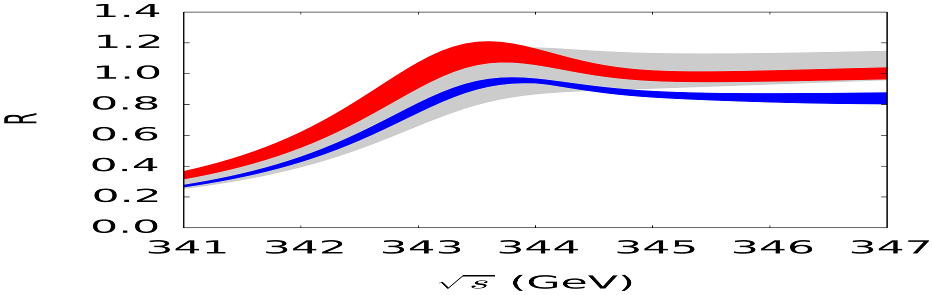
Top-pair production near threshold in e^+e^- collisions

Matthias Steinhauser | TTP Karlsruhe

15-19 September 2014, Durbach

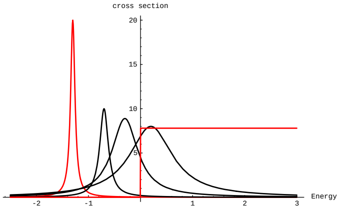


Outline

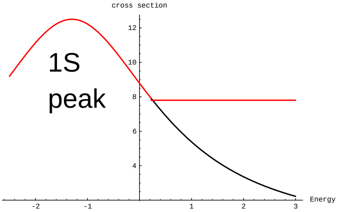
- $\sigma(e^+e^- \to t\bar{t} + X)$
  - Motivation for N^{3}LO
  - Framework
  - Results
- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$ to N^{3}LO

Results obtained in collaboration with
Martin Beneke, Yuichiro Kiyo, Peter Marquard, Alexander Penin, Jan Piclum,
Kurt Schuller, Dirk Seidel, Alexander Smirnov, Vladimir Smirnov

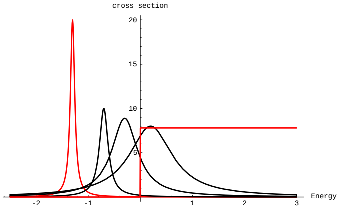
$$\text{“ } e^+e^- \to b\bar{b} \text{ ”}$$



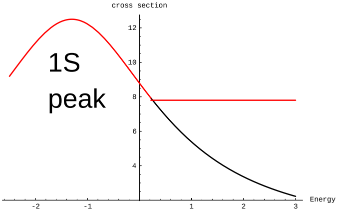
$$\text{“ } e^+e^- \to t\bar{t} \text{ ”}$$

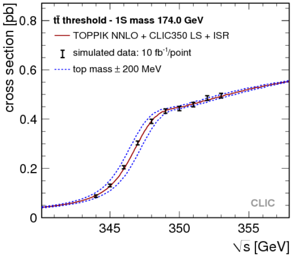


“ $e^+e^- \to b\bar{b}$ ”



“ $e^+e^- \to t\bar{t}$ ”





$\delta m_t \sim 100$ MeV

$\delta \Gamma_t \sim 30$ MeV

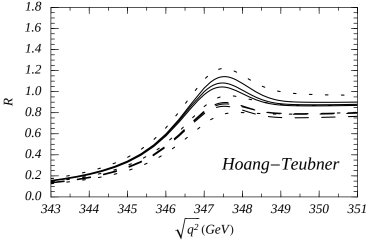
$\delta \alpha_s \sim 0.001$

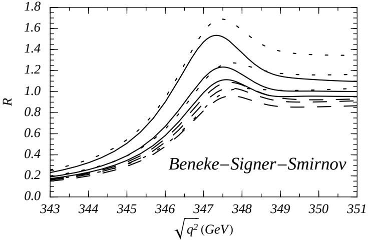
$\delta y_t \sim 30\%$

[Martinez,Miquel'02; Seidel,Simon,Tesar,Poss'13; Horiguchi,Ishikawa,Suehara, Fujii,Sumino,Kiyo,Yamamoto'14]

$e^+e^- \to t\bar{t}$ at NNLO: 2000

[Hoang,Beneke,Melnikov,Nagano,Ota,Penin,Pivovarov,Signer,Smirnov,Sumino,Teubner,Yakovlev,Yelkhovsky'00]



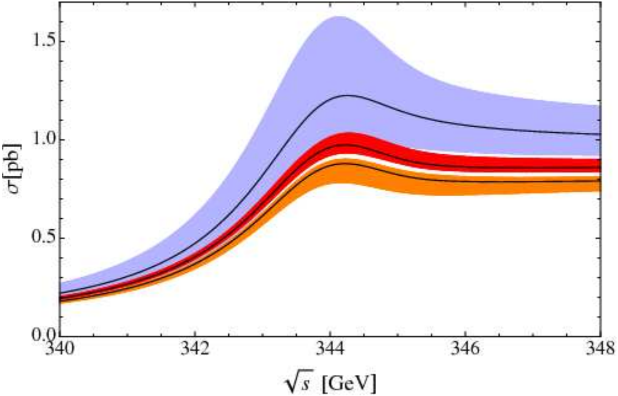


- stabilization of peak position (“threshold mass”)
- no stability in normalization of peak
- large differences between different groups

$e^+e^- \to t\bar{t}$ at NNLL: resum $(\alpha_s \ln v)^n$

[Hoang,Stahlhofen'14]

LL, NLL, NNLL,



- NNLL not complete
- no overlap of bands from μ variation
- $\delta\sigma/\sigma = \pm 5\%$ (peak)
- see also: [Hoang,Manohar,Stewart,Teubner'02; Pineda,Signer'07]

Framework: potential NRQCD

scales: mass, m: hard $\gg$ momentum, mv: soft $\gg$ energy, mv^2: ultrasoft $\gg$ Λ_{QCD}

potential quarks: $\begin{cases} E_{\vec{p}} \sim mv^2 \\ |\vec{p}\,| \sim mv \end{cases}$ $\qquad \dfrac{1}{E_{\vec{p}} - \frac{\vec{p}^{\,2}}{2m}}$

ultrasoft gluons: $\begin{cases} E_{\vec{k}} \sim mv^2 \\ |\vec{k}\,| \sim mv^2 \end{cases}$ $\qquad \dfrac{1}{E_{\vec{k}}^2 - \vec{k}^{\,2}}$

QCD ➪ NRQCD ➪ pNRQCD (potential non-relativistic QCD)

$\uparrow$ integrate out hard scale "m" from QCD [Caswell,Lepage'86; Bodwin,Braaten,Lepage'95]

$\uparrow$ integrate out all scales from NRQCD except potential quarks and ultrasoft gluons [Beneke,Smirnov'97; Pineda,Soto'98; Brambilla,Pineda,Soto,Vairo'00]

alternative formulation: velocity NRQCD (vNRQCD)

[Luke,Manohar,Rothstein'00; Hoang,Stewart'03]

Effective Hamiltonian to N^{3}LO

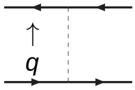
[Gupta,Radford'81,...,Manohar'97,...,Kniehl,Penin,Smirnov,Steinhauser'02,...,Beneke,Kiyo,Schuller'13]

$$\mathcal{H} = (2\pi)^3\delta(\vec{q})\left(\frac{\vec{p}^{\,2}}{m} - \frac{\vec{p}^{\,4}}{4m^3}\right) + C_c(\alpha_s)V_C(|\vec{q}|) + C_{1/m}(\alpha_s)V_{1/m}(|\vec{q}|)$$

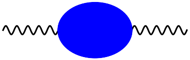
$$+\frac{\pi C_F\alpha_s(\mu)}{m^2}\left[C_\delta(\alpha_s) + C_p(\alpha_s)\frac{\vec{p}^{\,2}+\vec{p}^{\,\prime 2}}{2\vec{q}^{\,2}} + C_s(\alpha_s)\vec{S}^{\,2}\right]$$

Static potential:	$V_C(\lvert\vec{q}\rvert) = -\frac{4\pi C_F\alpha_s(\lvert\vec{q}\rvert)}{\vec{q}^2}$	C_c	3 loops
$1/m$ potential:	$V_{1/m}(\lvert\vec{q}\rvert) = \frac{\pi^2 C_F\alpha_s^2(\lvert\vec{q}\rvert)}{m\lvert\vec{q}\rvert}$	$C_{1/m}$	2 loops
"Breit" potential:	$\propto 1/m^2$	$C_{\delta,p,s}$	1 loop

$$\vec{q} = \vec{p}^{\,\prime} - \vec{p}$$



$\sigma_{\rm tot}(e^+e^- \to t\bar{t})$ in pNRQCD

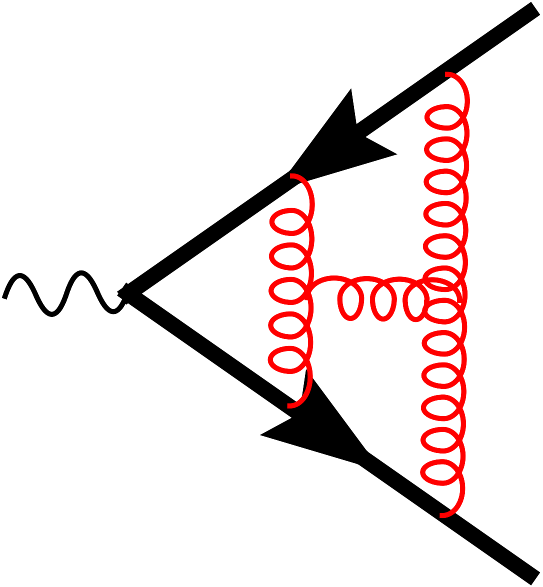


$$\sigma(s) = \sigma_0 \, \mathrm{Im}\Big[\Pi(q^2 = s + i\epsilon) + Z \text{ boson contr.} \Big]$$

$$\Pi = \frac{N_c}{2m_t^2} c_v \left[c_v - \frac{E}{m_t} \left(c_v + \frac{d_v}{3} \right) \right] G(E) + \ldots$$

➪ needed:

1. matching coefficients c_v 1 loop: [Källen, Sarby'55]
   2 loops: [Czarnecki,Melnikov'97; Beneke,Signer,Smirnov'97]; 3 loops: [Marquard,Piclum,Seidel,Steinhauser'06'08'14]
   and d_v 1 loop: [Luke,Savage'98]
2. $G(E)$ 3 loops: [Beneke,Kiyo,Penin'07; Beneke,Kiyo'08; Beneke,Kiyo,Schuller'13]



c_v to 3 loops

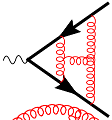
QCD $\longrightarrow$ NRQCD

$$j_v^\mu = \bar{Q}\gamma^\mu Q \qquad \longrightarrow \qquad \tilde{j}^i = \phi^\dagger \sigma^i \chi$$

$$j_v^i \qquad = \qquad c_v\, \tilde{j}^i + \frac{d_v(\mu)}{6m_Q^2}\phi^\dagger\sigma^i\vec{D}^2\chi + \ldots$$

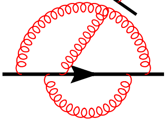
$$Z_2\Gamma_v \qquad = \qquad c_v\, \tilde{Z}_2\tilde{Z}_v^{-1}\tilde{\Gamma}_v + \ldots$$

Γ_v:



$\tilde{\Gamma}_v \equiv 1 \quad \tilde{Z}_2 \equiv 1$

Z_2:



$\tilde{Z}_v = 1 + \mathcal{O}(\alpha_s^2)$

[Melnikov,v.Ritbergen'00]
[Marquard,Mihaila,Piclum,Steinhauser'07]

c_V

$$c_V = 1 + \frac{\alpha_s}{\pi} c_V^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 c_V^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 c_V^{(3)} + \mathcal{O}(\alpha_s^4)$$

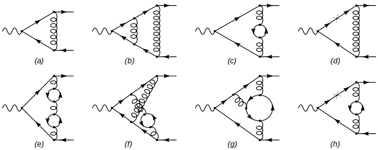
$c_V^{(1)}$ [Källen, Sarby'55]

$c_V^{(2)}$ [Czarnecki,Melnikov'97; Beneke,Signer,Smirnov'97]

$c_V^{(3),n_l}$ [Marquard,Piclum,Seidel,Steinhauser'06]

$c_V^{(3),n_h}$ [Marquard,Piclum,Seidel,Steinhauser'08]

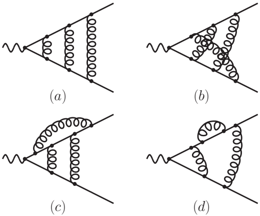
$c_V^{(3)}$ [Marquard,Piclum,Seidel,Steinhauser'14]



massive vertices

on-shell quarks: $q_1^2 = q_2^2 = M_Q^2$

$(q_1 + q_2)^2 = 4M_Q^2$



Results for c_v

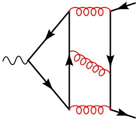
[Marquard,Piclum,Seidel,Steinhauser'14]

$$
\begin{aligned}
c_v(\mu = m_Q) \quad = \quad & 1 - 2.67\frac{\alpha_s}{\pi} + [-44.55 + 0.41 n_l]\left(\frac{\alpha_s}{\pi}\right)^2 \\
& + \left[-2091(2) + 120.66 n_l - 0.82 n_l^2\right]\left(\frac{\alpha_s}{\pi}\right)^3 \\
& + \text{singlet terms}
\end{aligned}
$$

- $\overline{\mathrm{MS}}$ scheme; $\mu = m_Q$
- large corrections
- singlet terms: small ($\leq 3\%$ of $c^{(2)}$)

  at 2 loops (for axial-vector, scalar, pseudo-scalar current)

  [Kniehl,Onishchenko,Piclum,Steinhauser'06]



Ingredients to $G(E)$

N^3LO $\psi(0)$, β_0^3 [Penin,Smirnov,Steinhauser'05]
N^3LO $\psi(0)$, $G(E)$, Coulomb [Beneke,Kiyo,Schuller'05]
N^3LO $\psi(0)$, ultra-soft [Beneke,Kiyo,Penin'07]
N^3LO $\psi(0)$, non-Coulomb [Beneke,Kiyo,Schuller'08]
N^3LO $G(E)$, ultra-soft [Beneke,Kiyo'08]

Review: [Beneke,Kiyo,Schuller'13]
Review: [Beneke,Kiyo,Schuller in progress]

- LO: Coulomb solution

$$\left(-\frac{\Delta}{m_t} + \frac{C_F \alpha_s}{r} - E \right) G_0(\vec{r}, \vec{r}\,', E) = \delta^{(3)}(\vec{r} - \vec{r}\,')$$

- NLO, NNLO, ... ⇨ perturbation theory in momentum space:

$$G(E) = \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \frac{d^{d-1}\vec{p}\,'}{(2\pi)^{d-1}} \Bigg[G_0(\vec{p}, \vec{p}\,'; E)$$
$$+ \int \frac{d^{d-1}\vec{p}_1}{(2\pi)^{d-1}} \frac{d^{d-1}\vec{p}\,'_1}{(2\pi)^{d-1}} G_0(\vec{p}, \vec{p}_1; E)\, \delta V(\vec{p}_1, \vec{p}\,'_1)\, G_0(\vec{p}\,'_1, \vec{p}\,'; E) + \ldots \Bigg]$$
$$+ \delta^{\rm us} G(E)$$

Ingredients to $G(E)$

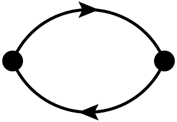
Review: [Beneke,Kiyo,Schuller'13]
Review: [Beneke,Kiyo,Schuller in progress]

- LO: Coulomb solution

$$\left(-\frac{\Delta}{m_t}+\frac{C_F\alpha_s}{r}-E\right)G_0(\vec{r},\vec{r}\,',E)=\delta^{(3)}(\vec{r}-\vec{r}\,')$$

- NLO, NNLO, ... ➪ perturbation theory in momentum space:

$$G(E)=\int\frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\frac{d^{d-1}\vec{p}\,'}{(2\pi)^{d-1}}\Bigg[G_0(\vec{p},\vec{p}\,';E)$$
$$+\int\frac{d^{d-1}\vec{p}_1}{(2\pi)^{d-1}}\frac{d^{d-1}\vec{p}_1'}{(2\pi)^{d-1}}G_0(\vec{p},\vec{p}_1;E)\,\delta V(\vec{p}_1,\vec{p}_1')\,G_0(\vec{p}_1',\vec{p}\,';E)+\ldots\Bigg]$$
$$+\delta^{\mathrm{us}}G(E)$$



Ingredients to $G(E)$

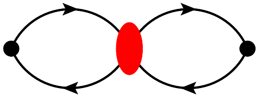
Review: [Beneke,Kiyo,Schuller'13]
Review: [Beneke,Kiyo,Schuller in progress]

- LO: Coulomb solution

$$\left(-\frac{\Delta}{m_t} + \frac{C_F \alpha_s}{r} - E \right) G_0(\vec{r}, \vec{r}\,', E) = \delta^{(3)}(\vec{r} - \vec{r}\,')$$

- NLO, NNLO, ... ⇨ perturbation theory in momentum space:

$$G(E) = \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \frac{d^{d-1}\vec{p}\,'}{(2\pi)^{d-1}} \Bigg[G_0(\vec{p}, \vec{p}\,'; E)$$
$$+ \int \frac{d^{d-1}\vec{p}_1}{(2\pi)^{d-1}} \frac{d^{d-1}\vec{p}_1'}{(2\pi)^{d-1}} G_0(\vec{p}, \vec{p}_1; E)\, \delta V(\vec{p}_1, \vec{p}_1')\, G_0(\vec{p}_1', \vec{p}\,'; E) + \dots \Bigg]$$
$$+ \delta^{\mathrm{us}} G(E)$$



Ingredients to $G(E)$

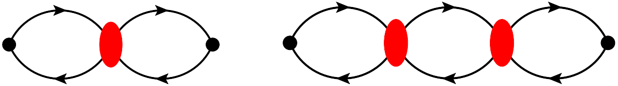
Review: [Beneke,Kiyo,Schuller'13]
Review: [Beneke,Kiyo,Schuller in progress]

- LO: Coulomb solution

$$\left(-\frac{\Delta}{m_t} + \frac{C_F\alpha_s}{r} - E\right) G_0(\vec{r},\vec{r}\,',E) = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

- NLO, NNLO, ... ⇨ perturbation theory in momentum space:

$$G(E) = \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\frac{d^{d-1}\vec{p}'}{(2\pi)^{d-1}}\Bigg[G_0(\vec{p},\vec{p}';E) + \int \frac{d^{d-1}\vec{p}_1}{(2\pi)^{d-1}}\frac{d^{d-1}\vec{p}_1'}{(2\pi)^{d-1}} G_0(\vec{p},\vec{p}_1;E)\,\delta V(\vec{p}_1,\vec{p}_1')\,G_0(\vec{p}_1',\vec{p}';E) + \ldots\Bigg] + \delta^{\mathrm{us}} G(E)$$



Ingredients to $G(E)$

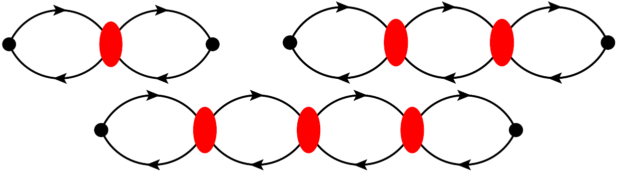
Review: [Beneke,Kiyo,Schuller'13]
Review: [Beneke,Kiyo,Schuller in progress]

- LO: Coulomb solution

$$\left(-\frac{\Delta}{m_t}+\frac{C_F\alpha_s}{r}-E\right)G_0(\vec{r},\vec{r}\,',E)=\delta^{(3)}(\vec{r}-\vec{r}\,')$$

- NLO, NNLO, ... ➪ perturbation theory in momentum space:

$$G(E)=\int\frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\frac{d^{d-1}\vec{p}\,'}{(2\pi)^{d-1}}\Bigg[G_0(\vec{p},\vec{p}\,';E)$$
$$+\int\frac{d^{d-1}\vec{p}_1}{(2\pi)^{d-1}}\frac{d^{d-1}\vec{p}_1^{\,\prime}}{(2\pi)^{d-1}}G_0(\vec{p},\vec{p}_1;E)\,\delta V(\vec{p}_1,\vec{p}_1^{\,\prime})\,G_0(\vec{p}_1^{\,\prime},\vec{p}\,';E)+\ldots\Bigg]$$
$$+\delta^{\rm us}G(E)$$



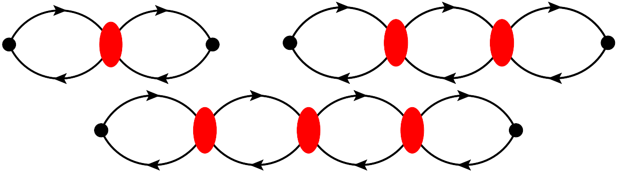
Ingredients to $G(E)$

Review: [Beneke,Kiyo,Schuller'13]
Review: [Beneke,Kiyo,Schuller in progress]

- LO: Coulomb solution

$$\left(-\frac{\Delta}{m_t}+\frac{C_F\alpha_s}{r}-E\right)G_0(\vec{r},\vec{r}\,',E)=\delta^{(3)}(\vec{r}-\vec{r}\,')$$

- NLO, NNLO, ... ⇨ perturbation theory in momentum space:

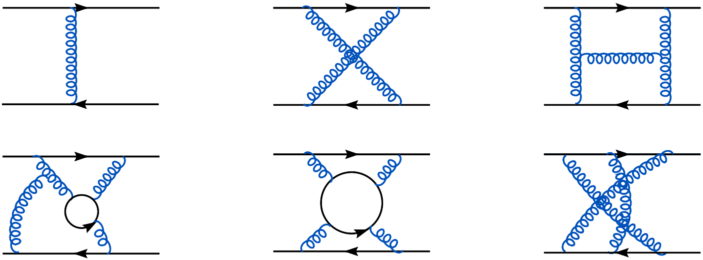


- δV

Static potential:	$V_C=-\frac{4\pi C_F\alpha_s}{\vec{q}^2}$	3 loops
$1/m$ potential:	$V_{1/m}=\frac{\pi^2 C_F\alpha_s^2}{m\|\vec{q}\|}$	2 loops
"Breit" potential:	$\propto 1/m^2$	1 loop

Static potential

$$V_C = -\frac{4\pi C_F \alpha_s(|\vec{q}\,|)}{|\vec{q}\,|^2}\left[1 + \frac{\alpha_s(|\vec{q}\,|)}{4\pi} a_1 + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^3 \left(a_3 + 8\pi^2 C_A^3 \ln\frac{\mu^2}{\vec{q}^{\,2}}\right) + \cdots\right]$$



[Appelquist,Politzer'75,Susskind'77] [Fischler'77;Biloire'80] [Peter'96;Schröder'98]

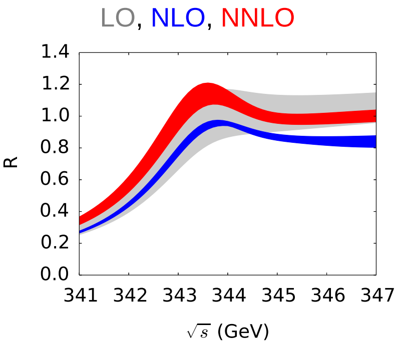
[Smirnov,Smirnov,Steinhauser'08; Smirnov,Smirnov,Steinhauser'09; Anzai,Kiyo,Sumino'09]

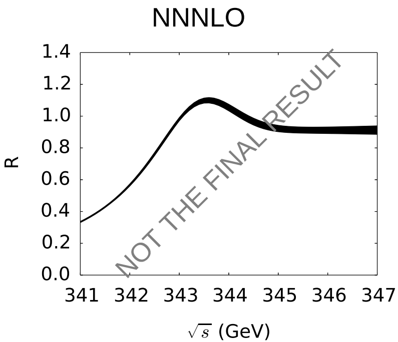
$1/m$ potential

$$V_{1/m} = \frac{\pi^2 C_F \alpha_s^2}{m|\vec{q}|}$$

- needed up to 2 loops [Kniehl,Penin,Smirnov,Steinhauser'01]
- $\mathcal{O}(\epsilon)$ term needed at 2 loops [Beneke,Kiyo,Marquard,Penin,Seidel,Steinhauser'14]

$\sigma_{\mathrm{tot}}(e^+e^- \to t\bar{t}\,)$



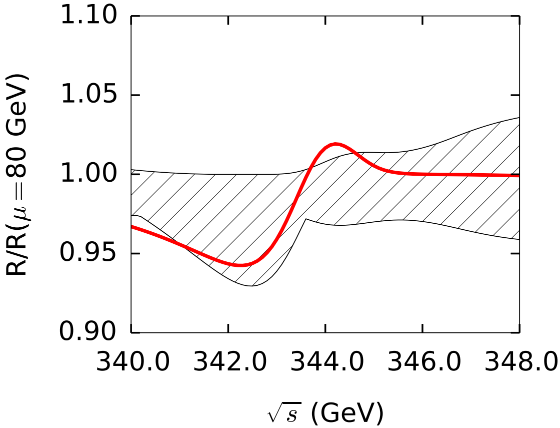


50 GeV $\leq \mu \leq$ 350 GeV

$m_t^{\mathrm{PS}} = 171.3$ GeV

[Beneke et al.]

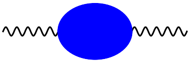
Parameter variation:
$m_t^{PS} \to m_t^{PS} + 50$ MeV



[Beneke et al.]

Residue Z_t

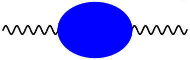
$$\left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right)\ \Pi(q^2) = i \int \mathrm{d}x\, e^{iqx} \langle 0|T j_\mu(x) j_\nu^\dagger(0)|0\rangle$$



$$\Pi(q^2) \stackrel{E\to E_n}{=} \frac{N_c}{2m_t^2} \frac{Z_n}{E_n - (E + i0)} + \ldots \qquad Z_t = \left[c_v^2 - \frac{E_1}{m_t} c_v \left(c_v + \frac{d_v}{3}\right)\right] |\psi_1(0)|^2$$

Residue Z_t

$$\left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right)\ \Pi(q^2) = i \int \mathrm{d}x\, e^{iqx} \langle 0 | T j_\mu(x) j_\nu^\dagger(0) | 0 \rangle$$



$$\Pi(q^2) \overset{E \to E_n}{=} \frac{N_c}{2m_t^2} \frac{Z_n}{E_n - (E + i0)} + \ldots \qquad Z_t = \left[c_v^2 - \frac{E_1}{m_t} c_v \left(c_v + \frac{d_v}{3}\right)\right] |\psi_1(0)|^2$$

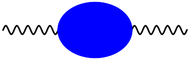
$$\begin{aligned}
Z_t &= \frac{(C_F m_t^{\mathrm{PS}} \alpha_s)^3}{8\pi} \left[1 + (-2.131 + 3.661 L)\, \alpha_s \right. \\
&\quad + \left(8.38 + 1.27 x_f - 7.26 \ln \alpha_s - 13.40 L + 8.93 L^2\right) \alpha_s^2 \\
&\quad + \left(5.46 + (-2.23 + 0.78 L_f)\, x_f + 2.21_{a_3} + 21.48_{b_2 \epsilon}\right. \\
&\quad + 37.53_{c_f} - 134.8(0.1)_{c_g} + (-9.79 - 44.27 L) \ln \alpha_s - 16.35 \ln^2 \alpha_s \\
&\quad \left.\left. + (53.17 + 4.66 x_f)\, L - 48.18 L^2 + 18.17 L^3\right) \alpha_s^3 + \mathcal{O}(\alpha_s^4)\right] \\
&= \frac{(C_F m_t^{\mathrm{PS}} \alpha_s)^3}{8\pi} \left[1 - 2.13 \alpha_s + 23.66 \alpha_s^2 - 113.0(0.1) \alpha_s^3 + \mathcal{O}(\alpha_s^4)\right]
\end{aligned}$$

$$x_f = \mu_f / (m_t^{\mathrm{PS}} \alpha_s), \quad L = \ln\left(\mu / (m_t^{\mathrm{PS}} C_F \alpha_s)\right), \quad L_f = \ln(\mu^2 / \mu_f^2)$$

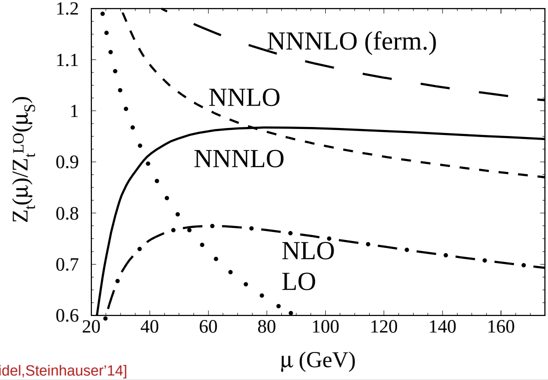
[Marquard,Piclum,Seidel,Steinhauser'14]

Residue Z_t

$$\left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right)\, \Pi(q^2) = i \int \mathrm{d}x\, e^{iqx} \langle 0|T j_\mu(x) j_\nu^\dagger(0)|0\rangle$$



$$\Pi(q^2) \overset{E\to E_n}{=} \frac{N_c}{2m_t^2} \frac{Z_n}{E_n - (E + i0)} + \dots \qquad Z_t = \left[c_v^2 - \frac{E_1}{m_t} c_v \left(c_v + \frac{d_v}{3}\right)\right] |\psi_1(0)|^2$$



[Marquard,Piclum,Seidel,Steinhauser'14]

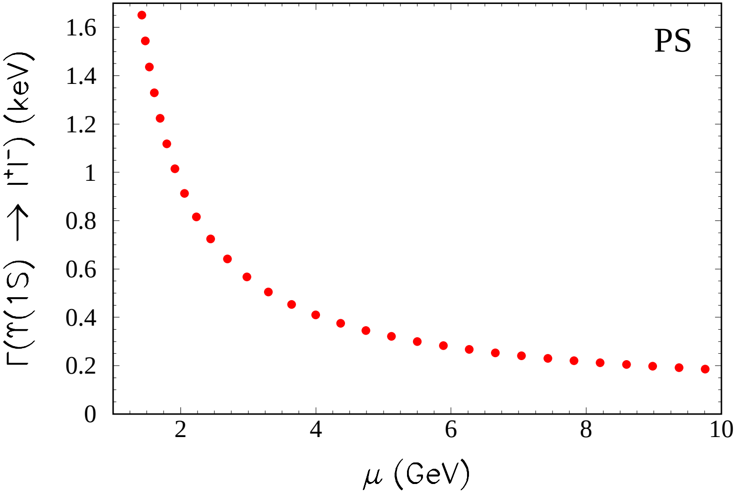
$$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$$

$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$

- $\Upsilon(1S)$ meson: simplest heavy quark bound state
- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)_{\rm exp} = 1.340(18)$ keV
- $$\boxed{\Gamma(\Upsilon(1S) \to \ell^+\ell^-) = \frac{4\pi\alpha^2}{9m_b^2} \left|\psi_1(0)\right|^2 c_v \left[c_v - \frac{E_1}{m_b}\left(c_v + \frac{d_v}{3}\right) + \ldots\right]}$$
- NLL: [Pineda'01]; NNLL-approx: [Pineda,Signer'07]; ...
- here: complete NNNLO

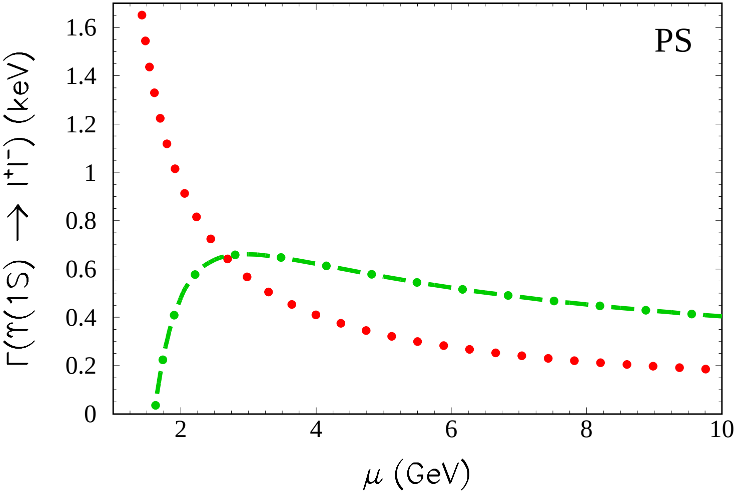
$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$: μ dependence

[Beneke,Kiyo,Marquard,Penin,Seidel,Steinhauser'14]



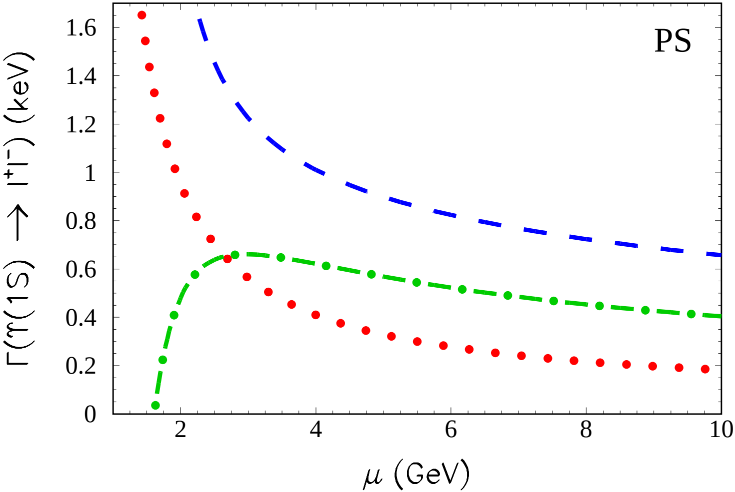
$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$: μ dependence

[Beneke,Kiyo,Marquard,Penin,Seidel,Steinhauser'14]



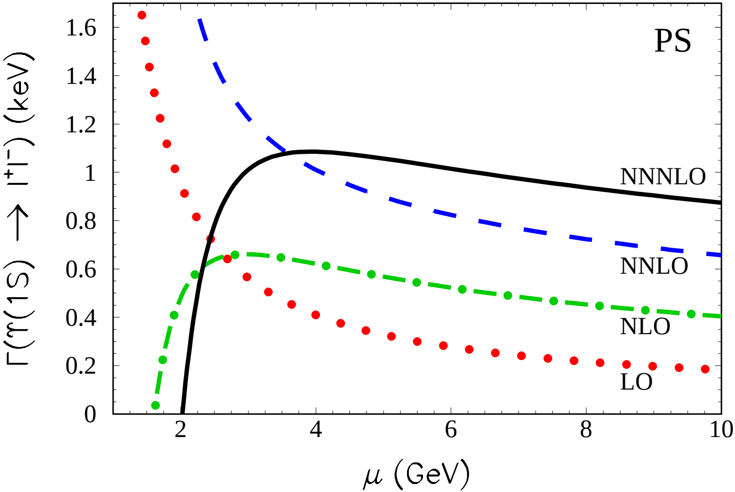
$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$: μ dependence

[Beneke,Kiyo,Marquard,Penin,Seidel,Steinhauser'14]



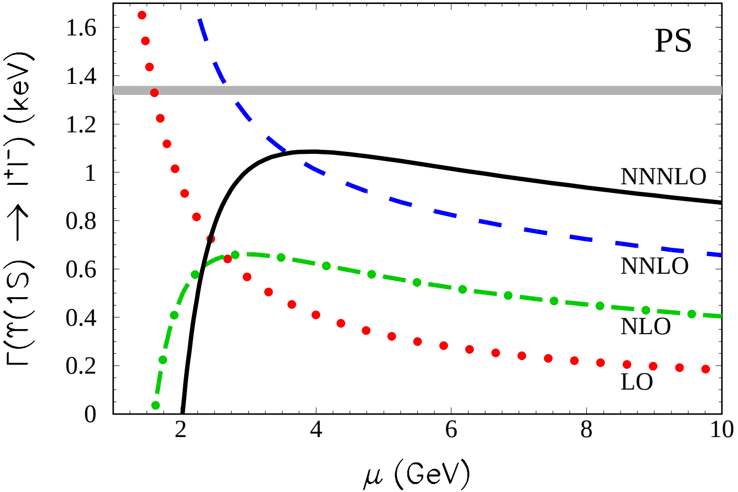
$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$: μ dependence

[Beneke,Kiyo,Marquard,Penin,Seidel,Steinhauser'14]

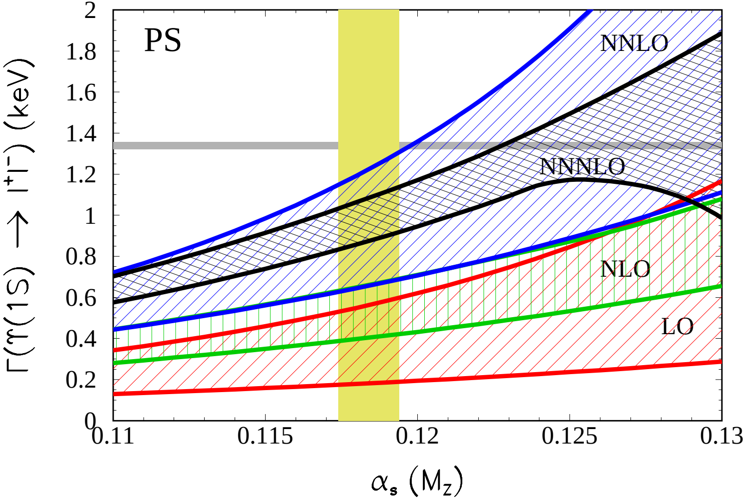


$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$: μ dependence

[Beneke,Kiyo,Marquard,Penin,Seidel,Steinhauser'14]



$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$: α_s dependence



Result

- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)|_{\rm pole} = [1.04 \pm 0.04(\alpha_s)^{+0.02}_{-0.15}(\mu)]$ keV
- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)|_{\rm PS} = [1.08 \pm 0.05(\alpha_s)^{+0.01}_{-0.20}(\mu)]$ keV

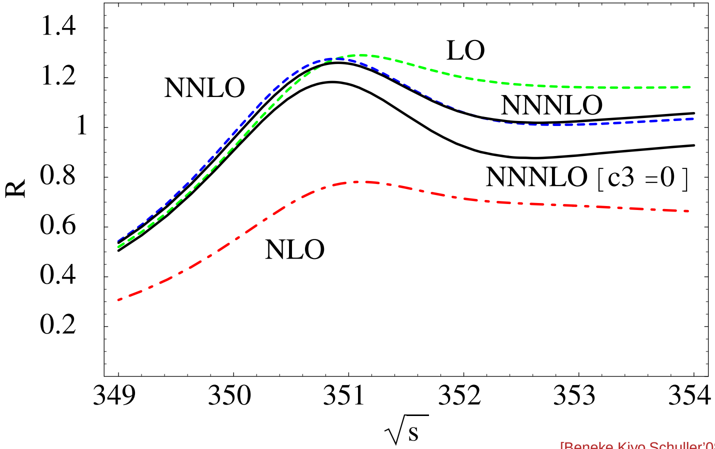
- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)|_{\rm exp} = [1.340 \pm 0.018]$ keV

- third-order perturbative result is $\approx 30\%$ too low
- possible explanation: sizeable non-perturbative contribution

Conclusions

- 3-loop matching coefficient c_v
- NNNLO corrections to $G(E)$
- $\sigma(e^+e^- \to \text{hadrons})$ to N^{3}LO
- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$ to N^{3}LO

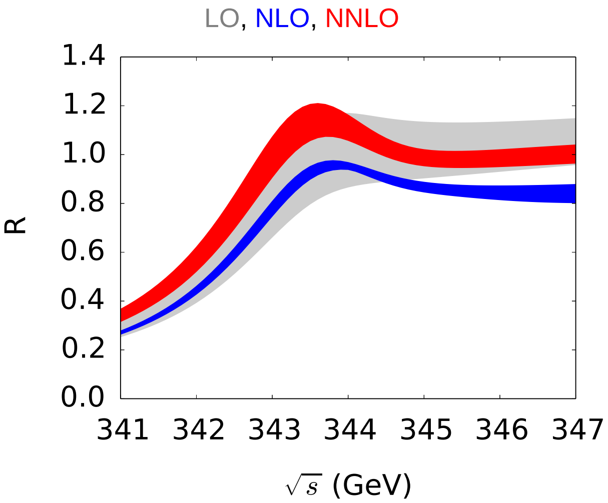
$e^+e^- \to t\bar{t}$ @ threshold



[Beneke,Kiyo,Schuller'08]

Only $c_v^{(3),n_l}$ included

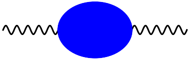
Application: height of $\sigma(e^+e^- \to t\bar{t})$



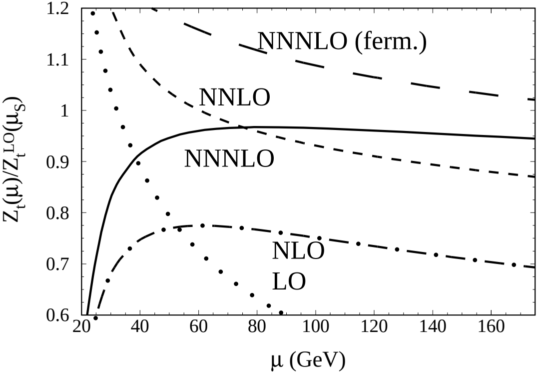
[Hoang,Beneke,Melnikov,Nagano,Ota,Penin,Pivovarov,Signer,Smirnov,Sumino,Teubner,Yakovlev,Yelkhovsky'00]

Application: height of $\sigma(e^+e^- \to t\bar{t})$

$$\left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right)\, \Pi(q^2) = i \int \mathrm{d}x\, e^{iqx} \langle 0|T j_\mu(x) j_\nu^\dagger(0)|0\rangle$$

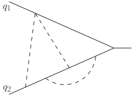


$$\Pi(q^2) \overset{E\to E_n}{=} \frac{N_c}{2m_Q^2} \frac{Z_n}{E_n - (E + i0)} + \ldots \qquad Z_t = \left[c_v^2 - \frac{E_1}{m_t} c_v \left(c_v + \frac{d_v}{3}\right)\right] |\psi_1(0)|^2$$



Some technical details ...

- automatic generation of Feynman diagrams, apply projector, reduce scalar products in numerator, ... ➪ (many) scalar integrals
- reduction to $\approx$ 100 master integrals with `CRUSHER` [Marquard,Seidel]
- compute MIs with `FIESTA` [Smirnov,Tentyukov'08; Smirnov,Smirnov,Tentyukov'11; Smirnov'13]



$$= \frac{e^{3\epsilon\gamma_E}}{m_Q^4}\left(\frac{\mu^2}{m_Q^2}\right)^{3\epsilon}$$

$$\left(\frac{0.411236(3)}{\epsilon^2} + \frac{3.4860(1)}{\epsilon} + 34.520(2) + 339.68(4)\epsilon + \mathcal{O}(\epsilon^2)\right)$$

- simpler integrals also known analytically
- add uncertainties of individual integrals in quadrature
- multiply final uncertainty by factor 5 ("5σ")

Checks

- finiteness
- n_l contribution
- gauge parameter independence (ξ^1)
- change basis of MIs

coefficient	default basis	alternative basis
$[C_F^3]$ c_{FFF}	$36.55(0.11)$	$36.61(2.93)$
$[C_F^2 C_A]$ c_{FFA}	$-188.10(0.17)$	$-188.04(2.91)$
$[C_F C_A^2]$ c_{FAA}	$-97.81(0.08)$	$-97.76(2.05)$
$c_v^{(3)}$ $(n_l = 4)$	$-1621.7(0.4)$	$-1621(23)$
$c_v^{(3)}$ $(n_l = 5)$	$-1508.4(0.4)$	$-1507(23)$

(factor 5 not yet included)

$\tilde{Z}_v$

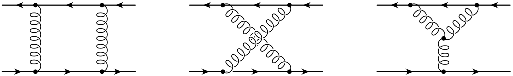
$$
\begin{aligned}
\tilde{Z}_v \;=\;& 1 + \left(\frac{\alpha_s^{(n_l)}(\mu)}{\pi}\right)^2 \frac{C_F\pi^2}{\epsilon}\left(\frac{1}{12}C_F + \frac{1}{8}C_A\right) + \left(\frac{\alpha_s^{(n_l)}(\mu)}{\pi}\right)^3 C_F\pi^2 \\
&\times \Bigg\{ C_F^2\left[\frac{5}{144\epsilon^2} + \left(\frac{43}{144} - \frac{1}{2}\ln 2 + \frac{5}{48}L_\mu\right)\frac{1}{\epsilon}\right] \\
&+ C_F C_A\left[\frac{1}{864\epsilon^2} + \left(\frac{113}{324} + \frac{1}{4}\ln 2 + \frac{5}{32}L_\mu\right)\frac{1}{\epsilon}\right] \\
&+ C_A^2\left[-\frac{1}{16\epsilon^2} + \left(\frac{2}{27} + \frac{1}{4}\ln 2 + \frac{1}{24}L_\mu\right)\frac{1}{\epsilon}\right] \\
&+ Tn_l\left[C_F\left(\frac{1}{54\epsilon^2} - \frac{25}{324\epsilon}\right) + C_A\left(\frac{1}{36\epsilon^2} - \frac{37}{432\epsilon}\right)\right] \\
&+ C_F T n_h \frac{1}{60\epsilon}\Bigg\} + \mathcal{O}(\alpha_s^4)
\end{aligned}
$$

$$L_\mu = \ln \mu^2/m^2$$

$$\boxed{Z_2\Gamma_v = c_v \tilde{Z}_2 \tilde{Z}_v^{-1}\tilde{\Gamma}_v}$$

- analytic result from [Beneke,Signer,Smirnov'98; Kniehl,Penin,Smirnov,Steinhauser'02; Marquard,Piclum,Seidel,Steinhauser'06; Beneke,Kiyo,Penin'07]
- (numerical) agreement better than 1%

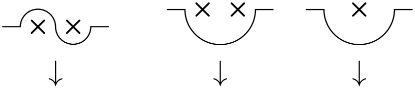
$1/m$ potential: 1-loop calculation



fermion propagator: $$\frac{1}{p_0 - \frac{\vec{p}^{\,2}}{2m} \pm i\epsilon}$$

- soft region: $(p_0, |\vec{p}\,|) \sim (mv, mv)$ ➩ expand in $\frac{\vec{p}^{\,2}}{2m}$
  potential region: $(p_0, |\vec{p}\,|) \sim (mv^2, mv)$ ➩ no expansion allowed

- poles in p_0 plane:



no expansion possible

expansion possible

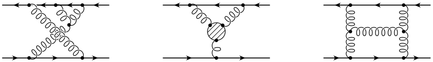
$1/m$ potential: 2 loops

[Kniehl,Penin,Smirnov,Steinhauser'01; Penin,Smirnov,Steinhauser'13; Beneke,Kiyo,Marquard,Penin,Seidel,Steinhauser'14]

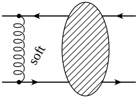
classify according to colour factor: $C_F^2 C_A$, $C_F^2 T n_l$, $C_F C_A^2$, $C_F C_A T n_l$

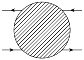
- "most non-abelian": $C_F C_A^2$, $C_F C_A T n_l$

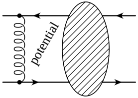
  ➪ expansion possible

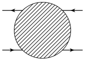


- "reducible" $C_F^2 C_A$, $C_F^2 T n_l$



➪ expand  up to $1/m$ ➪ soft integration



➪ expand  up to $1/m^2$ ➪ potential integration $\sim m$

- for $G(E)$ to NNNLO: $\mathcal{O}(\epsilon)$ part is needed at 2 loops

$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$

- $\Upsilon(1S)$ meson: simplest heavy quark bound state
- $\Gamma(\Upsilon(1S) \to \ell^+\ell^-)_{\rm exp} = 1.340(18)$ keV
- $\boxed{\Gamma(\Upsilon(1S) \to \ell^+\ell^-) = \frac{4\pi\alpha^2}{9m_b^2} \left|\psi_1(0)\right|^2 c_v \left[c_v - \frac{E_1}{m_b}\left(c_v + \frac{d_v}{3}\right) + \ldots\right]}$
- d_v: matching constant of sub-leading $b\bar{b}$ current
- $\psi_1(0)$ wave function of the $(b\bar{b})$ system $\quad \left|\psi_1^{\rm LO}(0)\right|^2 = \frac{8m_b^3\alpha_s^3}{27\pi}$
- $M_{\Upsilon(1S)} = 2m_b + E_1 \quad E_1^{p,{\rm LO}} = -(4m_b\alpha_s^2)/9$
- many building blocks necessary; most recent ones:
  - c_v [Marquard,Piclum,Seidel,Steinhauser'14]
  - $\psi_1(0)$: ultrasoft contribution [Beneke,Kiyo,Penin'07]
  - $\psi_1(0)$: single- and double-potential insertions [Beneke,Kiyo,Schuller'08'13]
  - $\mathcal{O}(\epsilon)$ term of 2-loop $1/(m_b r^2)$ pNRQCD potential [Peinin,Smirnov,Steinhauser'13]
- NLL: [Pineda'01]; NNLL-approx: [Pineda,Signer'07]; ...

Result

$$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)|_{\rm pole} =$$

$$\frac{2^5\alpha^2\alpha_s^3 m_b}{3^5}\Big[1 + \alpha_s\left(-2.003 + 3.979\,L\right) + \alpha_s^2\left(9.05 - 7.44\,\ln\alpha_s - 13.95\,L + 10.55\,L^2\right)$$

$$+ \alpha_s^3\left(-0.91 + 4.78_{a_3} + 22.07_{b_2\epsilon} + 30.22_{c_f} - 134.8(1)_{c_g} - 14.33\,\ln\alpha_s - 17.36\,\ln^2\alpha_s\right.$$

$$\left.+ \left(62.08 - 49.32\,\ln\alpha_s\right)L - 55.08\,L^2 + 23.33\,L^3\right) + \mathcal{O}(\alpha_s^4)\Big]$$

$$\stackrel{\mu=3.5\text{ GeV}}{=} \frac{2^5\alpha^2\alpha_s^3 m_b}{3^5}\Big[1 + 1.166\alpha_s + 15.2\alpha_s^2 + \left(66.5 + 4.8_{a_3}\right.$$

$$\left.+ 22.1_{b_2\epsilon} + 30.2_{c_f} - 134.8(1)_{c_g}\right)\alpha_s^3 + \mathcal{O}(\alpha_s^4)\Big]$$

$$= \frac{2^5\alpha^2\alpha_s^3 m_b^{\rm pole}}{3^5}\left[1 + 0.28 + 0.88 - 0.16\right]$$

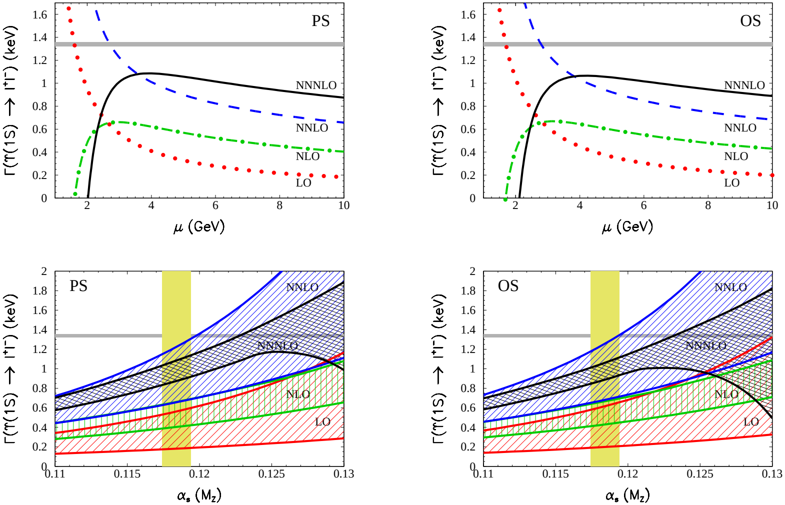
$$= \left[1.04 \pm 0.04(\alpha_s)^{+0.02}_{-0.15}(\mu)\right]\text{ keV}$$

$$= \frac{2^5\alpha^2\alpha_s^3 m_b^{\rm PS}}{3^5}\left[1 + 0.37 + 0.95 - 0.04\right]$$

$$= \left[1.08 \pm 0.05(\alpha_s)^{+0.01}_{-0.20}(\mu)\right]\text{ keV}$$

$$L = \ln\left[\mu/(m_b C_F \alpha_s)\right],\ C_F = 4/3$$

$\Gamma(\Upsilon(1S) \to \ell^+\ell^-)$: PS vs. OS



Non-perturbative contribution

Karlsruhe Institute of Technology

1. $\delta_{\rm np}|\psi_1(0)|^2 = |\psi_1^{\rm LO}(0)|^2 \times 17.54\pi^2 K$ [Leutwyler'81,Voloshin'82]

- $K = \frac{\langle \frac{\alpha_s}{\pi} G^2 \rangle}{m_b^4 (\alpha_s C_F)^6}$
- $\langle \frac{\alpha_s}{\pi} G^2 \rangle = 0.012$ GeV^4 ➪ $\delta_{\rm np}\Gamma_{\ell\ell}(\Upsilon(1S)) = 1.67_{\rm pole}/2.20_{\rm PS}$ keV

? value of $\langle \frac{\alpha_s}{\pi} G^2 \rangle$
? scale of α_s
? dimension-6 condensate contribution [Pineda'97]

Non-perturbative contribution

1. $\delta_{\rm np}|\psi_1(0)|^2 = |\psi_1^{\rm LO}(0)|^2 \times 17.54\pi^2 K$ [Leutwyler'81,Voloshin'82]
   - $K = \frac{\langle \frac{\alpha_s}{\pi} G^2 \rangle}{m_b^4(\alpha_s C_F)^6}$
2. $M_{\Upsilon(1S)} = 2m_b + E_1^{\rm p} + \frac{624\pi^2}{425} m_b(\alpha_s C_F)^2 K$
   - use PS scheme ➪ perturbation theory converges [Beneke,Kiyo,Schuller'05]
   - charm effects [Hoang'00]

➪ $\delta_{\rm np}\Gamma_{\ell\ell}(\Upsilon(1S)) = \frac{4\alpha^2\alpha_s}{9} \frac{17.54\times 425}{3744} \delta M_{\Upsilon(1S)}^{\rm np}$

$\approx [1.28^{+0.17}_{-0.18}(\alpha_s) \pm 0.42(m_b)^{+0.20}_{-0.57}(\mu) \pm 0.12(m_c)]$ keV

? $m_b^{\overline{\rm MS}} = 4.163$ GeV $\to 4.203$ GeV ➪ $\delta_{\rm np}\Gamma_{\ell\ell}(\Upsilon(1S)) \approx 0.3$ keV

Non-perturbative contribution

1. $\delta_{\rm np}|\psi_1(0)|^2 = |\psi_1^{\rm LO}(0)|^2 \times 17.54\pi^2 K$ [Leutwyler'81,Voloshin'82]
   - $K = \frac{\langle \frac{\alpha_s}{\pi} G^2 \rangle}{m_b^4 (\alpha_s C_F)^6}$
2. $M_{\Upsilon(1S)} = 2m_b + E_1^{\rm p} + \frac{624\pi^2}{425} m_b (\alpha_s C_F)^2 K$
   - use PS scheme ➪ perturbation theory converges [Beneke,Kiyo,Schuller'05]
   - charm effects [Hoang'00]

➪ $\delta_{\rm np}\Gamma_{\ell\ell}(\Upsilon(1S)) = \frac{4\alpha^2\alpha_s}{9} \frac{17.54\times 425}{3744} \delta M^{\rm np}_{\Upsilon(1S)}$

$\approx [1.28^{+0.17}_{-0.18}(\alpha_s) \pm 0.42(m_b)^{+0.20}_{-0.57}(\mu) \pm 0.12(m_c)]\,\text{keV}$

? $m_b^{\overline{\rm MS}} = 4.163\text{ GeV} \to 4.203\text{ GeV}$ ➪ $\delta_{\rm np}\Gamma_{\ell\ell}(\Upsilon(1S)) \approx 0.3\text{ keV}$

Summary: perturbation theory: solid prediction

non-perturbative contribution: unclear