\partial\Gamma}{\partial\eta} = \langle\mathcal{O}\rangle_{\mathrm{MC}}, \qquad \bar{g}^{\,2}(L) = g_0^2 + c_{1\,\mathrm{loop}}\,g_0^4 + \ldots$$

important features of the SF-coupling:

- $\bar{g}^2$ given by a simple observable [no fit etc.]
- nonperturbatively defined for all $L$
- $L$ is the only scale (beside cutoff $a$)
  - quark masses can be tuned to $m_{\rm PCAC}=0$ due to Dirichlet
  - $\Rightarrow$ SF = massless scheme
- well defined PT [unlike torus, no non-Gaussian modes]
  - two loops known for Wilson action
  - $\Rightarrow$ 2-loop relation with $\overline{\rm MS}$, 3-loop $\beta$-function for SF
- Wilson fermions (no $\chi$ symmetry), boundary conditions:
  O($a$) cutoff effects, but NP improvement ($\to a^2$) feasible

# History: $N_f = 0$ or quenched approximation




$N_f = 0$, Capitani et al.(1999)
[pre-SFB!]
$u_{\mathrm{max}} = 3.48$, $L_{\mathrm{max}}/r_0 = 0.718(16)$

$\Lambda^{(0)}_{\overline{\mathrm{MS}}} = 238(19)\mathrm{MeV}$

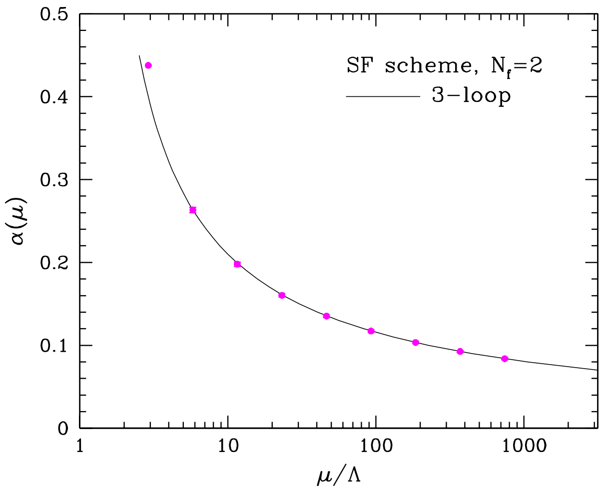
- $r_0$: Sommer scale from the static potential ($r_0 \approx 0.5$fm): $r_0^2 V'(r_0) = 1.65$

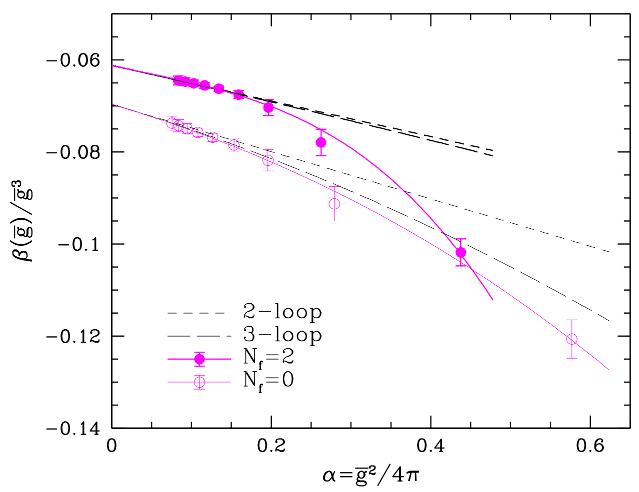
$$\Lambda_{\mathrm{SF}} = L^{-1}(b_0 \bar{g}^2(L))^{-b_1/2b_0^2} \mathrm{e}^{-1/(2b_0\bar{g}^2(L))} \exp\left[ -\int_0^{\bar{g}(L)} \left\{ \frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right\} dx \right]$$

- r.h.s. $L$-independent with exact NP $\beta$, small $L$ needed for 3-loop $\beta$

# More recent: $N_f = 2$

Della Morte et al. (2005):



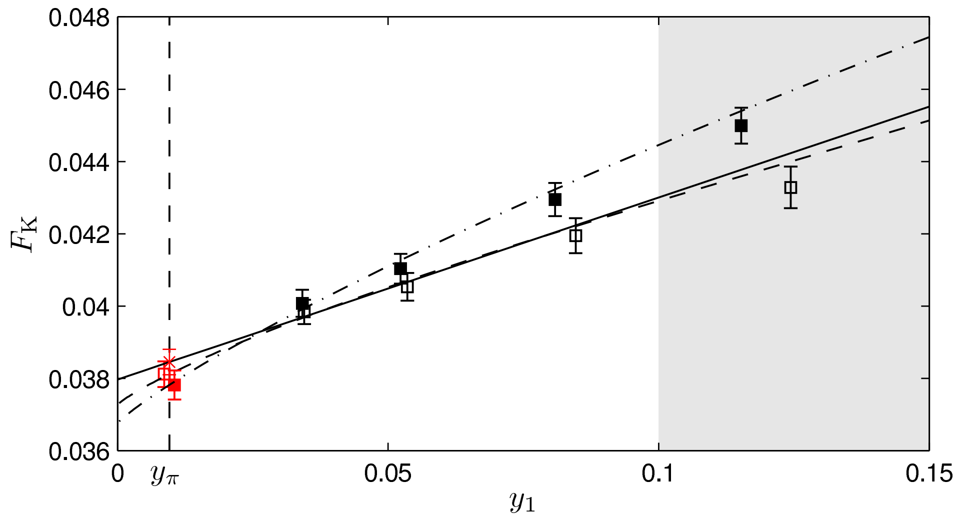




- degenerate quark doublet, $[\det(D + m_l)]^2 > 0$ as (hybrid-)MC weight
- NP renormalization and $\mathrm{O}(a)$ improvement and 2-loop PT impr. $\Sigma \to \sigma$ limit
- $\beta(g)$ from (interpolated) $\sigma(u)$, all for SF scheme
- computation mainly APE[next]
- overall scale only provisional in terms of $r_0$, finally completed $\to$

## $N_f = 2$ continued

Fritzsch et al. (2012)

- $u_{\max} = 4.484$, $L_{\max}/r_0 = 0.799(21)$, $L_{\max} f_K = 0.315(8)(2)$
- $\Lambda^{(2)}_{\overline{\mathrm{MS}}} = 310(20)\mathrm{MeV}$ [$f_K = 155\,\mathrm{MeV}$ input]
- same configurations $\Rightarrow M_s = 138(3)(1)$ MeV, $\overline{m}_s^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 102(3)(1)\mathrm{MeV}$
- strange quark still quenched!
- special adapted control of autocorrelations ('topological freezing') ($\rightarrow$talk R. Sommer)
- computation mainly Jülich, CLS




two chiral extrapolations of $F_K = a\, f_K$

$y_1 = m_\pi^2/(8\pi^2 f_K^2)$

open: $m_K$ kept constant

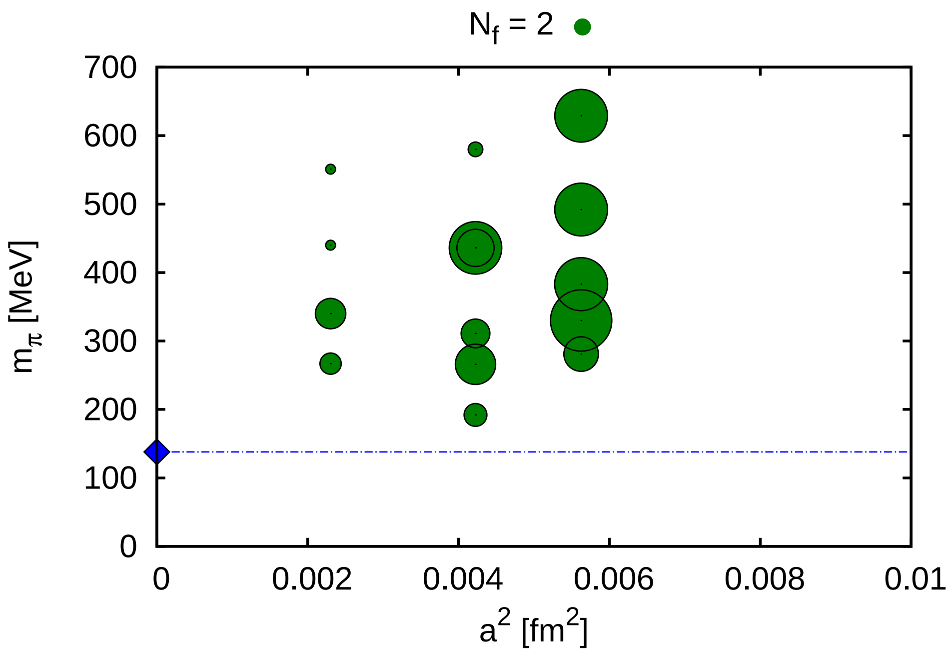
closed: $m_s$ kept constant

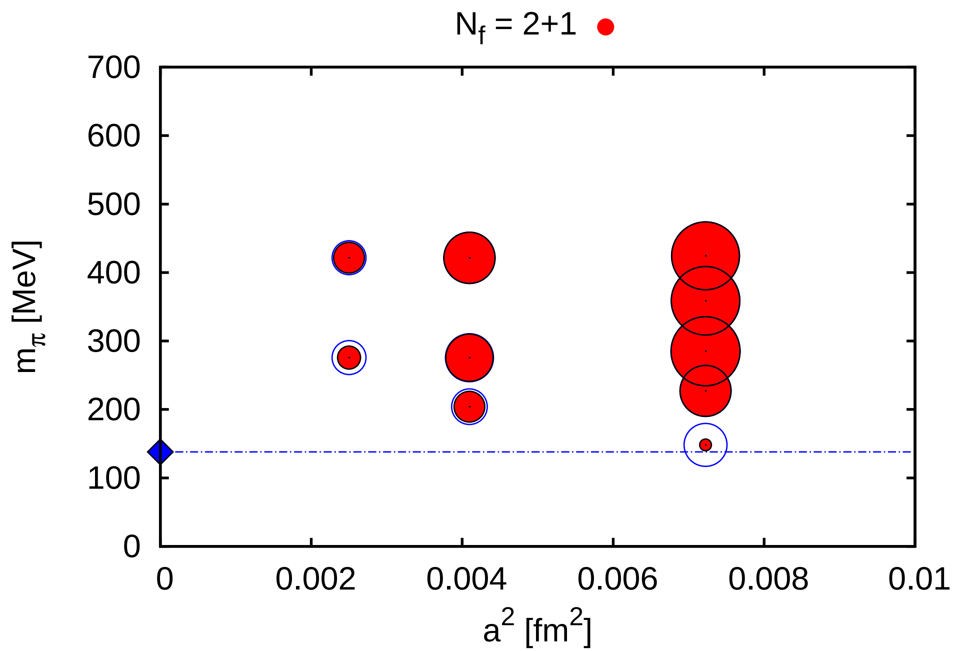
# Present: $N_f = 3$

- $N_f = 3$: soon contact with phenomenology, $\Lambda^{(5)}_{\overline{\rm MS}}$
- matching $3 \to 4$ by PT at first, check later, $4 \to 5$ PT certain
- why not $N_f = 4$ immediately? $\leftarrow$ multiple users of CLS large volume configurations, decision after debate
- indications: charm loops = very small effect ($\to$talk R. Sommer)

# Scale setting, large volume

runs still under way: blobsize area $\propto$ statistics $[\tau_{\rm exp}]$ (blue: target=150)



# New tool: gradient flow couplings (Lüscher, 2010)

$\Rightarrow$ whole new class of finite observables in QFT
$\Rightarrow$ new NP $\bar{g}^2(L)$, offers higher precision than SF at low energies $L^{-1}$

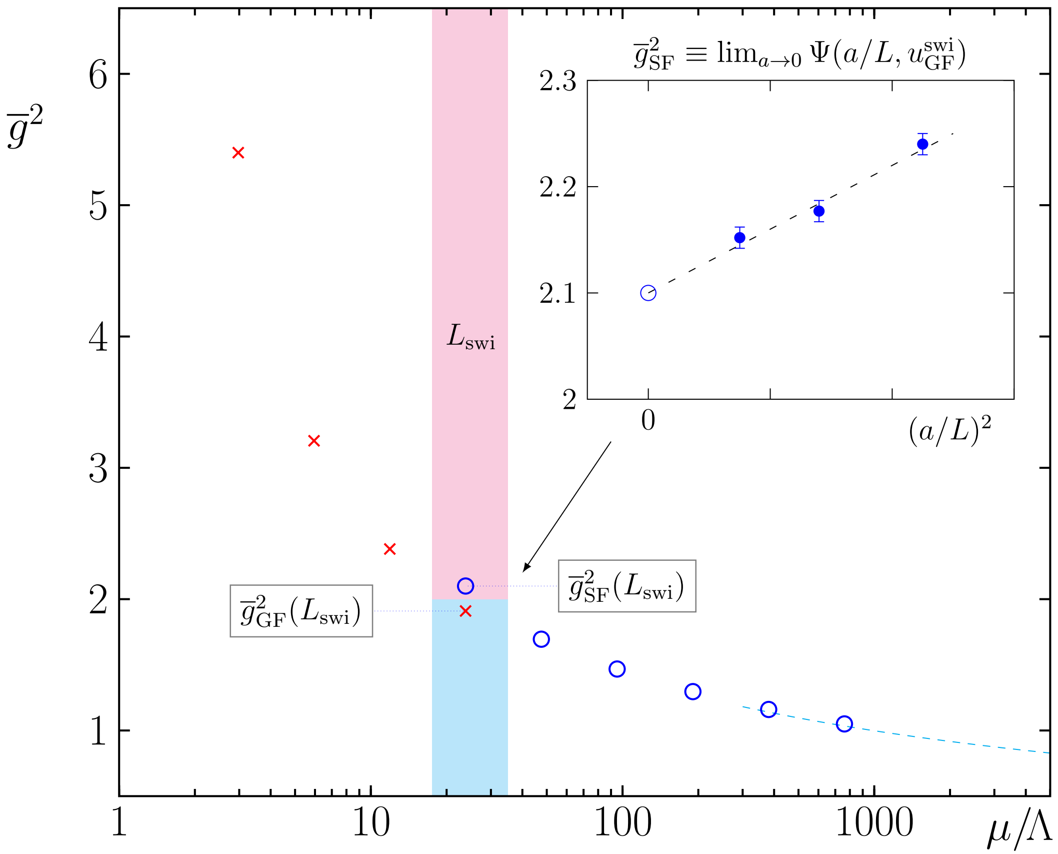
errors under stepscaling:

- one loop: $L\frac{\partial g}{\partial L} = b_0 g^3 \Rightarrow \delta g^{-2} \propto \frac{\delta L}{L}$ our model for error propagation
- combine absolute errors of $u_{\mathrm{max}}^{-2} \to u_1^{-2} \to \ldots \to u_n^{-2}$ for relative error in $\Lambda/M$
- achievable absolute precision in $\bar{g}^{-2}(L)$ counts
- around $\bar{g}^2 \gtrsim 2$: $10^3$ ($10^5$) configs. for GF (SF) coupling for $\delta\bar{g}^{-2} \sim 10^{-3}$
- but scaling for $\bar{g}^2 \searrow 0$: $\delta\bar{g}_{\mathrm{SF}}^{-2} \propto \bar{g}_{\mathrm{SF}}^{0}$ while $\delta\bar{g}_{\mathrm{GF}}^{-2} \propto \bar{g}_{\mathrm{GF}}^{-2}$

new strategy:

- run GF from hadronic to intermediate scale (all NP)
- relate GF, SF at intermediate scale (NP) [scheme ‘switch’ step]
- run SF from intermediate to PT

sketch of the ‘switch’:



# What is a flow coupling?

- for any $A_\mu(x)$ from Monte Carlo ensemble ($\leftrightarrow$path integral)
- evolve a trajectory $B_t$ out of $B_{t=0} = A$ by solving:

$$\frac{dB_t(x)}{dt} = -\frac{\delta S_{\rm YM}[B_t]}{\delta B_t(x)} \quad \Rightarrow \quad \frac{d}{dt} S_{\rm YM}[B_t] \leqslant 0 \quad \text{smoothing (steepest descent)}$$

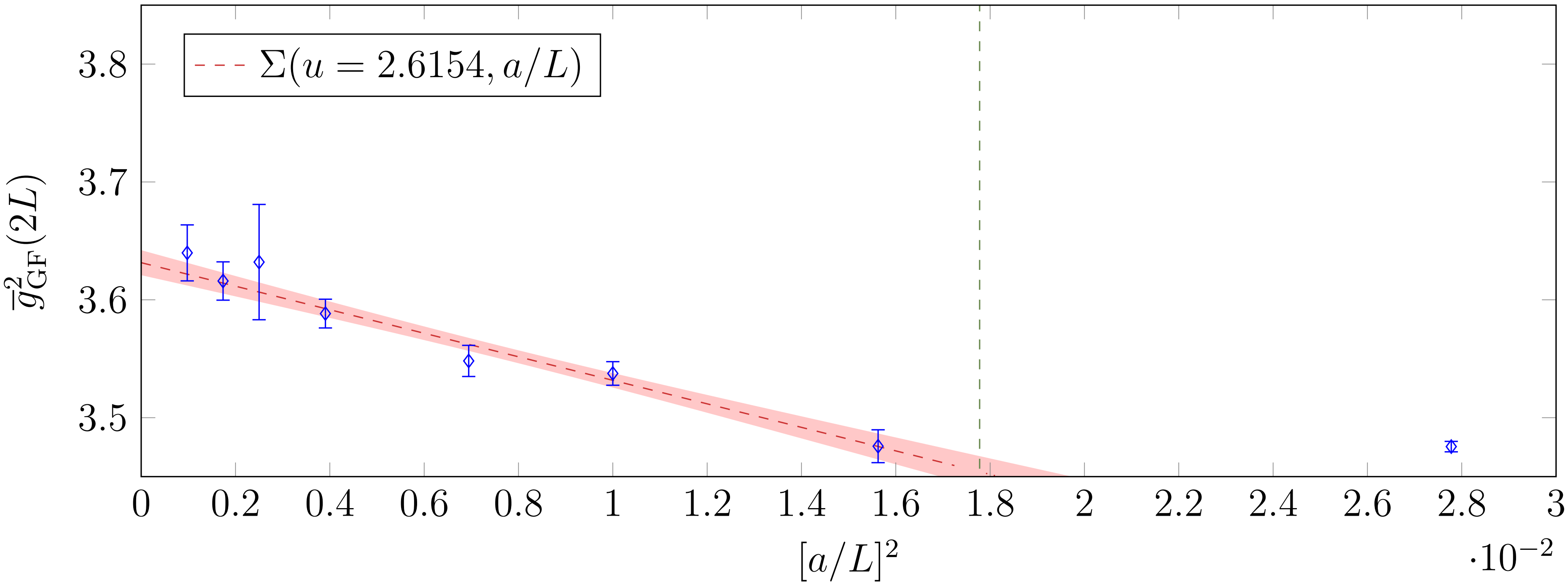
- theorem (Lüscher, Weisz 2011): finite correlations at finite $t$ (all orders PT)
- in particular the action density

$$-\frac{t^2}{2}\langle \mathrm{tr}\, G_{t,\mu\nu}^2 \rangle = \frac{3}{16\pi^2}\, g_{\overline{\rm MS}}^2(\mu) + c_1\, g_{\overline{\rm MS}}^4(\mu) + \ldots, \qquad \mu = 1/\sqrt{8t}$$

combine Gradient Flow with:

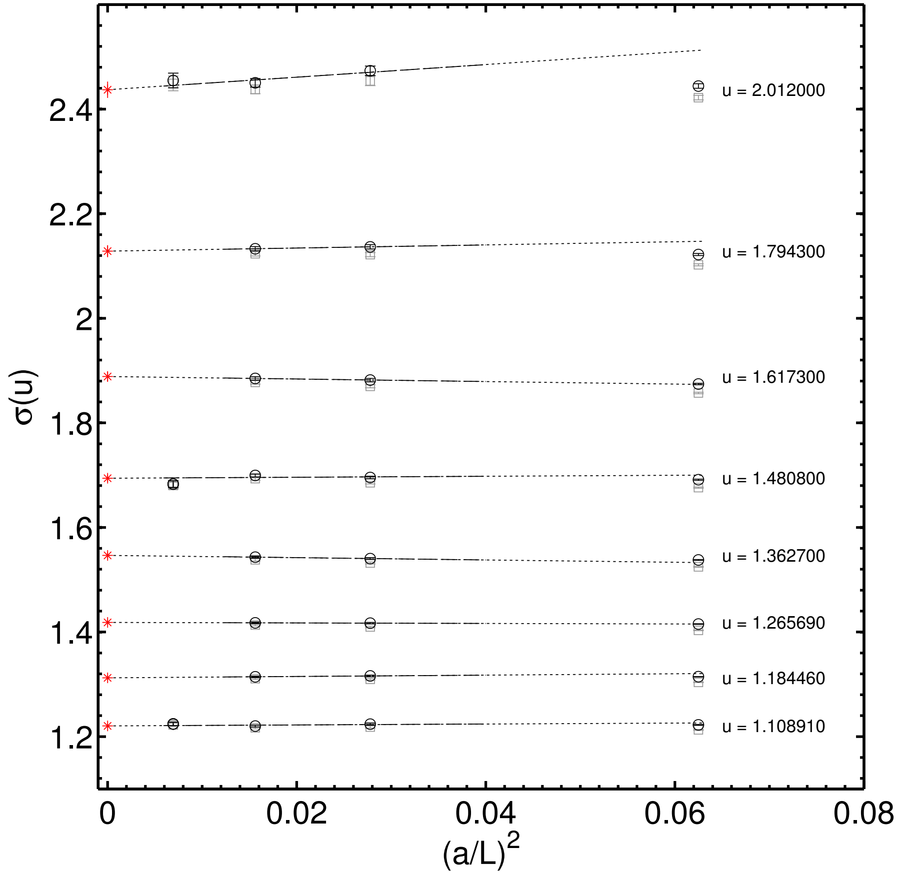
- finite volume $L^4$, SF boundary conditions (no zero modes, $m_f = 0$ possible)
- $\Rightarrow$ NP coupling $\bar{g}_{\rm GF}(L)$ with $\sqrt{8t} = cL$ (one scale, e.g. $c = 0.3$ fixed)

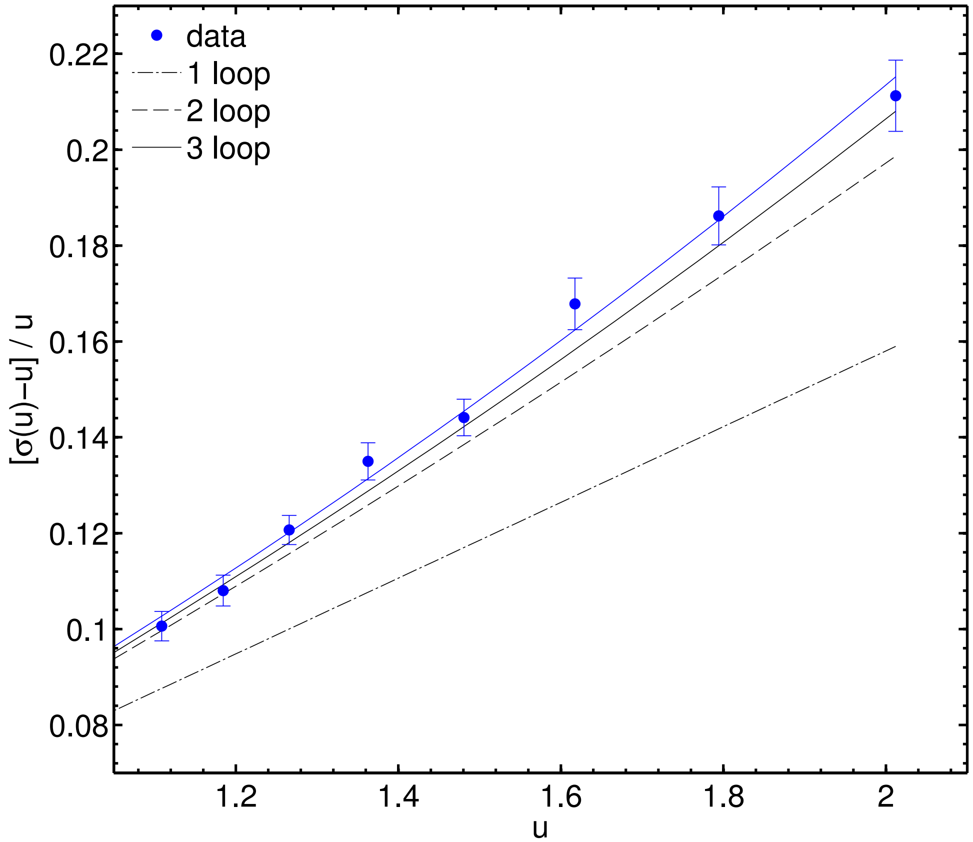
Example of an SSF for $\bar{g}^2_{\mathrm{GF}}$ in pure gauge theory:



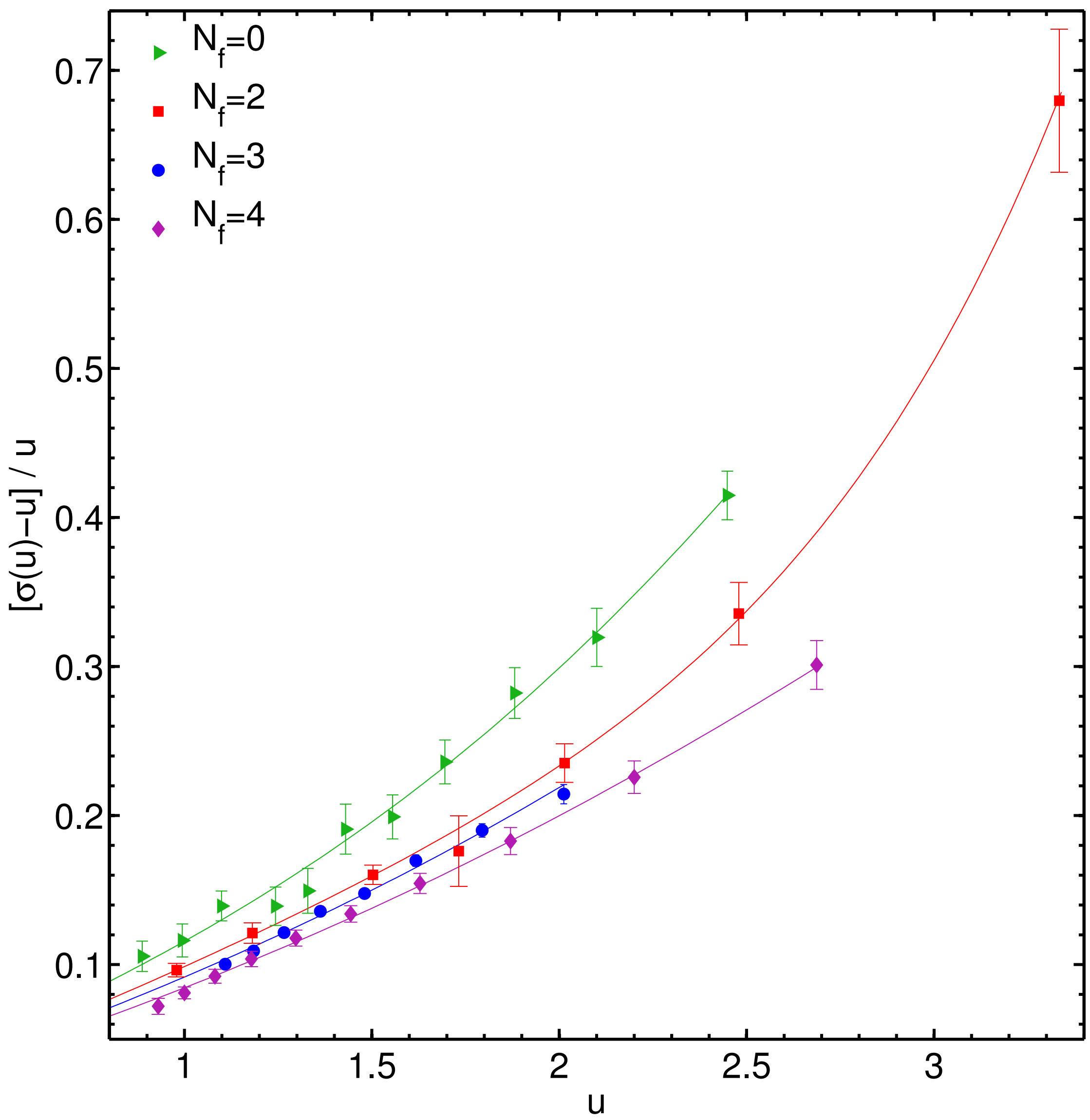

- cutoff effects not so small $\Rightarrow$ improve (Ramos, Sint)
- many knobs: underlying action, GF action, discretiztion $G_{\mu\nu}$, value $c$,...
- about to be finalized for $N_f = 3$: 'Zeuthen flow'

SF coupling results for $N_f = 3$:



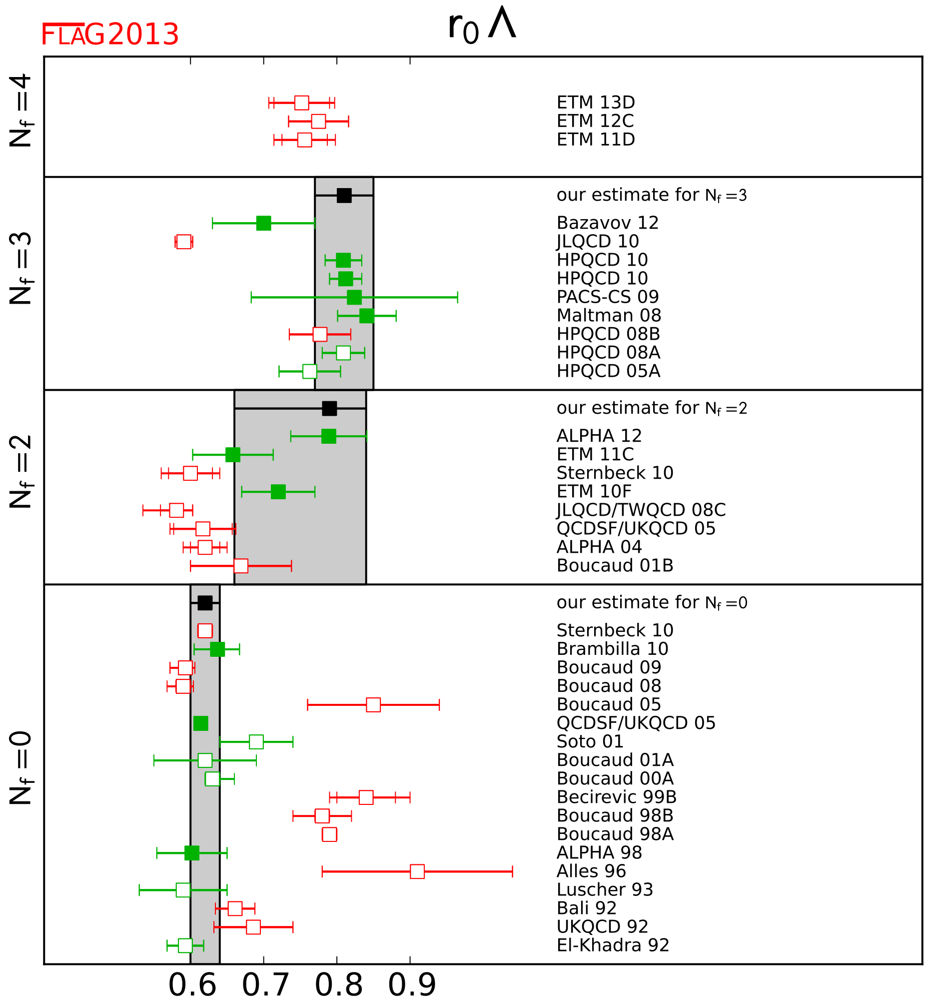

- data presumably sufficient, SF now at smallish coupling only
- still in progress: precision estimate for ren. factor $Z_A$ ($\chi$ rotated SF, Sint)



$N_f$=0
$N_f$=2
$N_f$=3
$N_f$=4
0.7
0.6
0.5
0.4
0.3
0.2
0.1
[σ(u)–u] / u
1
1.5
2
2.5
3
u

# Epilogue

- very many details suppressed here [which eat all the time]
- codes/resources: GHMC(Tao, Ape) $\rightarrow$ DD-HMC/MP-HMC $\rightarrow$openQCD (IBM, Cray)
- not quite there yet, but 2015: put it all together! ($N_f = 3$)
- also ongoing: improvement with charm quark (PhD Felix Stollenwerk)
  - problem: $\mathrm{O}(am_{\,\mathrm{charm}})$ much larger than $\mathrm{O}(am_{\,\mathrm{light}})$
  - NP improvement required, technically involved
  - goal $N_f = 3+1$ scheme to avoid/check PT matching $3 \rightarrow 4$
- many benefits to the wider community emerged from B2/ALPHA Collaboration
- **thank you TR9 family for super-enjoyable interaction over 12 years!!**



FLAG2013
$r_0\,\Lambda$
$N_f=4$
$N_f=3$
$N_f=2$
$N_f=0$
ETM 13D
ETM 12C
ETM 11D
our estimate for $N_f=3$
Bazavov 12
JLQCD 10
HPQCD 10
HPQCD 10
PACS-CS 09
Maltman 08
HPQCD 08B
HPQCD 08A
HPQCD 05A
our estimate for $N_f=2$
ALPHA 12
ETM 11C
Sternbeck 10
ETM 10F
JLQCD/TWQCD 08C
QCDSF/UKQCD 05
ALPHA 04
Boucaud 01B
our estimate for $N_f=0$
Sternbeck 10
Brambilla 10
Boucaud 09
Boucaud 08
Boucaud 05
QCDSF/UKQCD 05
Soto 01
Boucaud 01A
Boucaud 00A
Becirevic 99B
Boucaud 98B
Boucaud 98A
ALPHA 98
Alles 96
Luscher 93
Bali 92
UKQCD 92
El-Khadra 92
0.6
0.7
0.8
0.9