Hard scattering cross sections in QCD

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Helmholtz Alliance Physics at the Terascale School on Parton Distribution Functions, Nov 13, 2008, Zeuthen

Highest energies at colliders until 201x

Energy frontier

Search for Higgs boson, new massive particles at highest energies

$E = m c^2$

Hadron colliders

- Proton–(anti)-proton collisions reach TeV-scale
 - Tevatron $\sqrt{S} = 1.96$ TeV (until 2009), LHC with $\sqrt{S} = 14$ TeV
- Proton: composite multi-particle bound state
 - collider: "wide-band beams" of quarks and gluons
- Protons are heavy
 - no significant synchrotron radiation \sim

$$\left(\frac{E}{m}\right)^4/r$$





Higgs production at LHC



 Branching ratios for decay of Standard Model Higgs

- High-multiplicity final states
 - typical SM process is accompanied by radiation of multiple jets
- Example: Higgs-strahlung
 - channel $q\bar{q} \rightarrow W(Z)H$ (third largest rate at LHC)

• dominant decay $H \rightarrow b\bar{b}$



Supersymmetry

- Pair-production of supersymmetric particles (*R*-parity)
 - lightest supersymmetric particle (LSP) must be absolutely stable
- MSSM spectrum
 - typical signature: multiple jets, leptons and missing energy
- Example: neutralino production $\tilde{N}_{1,2}^0$
 - electric and color-neutral (dark matter candidate)



Large extra dimensions

- Spectrum of first Kaluza-Klein excitations
 - effective mass \simeq (compactification radius)⁻¹, $m^{(n)} \simeq 1/R$
- Pair-production of excited KK-modes in interactions
 - phenomenology: missing energy in subsequent chain decays





Theoretical predictions for the LHC

Challenge

Solve master equation

new physics = data – Standard Model

- New physics searches require understanding of SM background
- LHC explores the energy frontier
 - theory has to match or exceed accuracy of LHC data

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Tools

- LHC is a QCD machine
 - perturbative QCD is essential and established part of toolkit (we no longer "test" QCD)
- Electroweak corrections important for precision predictions

- Large rates expected for many Standard Model processes
- $\sigma_W \sim 150~{
 m nb}$

■ $BR(W \rightarrow e + \mu) \sim 20\%$ 10 fb⁻¹ gives 300M leptonic events rate(10³³ cm⁻² s⁻¹) ~ 30 Hz rate(10³⁴ cm⁻² s⁻¹) ~ 300 Hz



- Large rates expected for many Standard Model processes
- $\sigma_Z \sim 50~{
 m nb}$

• $BR(W \rightarrow ee + \mu\mu) \sim 6.6\%$ 10 fb⁻¹ gives 33M leptonic events rate(10³³ cm⁻² s⁻¹) ~ 3.5 Hz rate(10³⁴ cm⁻² s⁻¹) ~ 35 Hz



- Large rates expected for many Standard Model processes
- $\sigma_{tar{t}}\sim 800~{
 m pb}$

■ $BR(W \rightarrow e + \mu) \sim 30\%$ 10 fb⁻¹ gives 2.4M leptonic events rate(10³³ cm⁻² s⁻¹) ~ 0.2 Hz rate(10³⁴ cm⁻² s⁻¹) ~ 2 Hz



- Large rates expected for many Standard Model processes
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- New physics signals
 - cross section predictions $\sigma_{\rm new \ physics} \sim \mathcal{O}(1-10) \ {\rm pb}$
 - superpartners in MSSM (neutralinos, charginos, squarks, gluinos, ...), KK modes
 - searches often assume 100 fb $^{-1}$



Perturbative QCD at colliders

- Hard hadron-hadron scattering
 - constituent partons from each incoming hadron interact at short



separate sensitivity to dynamics from different scales

$$\sigma_{pp\to X} = \sum_{ijk} f_i(\mu^2) \otimes f_j(\mu^2) \otimes \hat{\sigma}_{ij\to k} \left(\alpha_s(\mu^2), Q^2, \mu^2 \right) \otimes D_{k\to X}(\mu^2)$$

• factorization scale μ , subprocess cross section $\hat{\sigma}_{ij \to k}$ for parton types *i*, *j* and hadronic final state *X*

Hard scattering cross section

- Standard approach to uncertainties in theoretical predictions
 - variation of factorization scale μ : $\frac{d}{d \ln \mu^2} \sigma_{pp \to X} = \mathcal{O}(\alpha_s^{l+1})$

$$\sigma_{pp\to X} = \sum_{ijk} f_i(\mu^2) \otimes f_j(\mu^2) \otimes \hat{\sigma}_{ij\to k} \left(\alpha_s(\mu^2), Q^2, \mu^2 \right) \otimes D_{k\to X}(\mu^2)$$

- Parton cross section $\hat{\sigma}_{ij \rightarrow k}$ calculable pertubatively in powers of α_s
 - constituent partons from incoming protons interact at short distances of order $\mathcal{O}(1/Q)$
- Parton luminosity $f_i \otimes f_j$
 - convolution of parton distribution functions
 - quarks/gluons carry fraction x of proton momentum
- Final state X: hadrons, mesons, jets, ...
 - fragmentation function $D_{k\to X}(\mu^2)$ or jet algorithm
 - interface with showering algorithms (Monte Carlo)



Accuracy of perturbative predictions for σ_{had}

- LO (leading order)
 - Automated tree level calculations in Standard Model, MSSM, ... (Madgraph, Sherpa, Alpgen, CompHEP, ...)
 - LO + parton shower
 - String inspired techniques
- NLO (next-to-leading order)
 - Analytical (or numerical) calculations of diagrams yield parton level Monte Carlos (NLOJET++, MCFM, ...)
 - NLO + parton shower (MC@NLO, VINCIA)
- NNLO (next-to-next-to-leading order)
 - selected results known (mostly inclusive kinematics)
- N³LO (next-to-next-to-next-to-leading order)
 - very few

Inelastic electron-proton scattering



• Virtuality of photon: resolution $Q^2 \equiv -q^2 = 4EE' \sin^2(\theta/2)$

• Bjorken variable: inelasticity $x = \frac{Q^2}{2P \cdot q} < 1$

• Cross section (X inclusive): proton structure function F_i^p

$$(E - E')\frac{d\sigma}{d\Omega \, dE'} \stackrel{\text{lab}}{=} \frac{\alpha^2 \cos^2 \frac{\theta}{2}}{4E^2 \sin^4 \frac{\theta}{2}} \left\{ F_2^p(x, Q^2) + \tan^2 \frac{\theta}{2} F_1^p(x, Q^2) \right\}$$

Mott-scattering (point-like)

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Mott-scattering (point-like)

■ Deep-inelastic scattering (Bjorken limit: $Q^2 \rightarrow \infty$ and x fixed) Parton modell (quasi-free point-like constituents, incoherence)

 $F_2(x,Q^2) \simeq F_2(x) = \sum_{i} e_i^2 x f_i(x)$

• $xf_i(x)$ distribution for momentum fraction x of parton i

QCD corrections in deep-inelastic scattering



- Structure function F_2 (up to terms $\mathcal{O}(1/Q^2)$)
 - Renormalization/factorization scale $\mu = \mathcal{O}(Q)$

$$x^{-1}F_2^{p}(x,Q^2) = \sum_i \int_x^1 \frac{d\xi}{\xi} c_{2,i}\left(\frac{x}{\xi}, \alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) f_i^{p}(\xi,\mu^2)$$

Coefficient functions ca

$$c_a = \alpha_s^{n_a} \left[c_a^{(0)} + \alpha_s c_a^{(1)} + \alpha_s^2 c_a^{(2)} + \dots \right]$$

NLO: standard approximation (large uncertainties)

Radiative corrections in a nutshell

- Leading order
 - partonic structure function

$$\hat{F}_{2,q}^{(0)} = e_q^2 \delta(1-x)$$





- virtual correction (infrared divergent; proportional to Born)
- dimensional regularization $D = 4 2\epsilon$

$$\hat{F}_{2,q}^{(1),v} = e_q^2 C_F \frac{\alpha_s}{4\pi} \,\delta(1-x) \,\left(\frac{\mu^2}{Q^2}\right)^\epsilon \,\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \zeta_2 + \mathcal{O}(\epsilon)\right)$$

Next-to-leading order



- add real and virtual corrections $\hat{F}_{2,q}^{(1)} = \hat{F}_{2,q}^{(1),r} + \hat{F}_{2,q}^{(1),v}$
- collinear divergence remains splitting functions $P_{qq}^{(0)}$

 $+\mathcal{O}(\epsilon)$

$$\hat{F}_{2,q}^{(1)} = e_q^2 C_F \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left\{ \frac{1}{\epsilon} \left(\frac{4}{1-x} - 2 - 2x + 3\delta(1-x)\right) + 4\frac{\ln(1-x)}{1-x} - 3\frac{1}{1-x} - (9 + 4\zeta_2)\delta(1-x) - 2(1+x)(\ln(1-x) - \ln(x)) - 4\frac{1}{1-x}\ln(x) + 6 + 4x + 4x + 3\beta \right\}$$

Hard scattering cross sections in QCD - p.16

- Structure of NLO correction
 - absorb collinear divergence $P_{qq}^{(0)}$ in renormalized parton distributions

$$\hat{F}_{2,q}^{(1),bare} = e_q^2 \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left\{ \frac{1}{\epsilon} P_{qq}^{(0)}(x) + c_{2,q}^{(1)}(x) + \mathcal{O}(\epsilon) \right\}$$

$$q^{ren}(\mu_F^2) = q^{bare} - e_q^2 \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} P_{qq}^{(0)}(x) \left(\frac{\mu^2}{\mu_F^2}\right)^{\epsilon}$$

• partonic (physical) structure function at factorization scale μ_F

$$\hat{F}_{2,q} = e_q^2 \left(\delta(1-x) + \frac{\alpha_s}{4\pi} \left\{ c_{2,q}^{(1)}(x) - \ln\left(\frac{Q^2}{\mu_F^2}\right) P_{qq}^{(0)}(x) \right\} \right)$$

Parton distributions in proton

- Evolution equations for parton distributions f_i
 - splitting functions $P = \alpha_s P^{(0)} + \alpha_s^2 P^{(1)} + \alpha_s^3 P^{(2)} + \dots$ (calculable in perturbative QCD)

$$\frac{d}{d\ln\mu^2} f_i(x,\mu^2) = \sum_k \left[P_{ik}(\alpha_s(\mu^2)) \otimes f_k(\mu^2) \right](x)$$

parton splitting in leading order











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parton splitting in leading order











Parton luminosity at LHC

- Precision HERA data on F₂ covers most of the LHC *x*-range
- Scale evolution of PDFs in Q over two to three orders
- Sensitivity at LHC
 - 100 GeV physics: small-x, sea partons
 - TeV scales: large-x
 - rapidity distributions probe extreme *x*-values
- Stable evolution in QCD
 - splitting functions to NNLO
 S.M. Vermaseren, Vogt '04



Our calculation in deep-inelastic scattering



Our calculation in deep-inelastic scattering



- more than 10 FTE years and a few CPU years
 - computer algebra updates: \rightarrow 3.1 \rightarrow 3.2 \rightarrow ...
 - $> 10^5$ tabulated symbolic integrals (> 3GB)

Sven-Olaf Moch

Splitting functions for a quarter of a century

$$\begin{split} P_{\rm ns}^{(0)}(x) &= C_F(2p_{\rm qq}(x) + 3\delta(1-x)) \\ P_{\rm ps}^{(0)}(x) &= 0 \\ P_{\rm qg}^{(0)}(x) &= 2n_f p_{\rm qg}(x) \\ P_{\rm gq}^{(0)}(x) &= 2C_F p_{\rm gq}(x) \\ P_{\rm gg}^{(0)}(x) &= C_A \Big(4p_{\rm gg}(x) + \frac{11}{3}\delta(1-x) \Big) - \frac{2}{3}n_f \delta(1-x) \end{split}$$

1973



"for the discovery of asymptotic freedom in the theory of the strong interaction"





David J. Gross

H. David Politzer Frank V

Frank Wilczek

 $P_{\rm ns}^{(1)+}(x) = 4C_A C_F \left(p_{\rm qq}(x) \left[\frac{67}{18} - \zeta_2 + \frac{11}{6} H_0 + H_{0,0} \right] + p_{\rm qq}(-x) \left[\zeta_2 + 2H_{-1,0} - H_{0,0} \right] \right]$ $+\frac{14}{3}(1-x)+\delta(1-x)\left[\frac{17}{24}+\frac{11}{3}\zeta_{2}-3\zeta_{3}\right]-4C_{F}n_{f}\left(p_{qq}(x)\left[\frac{5}{9}+\frac{1}{3}H_{0}\right]+\frac{2}{3}(1-x)\right)$ $+\delta(1-x)\left[\frac{1}{12}+\frac{2}{2}\zeta_{2}\right]+4C_{F}^{2}\left(2p_{qq}(x)\left[H_{1,0}-\frac{3}{4}H_{0}+H_{2}\right]-2p_{qq}(-x)\left[\zeta_{2}+2H_{-1,0}\right]\right]$ $-H_{0,0}$ - $(1-x)\left[1-\frac{3}{2}H_{0}\right] - H_{0} - (1+x)H_{0,0} + \delta(1-x)\left[\frac{3}{2}-3\zeta_{2}+6\zeta_{3}\right]$ $P_{\rm ns}^{(1)-}(x) = P_{\rm ns}^{(1)+}(x) + 16C_F\left(C_F - \frac{C_A}{2}\right)\left(p_{\rm qq}(-x)\left[\zeta_2 + 2H_{-1,0} - H_{0,0}\right] - 2(1-x)\right)$ $-(1+x)H_0$ $P_{\rm ps}^{(1)}(x) = 4C_F n_f \left(\frac{20}{9} \frac{1}{x} - 2 + 6x - 4H_0 + x^2 \left[\frac{8}{2}H_0 - \frac{56}{9}\right] + (1+x) \left[5H_0 - 2H_{0,0}\right]\right)$ $P_{qg}^{(1)}(x) = 4C_A n_f \left(\frac{20}{9}\frac{1}{x} - 2 + 25x - 2p_{qg}(-x)H_{-1,0} - 2p_{qg}(x)H_{1,1} + x^2 \left[\frac{44}{3}H_0 - \frac{218}{9}\right]\right)$ $+4(1-x)\left[H_{0,0}-2H_{0}+xH_{1}\right]-4\zeta_{2}x-6H_{0,0}+9H_{0}\right)+4C_{F}n_{f}\left(2p_{qg}(x)\left[H_{1,0}+H_{1,1}+H_{2}+2H_{1,0}+H_{2}+2H_{2$ $-\zeta_{2}$ + 4x² $\left[H_{0} + H_{0,0} + \frac{5}{2}\right]$ + 2(1 - x) $\left[H_{0} + H_{0,0} - 2xH_{1} + \frac{29}{4}\right] - \frac{15}{2} - H_{0,0} - \frac{1}{2}H_{0}$ $P_{\rm gq}^{(1)}(x) = 4C_A C_F \left(\frac{1}{x} + 2p_{\rm gq}(x) \left[H_{1,0} + H_{1,1} + H_2 - \frac{11}{6}H_1\right] - x^2 \left[\frac{8}{3}H_0 - \frac{44}{9}\right] + 4\zeta_2 - 2$ $-7H_0 + 2H_{0,0} - 2H_1x + (1+x)\left[2H_{0,0} - 5H_0 + \frac{37}{2}\right] - 2p_{gq}(-x)H_{-1,0} - 4C_F n_f\left(\frac{2}{2}x\right)$ $-p_{gg}(x)\left[\frac{2}{2}H_{1}-\frac{10}{2}\right]+4C_{F}^{2}\left(p_{gg}(x)\left[3H_{1}-2H_{1,1}\right]+(1+x)\left[H_{0,0}-\frac{7}{2}+\frac{7}{2}H_{0}\right]-3H_{0,0}\right]$ $+1-\frac{3}{2}H_0+2H_1x$ $P_{gg}^{(1)}(x) = 4C_A n_f \left(1 - x - \frac{10}{9} p_{gg}(x) - \frac{13}{9} \left(\frac{1}{x} - x^2\right) - \frac{2}{2} (1 + x) H_0 - \frac{2}{2} \delta(1 - x)\right) + 4C_A^2 \left(27 - \frac{10}{9} \left(\frac{1}{x} - \frac{10}{9}\right) + \frac{10}{9} \left(\frac{1}{3}\right) + \frac{1$ $+(1+x)\left[\frac{11}{3}H_{0}+8H_{0,0}-\frac{27}{2}\right]+2p_{gg}(-x)\left[H_{0,0}-2H_{-1,0}-\zeta_{2}\right]-\frac{67}{9}\left(\frac{1}{x}-x^{2}\right)-12H_{0}$ $-\frac{44}{2}x^{2}H_{0}+2p_{gg}(x)\left[\frac{67}{18}-\zeta_{2}+H_{0,0}+2H_{1,0}+2H_{2}\right]+\delta(1-x)\left[\frac{8}{2}+3\zeta_{3}\right]+4C_{F}n_{f}\left(2H_{0}-\zeta_{2}+H_{0,0}+2H_{1,0}+2H_{2}\right]+\delta(1-x)\left[\frac{8}{2}+3\zeta_{3}\right]$ $+\frac{2}{3}\frac{1}{x}+\frac{10}{3}x^2-12+(1+x)\left[4-5H_0-2H_{0,0}\right]-\frac{1}{2}\delta(1-x)\right).$ 980

NNLO splitting functions S.M. Vermaseren, Vogt

 $P_{ps}^{(2)}(x) = 16C_A C_F n_f \left(\frac{4}{2}(\frac{1}{x}+x^2) \left[\frac{13}{2}H_{-1,0} - \frac{14}{9}H_0 + \frac{1}{2}H_{-1}\zeta_2 - H_{-1,-1,0} - 2H_{-1,0,0}\right]\right)$ $-H_{-1,2}\left] + \frac{2}{3}\left(\frac{1}{r} - x^2\right)\left[\frac{16}{3}\zeta_2 + H_{2,1} + 9\zeta_3 + \frac{9}{4}H_{1,0} - \frac{676}{216} + \frac{571}{22}H_1 + \frac{10}{3}H_2 + H_1\zeta_2 - \frac{1}{6}H_{1,1}\right]$ $-3H_{1,0,0} + 2H_{1,1,0} + 2H_{1,1,1} + (1-x) \left[\frac{182}{9}H_1 + \frac{158}{38} + \frac{397}{36}H_{0,0} - \frac{13}{2}H_{-2,0} + 3H_{0,0,0} \right]$ $+\frac{13}{6}H_{1,0}+3xH_{1,0}+H_{-3,0}+H_{-2}\zeta_{2}+2H_{-2,-1,0}+3H_{-2,0,0}+\frac{1}{2}H_{0,0}\zeta_{2}+\frac{1}{2}H_{1}\zeta_{2}-\frac{9}{4}H_{1,0,0}$ $-\frac{3}{4}H_{1,1}+H_{1,1,0}+H_{1,1,1}\Big]+(1+x)\Big[\frac{7}{12}H_0\zeta_2+\frac{31}{6}\zeta_3+\frac{91}{18}H_2+\frac{71}{12}H_3+\frac{113}{18}\zeta_2-\frac{426}{27}H_0$ $\frac{4}{5}H_{2,0} + \frac{16}{2}H_{-1,0} + 6xH_{-1,0} + \frac{31}{6}H_{0,0,0} - \frac{17}{6}H_{2,1} + \frac{117}{20}\zeta_2^2 + 9H_0\zeta_3 + \frac{5}{2}H_{-1}\zeta_2 + 2H_{2,1,0}$ $+\frac{1}{2}H_{-1,0,0}-2H_{-1,2}+H_2\zeta_2-\frac{7}{2}H_{2,0,0}+H_{-1,-1,0}+2H_{2,1,1}+H_{3,1}-\frac{1}{2}H_4\Big]+5H_{-2,0}+H_{2,1}$ $-4\zeta_{3}-H_{0}\zeta_{2}+H_{3}+H_{2,0}-6H_{-2,0}])+16C_{F}n_{f}^{-2}\left(\frac{2}{27}H_{0}-2-H_{2}+\zeta_{2}+\frac{2}{3}x^{2}\left[H_{2}-\zeta_{2}+3\right]\right)$ $-\frac{19}{6}H_0$ + $\frac{2}{9}(\frac{1}{x}-x^2)$ H_{1,1} + $\frac{5}{3}H_1$ + $\frac{2}{3}$ + $(1-x)\left[\frac{1}{6}H_{1,1}-\frac{7}{6}H_1 + xH_1 + \frac{35}{27}H_0 + \frac{185}{54}\right]$ $+\frac{1}{2}(1+x)\left[\frac{4}{2}H_2-\frac{4}{2}\zeta_2+\zeta_3+H_{2,1}-2H_3+2H_0\zeta_2+\frac{29}{4}H_{0,0}+H_{0,0,0}\right]\right)+16C_F^2n_f\left(\frac{85}{12}H_1-\frac{1}{2}H_1-\frac{$ $\begin{array}{c} 3 \\ -\frac{3}{2} H_{0,0} - H_{0,0,0} + \frac{533}{12} H_0 - \frac{10}{54} + \frac{73}{4} \zeta_2 - \frac{73}{4} H_2 + H_3 - 5 H_{2,0} - H_{2,1} - H_0 \zeta_2 + x^2 \Big[\frac{55}{12} \\ -\frac{85}{2} H_0 - \frac{29}{2} H_{0,0} - \frac{109}{6} - \frac{13}{4} H_0 + \frac{28}{9} \zeta_2 - \frac{28}{9} H_2 - \frac{16}{3} H_0 \zeta_2 + \frac{16}{3} H_3 + 4 H_{2,0} + \frac{4}{3} H_{2,1} - \frac{26}{3} \zeta_3 \\ \end{array}$ $\frac{12}{23}H_{0,0,0}\left] + \frac{4}{3}(\frac{1}{7}-x^2)\left[\frac{23}{12}H_{1,0}-\frac{523}{144}H_1-3\zeta_3+\frac{55}{16}+\frac{1}{2}H_{1,0,0}+H_{1,1}-H_{1,1,0}-H_{1,1,1}\right] \right]$ $+(1-x)\left[\frac{1}{2}H_{1,0,0}+\frac{7}{12}H_{1,1}-\frac{2743}{72}H_0-\frac{53}{12}H_{0,0}-\frac{251}{12}H_1-\frac{5}{4}\zeta_2+\frac{5}{4}H_2-\frac{8}{3}H_{1,0}+3xH_{1,0}-\frac{5}{12}H_1-\frac{$ $+3H_0\zeta_2 - 3H_3 - H_{1,1,0} - H_{1,1,1} + (1+x) \left[\frac{1669}{2164} + \frac{5}{2}H_{0,0,0} + 4H_{2,1} + 7H_{2,0} + 10x\zeta_3 - \frac{37}{10}\zeta_2^2 \right]$ $-7H_0\zeta_3 + 6H_{0,0}\zeta_2 - 4H_{0,0,0,0} + H_{2,0,0} - 2H_{2,1,0} - 2H_{2,1,1} - 4H_{3,0} - H_{3,1} - 6H_4$

$$\begin{split} \mu_{tt}^{(2)}(x) &= 16 \zeta_{u}\zeta_{r} n_{f} \left(\mu_{tt}(x) \frac{39}{29} H_{1}\zeta_{5} - 4H_{1,1,1} + 3H_{2,0,0} - \frac{15}{4} H_{1,2} + \frac{9}{4} H_{1,1,0} + 3H_{2,1,0} \right. \\ + H_{0}\zeta_{1} - 2H_{2,1,1} + 4H_{2}\zeta_{5} - \frac{172}{12} H_{0,0} - \frac{551}{72} H_{0,0} + \frac{64}{3}\zeta_{5} - \zeta_{5}^{2} - \frac{49}{7} H_{2} - \frac{3}{2} H_{1,0,0,0} - \frac{1}{3} H_{1,0,0,0} \\ - \frac{385}{72} H_{1,0} - \frac{31}{2} H_{1,1} - \frac{113}{112} H_{1} + \frac{49}{4} H_{2,0} \leq \frac{5}{2} H_{1,2} - \frac{69}{6} H_{0,0} + \frac{172}{12} H_{0,-} \frac{1259}{32} + \frac{2833}{216} H_{0} \\ + 6H_{2,1} - 3H_{1,-2,0,0} + 9H_{1}\zeta_{5} + 6H_{1,1,0,0} + 3H_{1,1,0,0} - 3H_{1,1,1} - 6H_{1,1,2} - 6H_{1,2} - 6H_{1,2} \end{split} \end{split}$$

 $-2H_{3,0} - \frac{13}{2}H_0\zeta_2 - 13H_{-3,0} - \frac{13}{2}H_{3,1} + \frac{15}{2}H_3 - \frac{2005}{64} + \frac{157}{4}\zeta_2 + 8\zeta_3 + \frac{1291}{432}H_1 + \frac{55}{12}H_{1,1}$ $+\frac{3}{2}H_{2}+\frac{1}{2}H_{2,1}+\frac{27}{4}H_{-1,0}-\frac{11}{2}H_{1,0,0}-8H_{2,0,0}-4\zeta_{2}^{2}+\frac{3}{2}H_{1,2}-H_{2,2}+\frac{5}{2}H_{1}\zeta_{2}+8H_{-1,-1,0}$ $+4H_{2,0}+\frac{3}{2}H_{2,1,1}-H_{-1}\zeta_{2}+7H_{2}\zeta_{2}+6H_{-2}\zeta_{2}+12H_{-2,-1,0}-6H_{-2,0,0}+x\Big[3H_{1,1,1}-H_{0,0}\zeta_{2}+2H_{-2,0,0}+2H_{-2,$ $+\frac{9}{2}H_{-1,0,0}-\frac{35}{2}H_{1,0}+2H_4+3H_{1,1,0}+H_{-1,2}$ + $+16C_4^{-2}C_F(x^2\left[\frac{2}{2}H_1\zeta_2-\frac{2105}{24}-\frac{77}{42}H_{0,0}\right]$ $-6H_3 + \frac{16}{3}\zeta_3 - 10H_{-1,0} - \frac{14}{3}H_{2,0} - \frac{2}{3}H_{-1}\zeta_2 - \frac{14}{3}H_{0,0,0} + \frac{104}{9}H_2 - \frac{4}{3}H_{1,0,0} + \frac{3}{9}H_{1,1}$ $\begin{array}{c} \overset{(a)}{=} 3 & \overset{(a)}{=} 1 & \overset{(a)}{=} 3 & \overset{(a)}{=} 1 & \overset{(a)}{=}$ $+\frac{395}{\pi}H_{0}-2H_{1,0}\zeta_{2}-H_{1,1}\zeta_{2}-\frac{55}{12}H_{1,1,0}+2H_{1,1,0,0}+4H_{1,1,1,0}+2H_{1,1,1,1}+4H_{1,1,2}-\frac{55}{12}H_{1,2}$ $+6H_{1,2,0} + 4H_{1,2,1} + 4H_{1,3} + 3H_{2,1,0} + 3H_{2,2} + p_{10}(-x) \left[\frac{23}{2}H_{-1}\zeta_3 + 5H_{-2}\zeta_2 + 2H_{-2,-1,0}\right]$ $+\frac{109}{12}H_{-1,0}+H_0\zeta_3+\frac{17}{5}\zeta_2^2+\frac{1}{6}H_1\zeta_2+2H_2\zeta_2-\frac{65}{24}H_{1,1}-\frac{19}{2}H_{-1,-1,0}-4H_{3,0}-3H_{2,0,0}$ $\begin{array}{c} 12 \\ -7H_{-2,0,0} - \frac{3}{2}H_{-1,2} + \frac{3379}{216}H_1 - 4H_{-2,2} - \frac{49}{6}H_{-1,0,0} - \frac{11}{2}H_{-1,0,0,0} - 13H_{-1,-1}\zeta_2 - 8H_{-1,3} \end{array}$
$$\begin{split} &-n - 2 _{00} 0 = \frac{2}{2} ^{-1} _{-1,2} + \frac{216}{216} ^{-1} _{-1} - n (-2 _{-2} - \frac{2}{6} ^{-1} _{-1,00} - \frac{2}{2} ^{-1} _{-1,00} - 3 ^{-1} _{-1,-1} _{-1,2} - 3 ^{-1} _{-1,-2} \\ &-6 H_{-1,-1,-1} - 1 + 12 H_{-1,-1,00} + 10 H_{-1,-1,2} + 10 H_{-1,0} _{5,2} + 5 H_{-1,-2,0} - 2 H_{-1,2,0} - 2 H_{-1,2,1} \\ &+ \frac{11}{6} H_0 \zeta_2 \bigg] + (1 - x) \bigg| \frac{4169}{2592} - 3 H_{-2,-1,0} - \frac{3}{2} H_{-2} \zeta_2 - \frac{128}{9} \zeta_2 - 4 H_{3,0} + \frac{26}{3} \zeta_3 - \frac{5}{2} H_{-2,0,0} \end{split}$$
 $-7H_{1}\zeta_{2} + \frac{97}{17}H_{1,0,0} + \frac{10}{3}H_{-1,0,0} + \frac{245}{17}H_{3} - 8H_{0,0,0,0} + (1+x) \left[4H_{3,1} - H_{2,1,1} + \frac{29}{6}H_{-1,2}\right]$ $+\frac{17}{6}H_{-2,0} - 12H_{2,0} - \frac{31}{12}H_{2,1} + \frac{1}{2}H_{2,0,0} - H_2\zeta_2 + \frac{61}{24}H_{1,0} - 4H_0\zeta_3 - \frac{13}{2}H_{-1}\zeta_2 - \frac{46}{2}H_{-1,-1,0}$ $+\frac{25}{4}H_4+\frac{93}{4}H_0\zeta_2-\frac{55}{9}H_{1,1}-\frac{71}{18}H_2+\frac{49}{18}H_{0,0}-\frac{13}{2}H_{0,0}\zeta_2-\frac{47}{40}\zeta_2^{-2}\Big]+\frac{6131}{2502}-\frac{31}{2}H_{-2}\zeta_2$ $-\frac{67}{40}\zeta_2^2 + \frac{29}{5}H_{-1,2} - H_{-1,0} + 8H_{-2,2} + 25H_0\zeta_2 + \frac{412}{9}H_1 + \frac{928}{9}H_0 + \frac{1}{4}H_4 - 65H_3 - 38H_{0,0}$ $-9H_{-3,0} - \frac{17}{3}H_{0,0,0} + x \Big[\frac{27}{2}H_{-1,0} - \frac{1}{2}H_{0,0,0,0} + \frac{3}{4}H_{0,0}\zeta_2 + \frac{1}{2}H_{-3,0} - 14H_{0,0,0} + \frac{1}{12}H_{1,1,1}$ $\begin{array}{c} 3\\ -\frac{43}{36}\zeta_{2}-\frac{1}{2}H_{2}\zeta_{2}+\frac{7}{72}H_{0}+\frac{749}{54}H_{1}+\frac{135}{4}\zeta_{3}+\frac{9}{24}H_{10}+\frac{43}{42}H_{1}\zeta_{2}-\frac{85}{82}H_{-1}\zeta_{2}-\frac{13}{3}H_{1,0}\\ +\frac{53}{21}H_{2}+\frac{39}{2}H_{1,1}-2H_{3,1}+\frac{13}{6}H_{-1,-1,0}+\frac{7}{4}H_{2,0,0}-4H_{1,1,0}-4H_{1,2}\Big] +16C_{F}\eta_{*}^{2}\Big(\frac{1}{9}-\frac{1}{9}\frac{1}{x}H_{1,0}+\frac{1}{3}H_{$ $+\frac{2}{6}x - \frac{1}{6}xH_1 + \frac{1}{6}p_{gq}(x)\left[H_{1,1} - \frac{5}{3}H_1\right] + 16C_F^2n_f\left(\frac{4}{6}x^2\left[H_{0,0} - \frac{11}{6}H_0 - \frac{7}{2} + H_{-1,0}\right]\right]$

 $\begin{array}{l} -6H_{1,3}+\frac{49}{4}\varsigma_2]+\rho_{4g}(-x)\left[\frac{17}{2}H_{-1}\varsigma_3-\frac{5}{2}H_{-1,-1,0}-\frac{5}{2}H_{-1,2}-\frac{9}{2}H_{-1,0}+\frac{5}{2}H_{-2,0}+\frac{3}{2}H_{-1,0,0}-\frac{5}{2}H_{-1,2}-\frac{9}{2}H_{-1,0}+\frac{5}{2}H_{-2,0}+\frac{3}{2}H_{-1,0,0}-\frac{5}{2}H_{-1,0}+\frac{1}{2}H_{ -6H_{-1,-1,-1,0}+6H_{-1,-1,0,0}+6H_{-1,-1,2}+9H_{-1,0}\zeta_2-9H_{-1,-1}\zeta_2-2H_{-1,2,0}-\frac{11}{2}H_{-1,0,0,0}$ $-6H_{-1,3}$ + $(\frac{1}{2}-x^2)\left[\frac{55}{12}-4\zeta_3+\frac{23}{0}H_{1,0}-\frac{4}{2}H_{1,1,0}\right]$ + $(\frac{1}{2}+x^2)\left[\frac{2}{2}H_{1,0,0}-\frac{371}{109}H_1+\frac{23}{0}H_{1,1}\right]$ $-\frac{1189}{108}H_1 - \frac{67}{3}H_{2,1} - 29H_{2,0} - \frac{949}{36}\zeta_2 - \frac{67}{2}H_{0,0,0} - \frac{122}{3}H_3 + \frac{215}{32} - \frac{3989}{48}H_0 + 2H_{-3,0} \Big]$ +(1+x) $H_{-1,0,0} - 10H_{-2}\zeta_2 + 6H_{-2,0,0} + 2H_{0,0}\zeta_2 - 9H_{-1,-1,0} - 7H_{-1,2} - 9H_{-2,0} - 2H_{3,1}$ $-4H_{-2,-1,0} - 4H_4 - 4H_{3,0} - 4H_{0,0,0,0} + \frac{37}{2}H_{-1,0} + \frac{5}{2}(1+x)H_{-1}\zeta_2 - 4H_{-2,0,0} + 2H_{0,0}\zeta_2$ $+H_2\zeta_2 - 3H_{1,1,0} + 2H_{0,0,0,0} + H_{-3,0} - 9H_{2,1,0} - \frac{9}{5}H_{2,1,1} + \frac{11}{3}H_{1,1,1} + \frac{19}{5}H_{2,0,0} + \frac{9}{5}H_{1,2}$ $-\frac{91}{2}H_0\zeta_5 + 8H_{-2}\zeta_2 + \frac{5}{2}H_{-1,-1,0} + \frac{5}{2}H_{-1,2} + \frac{9}{2}H_{-1,0} + \frac{39}{2}H_{-2,0} - \frac{473}{12}H_0\zeta_2 - \frac{1853}{12}H_{0,0}$ $\begin{array}{c} 2 & 0 & 0 \\ 2 & 0 & 0 \\ -\frac{27}{12} & 5 & 0 \\ -\frac{5}{12} & 5 \\ -\frac{5}{12} & -\frac{5}{12} & 0 \\ -\frac{17}{12} & 5 & -\frac{5}{12} & -\frac{169}{12} \\ -\frac{19}{12} & -\frac{169}{12} \\ -\frac{19}{12} & -\frac{169}{12} \\ -\frac{19}{12} & -\frac{19}{12} \\ -\frac{19}{$ $\frac{4}{12} + \frac{12}{12} + \frac{192}{12} + \frac{48}{12} + \frac{12}{12} + \frac{192}{12} + \frac{48}{12} + \frac{11}{12} + \frac{1$ $\frac{53}{18}H_0 + \frac{17}{6}H_{0,0} - \zeta_3 + \frac{11}{18}\zeta_2 - \frac{139}{108} + \frac{1}{3}p_{qg}(-x)H_{-1,0,0} - \frac{53}{169}(\frac{1}{x} - x^2) - \frac{2}{9}(1-x)\left[6H_{0,0,0} - \frac{1}{18}(1-x) + \frac{1}{18}(1-x)\right]$ $\frac{7}{6}xH_1 - H_{0,0} + \frac{7}{2}xH_{1,1} + \frac{7}{6}x(1+x)H_{-1,0} + \frac{7}{4}H_0 - \frac{19}{54}H_1 + H_{0,0,0} + \frac{5}{6}H_{1,1} + \frac{5}{9}H_{-1,0}$ $\frac{85}{216} + 16C_A^2 n_f \left(p_{qg}(x) \left[3H_{1,3} + \frac{31}{6}H_{1,0,0} - \frac{17}{2}H_{2,1} + \frac{7}{5}\zeta_2^2 - \frac{55}{12}H_{1,1,0} + \frac{31}{12}H_3 - \frac{31}{2}H_1\zeta_3 \right] \right)$ $\frac{216}{12} (1-5)^{-1} (1-5)^{ ^{72}$ 2 2 8 4 12 216 2 72 + 11 12 216 2 72 $-2H_{1,1,2} - 2H_{1,2,0}\Big] + p_{qg}(-x)\Big[H_{-1,-1}\zeta_2 - 2H_{-1,2} - 6H_{-1,-1,0} + H_{1,1,1} + 2H_{-2}\zeta_2 - H_{-2,0,0} + H_{-1,1,1} + 2H_{-2}\zeta_2 - H_{-2,0,0} + H_{-1,1,1} +$ $+\frac{727}{26}H_{-1,0}-H_{-1}\zeta_{2}-2H_{-2,2}-\frac{5}{2}H_{-1}\zeta_{3}-H_{-1,-2,0}+2H_{-1,-1,0,0}+2H_{-1,-1,2}-\frac{3}{2}H_{-1,0,0,0}$ $+6H_{-1,-1,-0}-2H_{-1,3}+2H_{-1,2,1}\Big]+(\frac{1}{x}-x^2)\Big[\frac{2}{3}H_{2,1}+\frac{32}{9}\zeta_2-2H_{1,0,0}+\frac{4}{3}H_{1,1,0}-\frac{10}{9}H_{1,1,1}-\frac{10}{9}H_{1,1}-\frac{10}{9}H$ $-\frac{8}{3}H_{-1,0,0} + \frac{3}{2}H_{1,0} + 6\zeta_3 + \frac{161}{26}H_1 - \frac{231}{108}\Big] + \frac{2}{3}(\frac{1}{2} + x^2)\Big[\frac{26}{3}H_{-1,0} - \frac{28}{9}H_0 - 2H_{-1,-1,0}\Big]$

 $+\frac{1}{3}p_{03}(x)\left[H_{1,2}-H_{1,0}-H_{1}\zeta_{2}+9\zeta_{3}+\frac{83}{12}H_{1,1}+2H_{-2,0}-\frac{7}{36}H_{1}+2H_{0}\zeta_{2}-\frac{1625}{48}+\frac{3}{2}H_{1,0,0}+\frac{3}{2}H_{1,0}+\frac{3}$ $+2H_{1,1,0} - \frac{5}{2}H_{1,1,1} \Big] + \frac{31}{18} \rho_{gq}(-x) \Big[\frac{95}{93}H_0 - \zeta_2 - H_{-1,0} \Big] + \frac{1}{3}(2-x) \Big[6H_{0,0,0,0} - H_3 - \frac{13051}{288} \Big] + \frac{1}{3}(2-x) \Big[\frac{1}{3}H_{0,0,0} - H_3 - \frac{1}{3}H_{0,0,0} \Big] + \frac{1}{3}H_{0,0,0} - H_3 - \frac{1}{3}H_{0,0,0} - H_3 - \frac{1}{3}H_{0,0,0} \Big] + \frac{1}{3}H_{0,0,0} - H_3 - \frac{1}{3}H_{0,0,0} - H_3 - \frac{1}{3}H_{0,0,0} - H_3 - \frac{1}{3}H_{0,0,0} \Big] + \frac{1}{3}H_{0,0,0} - H_3 - \frac{1}{3}H_{0,0,0} - H_0 - \frac{1}{3}H_{0,0,0} - \frac{1}{3}H_{0,0,0} - H_0 - \frac{1}{3}H_{0,0} - \frac{1}{3}H_{0,0,0} - H_0 - \frac{1}{3}H_{0,0} - \frac{1}{3}H_{0,$ $-\frac{13}{2}\zeta_{3}-4H_{-2,0}-H_{2,0}-\frac{1}{2}H_{1,0}-\frac{1}{2}H_{2,1}+2H_{0,0,0}-\frac{653}{24}H_{0,0}\right]+(1+x)\left[H_{0}\zeta_{2}-\frac{1187}{216}H_{0}\right]$ $\frac{2}{8}H_{2} - \frac{85}{18}H_{-1,0} - \frac{101}{18}\zeta_{2} - \frac{80}{27}H_{0} + \frac{23}{18}\zeta_{2} - \frac{1}{3}H_{1,1} + \frac{5}{4}xH_{1,1} - \frac{1}{9}H_{1} - \frac{37}{12}xH_{1} + \frac{210}{18}H_{-1,0}$ $+\frac{1501}{54}+H_0\zeta_2-H_{0,0,0}+\frac{101}{2}H_{0,0}-\frac{1}{2}H_{1,0}\Big)+16C_F{}^3\Big(p_{88}(x)\Big[3H_{1,1}\zeta_2+3H_1\zeta_2+\frac{7}{2}\zeta_2$ $-\frac{23}{2}H_{1,1}-8H_1\zeta_3-6H_{1,-2,0}-2H_{1,0}\zeta_2+3H_{1,1,0}-3H_{1,1,0,0}-H_{1,1,1,0}+2H_{1,1,1,1}-3H_{1,1,2}$ $-2H_{1,2,0} - 2H_{1,2,1} - \frac{9}{2}H_{1,1,1} - \frac{3}{2}H_{1,0,0} - \frac{47}{16} - \frac{47}{16}H_1 - \frac{15}{2}\zeta_3 + p_{BQ}(-x) [2H_{-1,-2,0}]$ $+6H_{-1,-1,0}+3H_{-1}\zeta_{2}+\frac{7}{4}H_{1,0}-\frac{16}{5}\zeta_{2}^{-2}-6H_{-1,0,0}-\frac{7}{2}H_{-1,0}+4H_{-1,-1,0,0}-2H_{-1,0}\zeta_{2}$ $-H_{-1,0,0,0}$ + (1-x) 9 $H_{1,0,0}$ + $H_{1,1,1}$ - $10H_1\zeta_2$ + $3H_0\zeta_3$ + $H_{2,2}$ - $H_2\zeta_2$ + $H_{0,0,0}$ + $5H_{2,0,0}$ $-4H_3 + H_{2,1,1} + 3H_{0,0}\zeta_2 + 3H_{3,1} - 3H_4 + \frac{211}{16}H_1 + \frac{49}{26}\zeta_2^{-2} + (1+x) \left[11\zeta_3 + \frac{1}{4}H_{1,1} + \frac{1}{4}H_{1,0} + \frac{1}{4}H_{1,1} + \frac{$ $+\frac{91}{12}H_0+36H_{-1,0}+8H_{-1,0,0}-14H_{-1,-1,0}-7H_{-1}\zeta_2+2H_{1,2}+4H_0\zeta_2-H_{2,1}+2H_{-2,0,0}$ $+5H_{-2,0} + \frac{11}{2}H_2 - 2H_{0,0,0,0} - 2H_{-1,-1,0} - H_{-1}\zeta_2 - \frac{13}{4}\zeta_2 + \frac{9}{4}H_{1,0} + \frac{9}{20}\zeta_2^2 + \frac{287}{32} + \frac{11}{16}H_1$ $+4H_{-1,0,0}+16H_{-3,0}-4H_{-2}\zeta_2-8H_{-2,-1,0}-5H_2\zeta_2+\frac{19}{4}H_2+H_{2,2}-\frac{35}{8}H_{0,0}+9H_0\zeta_3$ $+25H_{-2,0}+6H_{-2,0,0}+\frac{3}{2}x\Big[\frac{58}{3}\zeta_2-\frac{7}{3}H_1\zeta_2+4H_{1,1}-\frac{3}{2}H_{1,1,1}+\frac{5}{2}H_{1,0,0}-\frac{175}{96}+H_{3,1}+\frac{19}{3}\zeta_3$ $+2H_{2,0}-14H_0+H_{0,0}\zeta_2-H_{-1,0}-H_4-\frac{3}{2}H_{2,1}+\frac{1}{3}H_{2,1,1}+3H_{2,0,0}-\frac{5}{6}H_3-H_{1,2}-\frac{7}{6}H_0\zeta_2$ $+\frac{2}{2}H_{1,1,0}-\frac{29}{6}H_{0,0,0}-\frac{185}{8}H_{0,0}$

$$\begin{split} & p_{41}^{(2)}(x) = 1.6C_{45}C_{47}m_{1}'(z^{2}\left[\frac{4}{9}H_{2}+3H_{1,0}-\frac{97}{12}H_{1}+\frac{8}{3}H_{-2,0}-\frac{2}{3}H_{0}\xi_{2}+\frac{123}{12}H_{0}-\frac{16}{9}\xi_{2}+2H_{1}\right.\\ & -6H_{-1,0}+2H_{1,0}+\frac{122}{14}H_{0,0}-\frac{511}{21}]+p_{00}(x)\left[2\xi_{0}-\frac{55}{24}\right]+\frac{4}{3}\left[\frac{1}{x}-x^{2}\right]\left[\frac{12}{2}H_{1,0}-\frac{43}{8}H_{0}\right]\\ & -\frac{124}{14}H_{-}-\frac{693}{432}-\frac{1}{2}H_{21}+2H_{1}\xi_{2}+H_{0}\xi_{2}-2H_{1,00}+\frac{1}{3}H_{1}-H_{-1,1}-H_{-1,1}\right]-\frac{17}{12}H_{2}\\ & +6H_{-1,0}+8H_{0}\xi_{2}-6H_{-1,0}-\frac{53}{6}H_{0}\xi_{2}-\frac{49}{2}H_{0}+\frac{183}{4}\xi_{2}+\frac{511}{12}-\frac{1}{2}H_{2,0}-3H_{1,0}-4H_{0,0,0}\\ & -\frac{12}{12}H_{0,0}+\frac{43}{2}\zeta_{0}-H_{21}+\frac{97}{2}H_{1}-4\zeta_{2}-\frac{2}{2}H_{0}-8H_{0}-4H_{0,0,0}+\frac{1}{3}H_{0,0}+\frac{4}{3}\left[\frac{1}{x}-x^{2}\right]\left[\frac{1}{2}H_{-}-H_{2,0}\right] \end{split}$$

 $-2H_{-1,2} + H_1\zeta_2 + H_{-1}\zeta_2 + \frac{10}{2}H_2 + H_{1,1,1} + (1-x) \left[15H_{0,0,0,0} - 5H_2\zeta_2 - \frac{65}{2}\zeta_3 + \frac{11}{2}H_{1,1,1} + (1-x)\left[15H_{0,0,0,0} - 5H_2\zeta_2 + \frac{11}{2}H_{1,1,1} + (1-x)\left[15H_{0,0,0,0} - 5H_2\zeta_2 + \frac{11}{2}H_{1,1,1} + (1-x)\left[15H_{0,0,0,0} - 5H_2\zeta_2 - \frac{11}{2}H_{1,1} + \frac{11}{2}H_{1,1}$ $-\frac{3}{2}H_4 + \frac{5}{2}H_{0,0}\zeta_2 + H_{1,1,0} - \frac{31}{6}H_{2,0} + \frac{17}{12}H_{1,0} - \frac{551}{20}\zeta_2^2 - \frac{29}{4}H_{1,0,0} - \frac{113}{4}H_2 + \frac{18691}{22}H_0$ $+\frac{2243}{109}+\frac{265}{6}H_{-1,0,0}+\frac{33}{2}H_{2,0,0}+19H_{2,1}+\frac{31}{12}H_{1,1}+\frac{23}{2}H_{-2,0}-\frac{497}{36}\zeta_2+\frac{29}{6}H_1\zeta_2-\frac{143}{12}H_3$ $\frac{108}{6} - \frac{11}{6} H_{1,1,1} - \frac{19}{12} H_0 \zeta_2 + \frac{1223}{72} H_1 - \frac{43}{6} H_{0,0,0} - \frac{3011}{26} H_{0,0} \right] + (1+x) \left[8 H_{2,1,0} - 4 H_{-1,2} - 4 H_{ +7H_{-1,-1,0} - \frac{35}{3}H_{1,1,1} - 5H_{-2}\zeta_2 - 11H_{-2,0,0} + \frac{1}{2}H_{-1,0} + \frac{15}{2}H_{-1}\zeta_2 + 8H_{3,1} - 10H_{-2,-1,0}$ $+5H_2\zeta_2 + 4H_{2,1,1} - H_{-3,0} + 36H_0\zeta_3 - 5H_2\zeta_2 + 2H_{-1,2} + 6H_{-1,-1,0} - 6H_{2,1,0} - 3H_{2,1,1}$ $-11H_{0,0,0,0} - 5H_{3,1} + \frac{25}{4}H_{1,1,1} + \frac{13}{2}H_{-2}\zeta_2 + \frac{27}{2}H_{-2,0,0} + \frac{11}{2}H_{-3,0} + \frac{13}{2}H_2\zeta_2 - \frac{17}{4}H_{1,0,0}$ $+13H_{-2,-1,0} - \frac{17}{12}H_{1,1,1} - \frac{3}{4}H_4 - \frac{1}{4}H_{0,0}\zeta_2 + H_{1,2} + \frac{11}{2}H_{1,1,0} + \frac{79}{12}H_{2,0} + \frac{67}{8}H_{1,0} + \frac{263}{8}\zeta_2^2$
$$\begin{split} &+i3\pi z_{-} - 10 - \frac{1}{12} t r_{1,1} - \frac{1}{4} H_{1} - \frac{1}{4} H_{0,0} \zeta_{2} + H_{1,2} + \frac{1}{2} H_{1,0} + \frac{1}{12} H_{0,0} + \frac{t^{2}}{12} H_{0,0} + \frac{t^{2}}{12} H_{0,0} + \frac{t^{2}}{12} H_{0,0} - \frac{t^{2}}{12} H_{1,1} \\ &+ \frac{19}{12} \zeta_{2} + \frac{6t^{2}}{12} H_{1,0} - \frac{2}{12} H_{1,0} - 24H_{0,5} + H_{-} I\zeta_{2} - \frac{13375}{12} H_{0} - \frac{1889}{18} - 38H_{-} I_{0,0} - \frac{2}{1} H_{2,1} \\ &- \frac{7}{12} H_{2,0} - \frac{217}{12} H_{1,1} - \frac{7}{2} H_{-,2} - \frac{7}{12} L_{2,0} + \frac{7}{12} L_{2} + \frac{4}{12} H_{1,1} + \frac{17}{12} H_{0,2} + \frac{31}{14} H_{1} + 3H_{0,0,0} \end{split}$$
 $-\zeta_3 - 2H_{1,0,0} + \frac{7}{9}H_1 + \frac{77}{81}(\frac{1}{r} - x^2) + (1 - x)\left[\frac{1}{12}H_1 - \frac{6463}{432} - 4H_{0,0,0,0} - \frac{16}{3}H_{0,0,0} + \frac{7}{9}xH_{1,1}\right]$ $+\frac{7}{9}xH_2 + \frac{8}{9}xH_{1,0} - \frac{7}{6}x\zeta_2 \Big] - (1+x)\Big[\frac{3475}{216}H_0 + \frac{103}{12}H_{0,0}\Big]\Big) + 16C_F^{-2}n_f(p_{qg}(x)]7H_{1,3} + 7H_4$ $-2H_{-3,0}-7H_1\zeta_3+5H_{2,2}+6H_{3,0}+6H_{3,1}+H_{2,1,0}+4H_{2,0,0}+3H_{2,1}+2H_{2,1,1}+\frac{5}{2}H_{2,0}$ $+\frac{61}{9}H_2-\frac{61}{8}\zeta_2+\frac{87}{8}H_1+\frac{11}{2}H_{1,2}+\frac{61}{8}H_{1,1}+\frac{17}{2}H_{1,0}-7H_{0,0}\zeta_2+\frac{5}{2}H_{1,0,0}+\frac{5}{2}H_{1,1,0}-\frac{19}{2}\zeta_3$ $+\frac{81}{32}+\frac{11}{2}H_3-\frac{11}{2}H_0\zeta_2-\frac{7}{2}H_1\zeta_2+\frac{15}{2}H_{0,0,0}+\frac{87}{8}H_0+\frac{11}{5}\zeta_2^2+3H_{1,1,1}-5H_2\zeta_2-7H_0\zeta_3$ 52 2 2 2 6 5 +11H_{0,0} - 2H_{1,-2,0} - 7H_{1,0} ζ_2 + 3H_{1,0,0} - 5H_{1,1} ζ_2 + 4H_{1,1,0} + H_{1,1,0} + 2H_{1,1,1} + 5H_{1,1,2} $+6\mathbf{H}_{1,2,0}+6\mathbf{H}_{1,2,1}\Big]+4p_{\rm qg}(-\mathbf{x})\Big[\mathbf{H}_{0,0,0,0}-\mathbf{H}_{-2,0}+\mathbf{H}_{-1,-1,0}-\mathbf{H}_{-2,0,0}+\frac{1}{2}\mathbf{H}_{-1,-2,0}-\frac{5}{6}\mathbf{H}_{-1,0}\Big]$ $-\frac{5}{4}H_{-1,0,0} - \frac{1}{2}H_{-3,0} + \frac{1}{2}H_{-1}\zeta_2 + H_{-1,-1,0,0} - \frac{1}{4}H_{-1,0,0,0} + 2(1-x) \left[H_{2,1,0} - H_{2,0,0} - H_{2,2}\right]$ $-H_{3,1}-2H_{3,0}-2H_{-1}\zeta_2+H_{1,2}-H_{1,0,0}-H_{1,1,0}+H_2\zeta_2-\zeta_2^2+\frac{43}{9}H_2+\frac{49}{9}\zeta_2+\frac{13}{9}H_{1,1}$ $-\frac{33}{16}H_1+\frac{5}{2}H_{1,0}+\frac{7}{2}H_{0,0}\zeta_2+\frac{21}{4}\zeta_3+\frac{479}{64}-\frac{1}{2}H_{1,1,1}-\frac{1}{2}H_3+\frac{1}{4}H_{2,1}+\frac{1}{2}H_{2,1,1}+\frac{3}{2}H_0\zeta_2$ $+\frac{1}{2}H_0\zeta_3 - \frac{7}{2}H_4 + H_1\zeta_2 - \frac{19}{2}H_{0,0,0} - \frac{239}{16}H_{0,0} - \frac{405}{32}H_0 + 8(1+x)\left[H_{-1,-1,0} - H_{-1,0,0}\right]$

 $+12H_{0,0,0,0} - \frac{293}{108} + \frac{61}{6}H_0\zeta_2 - \frac{7}{2}H_{1,0} - \frac{857}{26}H_1 - 9H_0\zeta_3 + 16H_{-2,-1,0} - 4H_{-2,0,0} + 8H_{-2}\zeta_2$ $-\frac{13}{2}H_{1,0,0} + \frac{3}{4}H_{1,1} - H_{1,1,0} - H_{1,1,1} \Big] + (1+x) \Big[\frac{1}{6}H_{2,0} - \frac{95}{3}H_{-1,0} - \frac{149}{36}H_2 + \frac{3451}{108}H_0$ $+18H_{-1,-1,0} - H_{3,1} - 6H_4 - 4H_{-1,2} + 6H_{0,0}\zeta_2 + 8H_2\zeta_2 - 7H_{2,0,0} - 2H_{2,1,0} - 2H_{2,1,1} - 4H_{3,0}$ $-9H_{-1,0,0}\left]-\frac{241}{288}\delta(1-x)\right)+16C_A n_f^2\left(\frac{19}{54}H_0-\frac{1}{24}xH_0-\frac{1}{27}p_{gg}(x)+\frac{13}{54}(\frac{1}{x}-x^2)\left[\frac{5}{3}-H_1\right]\right]$ $+(1-x)\left[\frac{11}{72}H_1-\frac{71}{216}\right]+\frac{2}{9}(1+x)\left[\zeta_2+\frac{13}{12}xH_0-\frac{1}{2}H_{0,0}-H_2\right]+\frac{29}{288}\delta(1-x)\right)$ $+16C_{A}^{2}n_{f}\left(x^{2}\left[\zeta_{3}+\frac{11}{9}\zeta_{2}+\frac{11}{9}H_{0,0}-\frac{2}{3}H_{3}+\frac{2}{3}H_{0}\zeta_{2}+\frac{1639}{108}H_{0}-2H_{-2,0}\right]+\frac{1}{3}p_{gg}(x)\left[\frac{10}{3}\zeta_{2}+\frac{10}{3}H_{0}-\frac{2}{3}H_{0}+\frac{2}{3}H_{0}-\frac{2}{3}H_{0}+\frac{2}{3}H_{0}-\frac{2$ $-\frac{209}{27} - 8\zeta_3 - 2H_{-2,0} - \frac{1}{2}H_0 - \frac{10}{2}H_{0,0} - \frac{20}{3}H_{1,0} - H_{1,0,0} - \frac{20}{3}H_2 - H_3 \Big] + \frac{10}{9}p_{gg}(-x)\Big[\zeta_2 - \frac{10}{3}H_2 - \frac{10}{3}H_2$ $\begin{array}{c} 3\\ -\overline{3}^{2}H_{-1,0}+\frac{2}{3}H_{-1,2} \right] + (1-x) \Big[\frac{5}{6}H_{-2,0} + H_{-3,0} + 2H_{0,0,0} - \frac{269}{36}\zeta_{2} - \frac{4097}{216} - 3H_{-2}\zeta_{2} \\ -6H_{-2,-1,0} + 3H_{-2,0,0} - \frac{7}{2}H_{1}\zeta_{2} + \frac{677}{22}H_{1} + H_{1,0} + \frac{7}{4}H_{1,0,0} \Big] + (1+x) \Big[\frac{193}{36}H_{-2} - \frac{11}{2}H_{-1}\zeta_{2} \\ \end{array}$ $+\frac{39}{20}\zeta_{2}^{2}-\frac{7}{12}H_{3}-\frac{53}{9}H_{0,0}+\frac{7}{12}H_{0}\zeta_{2}-\frac{5}{2}H_{0,0}\zeta_{2}+5\zeta_{3}-7H_{-1,-1,0}+\frac{7}{7}H_{-1,0}+\frac{9}{2}H_{-1,0,0}$ $+2H_{-1,2} - 3H_2\zeta_2 - \frac{2}{2}H_{2,0} + \frac{3}{2}H_{2,0,0} + \frac{3}{2}H_4 \Big] + \frac{1}{6}\zeta_2 + 7H_{-2,0} + 2H_2 + \frac{458}{27}H_0 + H_{0,0}\zeta_2$ $\begin{array}{c} & - \sum_{n=0}^{\infty} - \sum_{n=$ $\begin{array}{c} +2^{2} \left[-4^{2} - 4^{$ $+\frac{134}{9}H_{1,0}+\frac{11}{6}H_{1,0,0}+8H_{1,2,0}+8H_{1,3}+\frac{134}{9}H_2-4H_2\zeta_2+8H_{3,1}+8H_{2,2}+\frac{11}{6}H_3+10H_{3,0}$
$$\begin{split} & + \frac{1}{9} \cdot \Pi b_1 + \frac{1}{6} \cdot \Pi \Pi b_0 + 3\pi \Pi \Pi b_2 + 3\pi \Pi b_1 + \frac{1}{9} \cdot \Pi b_2 - 3\pi \Pi b_2 + \pi \delta \Pi b_1 + \pi \delta$$
 $\begin{array}{l} -H_{0,0,0,0}+\frac{3}{4}H_{-2,0}-\frac{9}{4}H_{-1,0}\Big]-4H_{-1,-1,0}+5H_{0,0,0,0}+5H_{-1,0,0}-13H_{-2,0}+\frac{1}{2}H_{-1,0}\\ +44H_{-2,0,0}-\frac{113}{8}H_2-\frac{71}{8}\xi_2-\frac{3}{2}H_1-\frac{11}{2}H_{1,2}-\frac{33}{8}H_{1,1}-\frac{7}{2}H_{1,0}-\frac{7}{2}H_{0,0}\xi_2-\frac{5}{2}H_{1,0}\\ -\frac{5}{5}H_{1,1,0}-\frac{5}{2}H_2-\frac{57}{64}-\frac{9}{4}H_{1,1}-\frac{9}{4}H_1-\frac{5}{4}H_{2,1}-\frac{1}{4}H_{1,1}+\frac{1}{4}H_0\xi_2+H_2\xi_3+\frac{5}{2}H_0\zeta_3\\ +\frac{5}{5}\xi_2+\frac{7}{2}H_4-\frac{7}{2}H_1\xi_4+\frac{4}{4}H_{0,00}-\frac{39}{16}H_0+\frac{40}{16}H_0-H_{2,0,0}-H_{2,1,0}+H_{2,2}+H_{1,1}\\ +\frac{3}{4}H_{0,0}+H_{1,2}+\frac{1}{4}H_{2,0}+2H_{2,5}+4H_{2,-1,0}\end{array}$

 $P_{gq}^{(2)}(x) = 16C_{A}C_{F}n_{f}\left(\frac{2}{6}x^{2}\left[\frac{25}{6}H_{1}-\frac{131}{4}+3\zeta_{2}-H_{-1,0}-3H_{2}+H_{1,1}+\frac{125}{6}H_{0}-H_{0,0}\right]$ $+\frac{5}{6}p_{B9}(x)\left[H_{1,2}+H_{2,1}+\frac{967}{120}+\frac{251}{40}H_1-\frac{39}{10}H_{1,1}-3\zeta_3-\frac{2}{5}H_0\zeta_2-\frac{1}{5}H_1\zeta_2-\frac{4}{3}H_{1,0}+H_{1,1,0}\right]$ $-\frac{1}{\epsilon}H_{1,0,0} + H_{1,1,1} + \frac{2}{\epsilon}H_{2,0} + \frac{2}{2}p_{gq}(-x) \left[2H_{-1}\zeta_2 + \frac{7}{4}\zeta_2 + \frac{41}{12}H_{-1,0} - \frac{151}{72}H_0 + \frac{1}{2}H_{-2,0}\right]$ $+\frac{5}{2}H_2 + 2H_{-1,-1,0} - H_{-1,0,0} - H_{-1,2} + \frac{2}{2}(1-x)[H_{-2,0} + 2\zeta_3 - H_3] + (1+x)[\frac{179}{100}H_1]$ $+\frac{5}{6}\zeta_{2}+\frac{25}{9}H_{-1,0}-\frac{5}{26}H_{1,1}-\frac{167}{36}H_{0,0}-\frac{1}{3}H_{2,1}-\frac{4}{3}H_{0}\zeta_{2}\Big]-\frac{193}{77}+\frac{1}{4}H_{1}+\frac{1}{6}H_{-1,0}+4H_{2}$ $\frac{523}{144} - \frac{19}{126}H_2 + \frac{271}{108}H_0 - \frac{5}{6}H_{1,0}\right) + 16C_A C_F^2 \left(x^2 \left[\frac{7}{2} + \frac{173}{54}H_1 - 2\zeta_3 - \frac{2}{3}H_{1,1,1} - \frac{26}{9}H_{1,1}\right]\right)$ $-6H_2 + 2H_{2,1} + 6\zeta_2 + \frac{335}{54}H_0 - \frac{28}{9}H_{0,0} - \frac{8}{3}H_{0,0,0} + p_{gq}(x) \left[\frac{3}{2}H_1\zeta_3 + \frac{163}{32} - 5\zeta_2 + \frac{27}{4}\zeta_3\right]$ $+\frac{6503}{422}H_1+\frac{2}{9}H_{1,1}+\frac{35}{2}H_{1,1,1}+4H_2+\frac{9}{2}H_{2,1}+4H_{1,0,0}+2H_{2,0,0}-H_2\zeta_2+\frac{41}{12}H_{1,2}+H_{2,2}$ $\frac{491}{24}H_{1,0} + 3H_{2,0} - 2H_{2,1,1} - \frac{3}{2}H_{-1}\zeta_2 - \frac{59}{12}H_1\zeta_2 + 5H_{1,-2,0} + H_{1,0}\zeta_2 + \frac{5}{2}H_{1,0,0,0} - 2H_{1,1}\zeta_2$ $\frac{2^{24}}{12}H_{1,1,0} + 5H_{1,1,0,0} - 3H_{1,1,1,0} - 4H_{1,1,1,1} - H_{1,1,2} - 2H_{1,2,1} + H_{2,1,0} + p_{\beta q}(-x) \left[H_{-1,0}\right]$ $+H_{-1,0}\zeta_2 + \frac{3}{2}H_{-1,0,0} + \frac{27}{10}\zeta_2^2 - 3H_{-1,-1,0} - \frac{11}{2}H_{-1}\zeta_3 - 3H_{-1,-2,0} - \frac{3}{2}H_{-1,0,0,0} - 3H_{-1,2}$ $+5H_{-1,-1}\zeta_2 - 4H_{-1,-1,0,0} - 2H_{-1,-1,2} + 6H_{-1,-1,-1,0} + 2H_{-1,2,1}$ + (1-x) $\left[H_2\zeta_2 - H_{2,2}\right]$ $+\frac{23}{12}H_{1,0}-\frac{7061}{432}H_0-\frac{4631}{144}H_{0,0}-\frac{38}{3}H_{0,0,0}-H_{-3,0}-2H_{3,0}-\frac{4433}{432}H_1-2H_{2,0,0}-\frac{21}{2}H_{1,0,0}$ $\frac{12}{2}\zeta_{2}^{2}-\frac{432}{7}H_{1,2}+\frac{23}{2}H_{1}\zeta_{2}-4H_{0}\zeta_{3}]+(1+x)\left[\frac{49}{4}H_{3}-H_{-2,0}-\frac{55}{2}H_{0}\zeta_{2}-\frac{1}{2}H_{3,1}-\frac{1159}{24}\zeta_{2}\right]$ $\frac{5}{576} - \frac{151}{576} \zeta_3 - \frac{125}{18} H_{1,1} + \frac{1}{6} H_{1,1,1} - \frac{95}{9} H_2 + \frac{29}{6} H_{2,1} - \frac{171}{4} H_{-1,0} - 12 H_{-1,0,0} + 7 H_{-1} \zeta_2$ $+16H_{-1,-1,0} + \frac{5}{2}H_{2,0} + \frac{3}{2}H_{2,1,1} + 4H_{0,0,0,0} - \frac{35H_{-2,0} - \frac{179}{22}H_0}{122} + \frac{2041}{144}H_{0,0} - \frac{19}{6}H_{0,0,0}$

 $-\frac{413}{108}H_1 - \frac{11}{2}H_1\zeta_2 + \frac{33}{2}H_{1,0,0}\Big] + 11(\frac{1}{x} + x^2)\Big[\frac{71}{54}H_0 - \frac{1}{6}H_3 - \frac{389}{198}\zeta_2 - \frac{2}{3}H_{-2,0} - \frac{1}{2}H_{-1}\zeta_2$ $\frac{-158}{n}H_2 - \frac{53}{2}H_{-1}\zeta_2\Big] - 29H_{0,0} - \frac{40}{3}H_{0,0,0} + 27H_{-2,0} + \frac{41}{3}H_0\zeta_2 - 20H_3 - 24H_{2,0} + \frac{53}{6}\zeta_2$ $\frac{601}{19}H_0 + 24\zeta_3 + 2\zeta_2^2 + 27H_2 - 4H_{0,0}\zeta_2 - 16H_0\zeta_3 - 16H_{-3,0} + 28xH_{0,0,0,0} + \delta(1-x)\left[\frac{79}{32}\right]$ $-\zeta_2\zeta_3 + \frac{1}{\epsilon}\zeta_2 + \frac{11}{24}\zeta_2^2 + \frac{67}{\epsilon}\zeta_3 - 5\zeta_5 \right] + 16C_F n_f^2 \left(\frac{2}{6}x^2 \left[\frac{11}{\epsilon}H_0 + H_2 - \zeta_2 + 2H_{0,0} - 9\right] + \frac{1}{2}H_2$
$$\begin{split} & \frac{3^{2}5^{3}}{1} - \frac{5^{2}}{3} + \frac{3^{2}}{4} - \frac{4^{2}}{5} - \frac{5^{2}}{6} - \frac{5^{2}}{1} - \frac{5^{2}}{1} - \frac{1}{2} - \frac{7}{1} - \frac{7}{1}$$
 $-H_{2,1} - 2H_{2,0} + \frac{11}{144}\delta(1-x) + 16C_F^{-2}n_f(\frac{4}{3}x^2 \left[\frac{163}{16} + \frac{27}{8}H_0 + \frac{7}{2}H_{0,0} - H_{2,0} - \zeta_2 + \frac{9}{4}H_{1,0}\right]$ $-H_{2,1} + \frac{1}{2}H_{0,0,0} + \frac{85}{16}H_1 + H_2 - 2H_{-2,0} - \frac{3}{2}\zeta_3 \Big] + \frac{4}{3}(\frac{1}{z} - x^2) \Big[\frac{31}{16}H_1 - \frac{11}{16} - \frac{5}{4}H_{1,0} + \frac{1}{2}H_{1,0,0} + \frac{1}{2}H_{1,0,0$ $-H_1\zeta_2 - H_{1,1} + H_{1,1,0} + H_{1,1,1} + \zeta_3 + \frac{4}{3}(\frac{1}{\nu} + x^2) \left[H_{-1}\zeta_2 + 2H_{-1,-1,0} - H_{-1,0,0}\right] + \frac{215}{12}H_{0,0}$ $+\frac{20}{2}H_0 - \frac{131}{6} + 3H_{2,0} + \frac{205}{12}x\zeta_2 - 3H_{1,0} + H_{2,1} - \frac{85}{12}H_1 + \frac{11}{4}H_2 + 8H_{-2,0} + 2\zeta_2^2 - H_0\zeta_2$ $+H_3 + 6H_0\zeta_3 + 8H_{-3,0} - 4xH_{0,0,0} + (1-x)\left[\frac{107}{12}H_1 - \frac{5}{5}H_{1,0} - 4\zeta_2 + H_0\zeta_3 - 8H_{-2,-1,0}\right]$ $-4H_{-2}\zeta_{2} + 4H_{-2,0,0} - 4H_{1}\zeta_{2} + \frac{7}{2}H_{1,0,0} - \frac{7}{12}H_{1,1} + H_{1,1,0} + H_{1,1,1} + (1+x) \left[\frac{5}{4}H_{2} + \frac{33}{8}H_{1,1} + \frac{33}{8}$ $-\frac{99}{4}H_{0,0} - 8H_{2,0} - \frac{541}{24}H_0 - 4H_{2,1} - \frac{3}{2}H_{0,0,0} - 2x\zeta_3 + \frac{9}{2}\zeta_2^2 + 5H_0\zeta_2 - 5H_3 - 4H_{-1}\zeta_2$ $-8H_{-1,-1,0} + \frac{67}{2}H_{-1,0} + 4H_{-1,0,0} + 2H_{0,0}\zeta_2 - 2H_{0,0,0,0} - 4H_2\zeta_2 + 3H_{2,0,0} + 2H_{2,1,0}$ $+2H_{2,1,1}+H_{3,1}-2H_4 + \frac{1}{16}\delta(1-x)$.

2004

Perturbative stability of evolution

• Scale derivatives of quark and gluon distributions at $Q^2 \approx 30 \text{ GeV}^2$



Perturbative stability of evolution

• Scale derivatives of quark and gluon distributions at $Q^2 \approx 30 \text{ GeV}^2$



Expansion very stable except for very small momenta $x \leq 10^{-4}$ S.M. Vermaseren, Vogt '04

Strange asymmetry

- Probability of a splitting $q \rightarrow q'$ different from that of $q \rightarrow \overline{q}'$ at higher orders (starting at three loops)
 - dynamical generation of asymmetric sea $q \bar{q}$ Catani, De Florian, Rodrigo, Vogelsang '04
- Non-singlet distributions q^{\pm} and q^{v}
 - splitting function combinations $P_{\rm ns}^{\pm}$ and $P_{\rm ns}^{\rm v} = P_{\rm ns}^{-} + P_{\rm ns}^{\rm s}$

$$q_{\mathrm{ns},ik}^{\pm} = q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k)$$
 flavour asymmetries
 $q_{\mathrm{ns}}^{\mathrm{v}} = \sum_{r=1}^{n_f} (q_r - \bar{q}_r)$ total valence distribution

• New colour factor in $P_{\rm ns}^{(2)s}$ $d^{abc}d_{abc}/n_c = (n_c^2 - 1)(n_c^2 - 4)/n_c^2$



Vector boson production



- Kinematical variables (inclusive)
 - energy (cms) $s = Q^2$ (time-like)
 - scaling variable $x = M_{W^{\pm}/Z}^2/s$

Vector boson production



- Kinematical variables (inclusive)
 - energy (cms) $s = Q^2$ (time-like)
 - scaling variable $x = M_{W^{\pm}/Z}^2/s$

 20 years of measurements of W[±] and Z cross sections at hadron colliders

Universal aspects of radiative corrections

- Recall perturbative QCD:
 - calculation of observables as series in $\alpha_s \ll 1$
 - but: large logarithmic corrections, $\ln(...) \gg 1$ double logarithms (Sudakov)
- Soft/Collinear regions of phase space
 - double logarithms from singular regions in Feynman diagrams
 - propagator vanishes for: $E_g = 0$, soft $\theta_{qg} = 0$ collinear

$$\frac{1}{(p+k)^2} = \frac{1}{2p \cdot k} = \frac{1}{2E_q E_g (1 - \cos \theta_{qg})}$$

$$\alpha_s \int d^4 k \frac{1}{(p+k)^2} \longrightarrow \alpha_s \int dE_g \, d\theta_{qg} \, \frac{1}{2E_q E_g (1 - \cos \theta_{qg})}$$

$$\longrightarrow \alpha_s \, \ln^2(\dots)$$

Improved perturbation theory: resum logarithms to all orders

Sudakov logarithms in cross sections

Intuitive aspects of higher order corrections (e.g. Drell-Yan)



- at threshold for $\mu^+\mu^-$ -creation
 - strong Sudakov-supression inelastic tendency

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\sigma \sim \exp\left[-\alpha_s \ln^2(1 - 4m_\mu^2/s)\right]
```

 universal factor for parton splittings (leading log accuracy) modelling of MC parton showers

- Hadronic reaction $p\bar{p}$:
 - recall master equation

$$\sigma_{pp\to\mu^+\mu^-} = \sum_{ij} f_i \otimes f_j \otimes \hat{\sigma}_{ij\to\mu^+\mu^-}$$

initial partons: also Sudakov-supressed



$$\sigma_{ij\to\mu^+\mu^-} = \frac{\mu + \mu}{f_i \otimes f_j} = \frac{1}{\left(e^{-\alpha_s \ln^2(\dots)}\right)^2} = e^{+\alpha_s \mu}$$

Iarge double logarithms

Differential distributions (rapidity)



• W^{\pm}, Z -boson rapidity distribution with scale variation $m_{W,Z}/2 \le \mu \le 2m_{W,Z}$ Anastasiou, Petriello, Melnikov '05

Differential distributions (rapidity)



- W^{\pm}, Z -boson rapidity distribution with scale variation $m_{W,Z}/2 \le \mu \le 2m_{W,Z}$ Anastasiou, Petriello, Melnikov '05
- Reduction of theoretical uncertainties (renormalization / factorization scale) to level of 1% in NNLO QCD analysis Dissertori '05

Top quark production

Leading order Feynman diagrams





- NLO in QCD Nason, Dawson, Ellis '88; Beenakker, Smith, van Neerven '89; Mangano, Nason, Ridolfi '92; Bernreuther, Brandenburg, Si, Uwer '04; ...
 - accurate to $\mathcal{O}(15\%)$ at LHC

Much activity towards higher orders in QCD

- small-mass limit $m^2 \ll s, t, u$ for two-loop virtual corrections to $q\bar{q} \rightarrow t\bar{t}$ and $gg \rightarrow t\bar{t}$ S.M., Czakon, Mitov '07
- full mass dependence for two-loop virtual $q\bar{q} \rightarrow t\bar{t}$ Czakon '08
- analytic two-loop fermionic corrections for $q\bar{q} \rightarrow t\bar{t}$ Bonciani, Ferroglia, Gehrmann, Maitre, Studerus '08
- one-loop squared terms (NLO × NLO) Anastasiou, Mert Aybat '08;
 Kniehl, Merebashvili, Körner, Rogal '08



$$\sigma_{pp \to t\bar{t}} = \sum_{ij} f_i \otimes f_j \otimes \hat{\sigma}_{ij \to t\bar{t}}$$



Recall Drell-Yan process:
parton cross section Sudakov enhanced close to threshold $s \simeq 4m^2$

• Sudakov-type logarithms $\ln(\beta)$ with velocity of heavy quark $\beta = \sqrt{1 - 4m^2/s}$ at nth-order

 $\alpha_s^n \ln^{2n}(\beta) \longleftrightarrow \alpha_s^n \ln^{2n}(N)$

resummation in Mellin space (renormalization group equation)

 $\hat{\sigma}_{ij}^{N} = (1 + \alpha_s g_{01} + \ldots) \cdot \exp(G^N) + \mathcal{O}(N^{-1} \ln^n N)$

• Resummed G^N predicts fixed orders in perturbation theory

generating functional for towers of large logarithms

Sven-Olaf Moch

Total cross section at LHC



New results

- NNLO cross section for heavy-quark hadro-production near threshold (all powers of $\ln \beta$ and Coulomb corrections) S.M., Uwer '08
 - e.g. gg-fusion for $n_f = 5$ light flavors at $\mu = m$

$$\hat{\sigma}_{gg \to t\bar{t}}^{(1)} = \hat{\sigma}_{gg \to t\bar{t}}^{(0)} \left\{ 96 \ln^2 \beta - 9.5165 \ln \beta + 35.322 + 5.1698 \frac{1}{\beta} \right\}$$

$$\hat{\sigma}_{gg \to t\bar{t}}^{(2)} = \hat{\sigma}_{gg \to t\bar{t}}^{(0)} \left\{ 4608 \ln^4 \beta - 1894.9 \ln^3 \beta + \left(-3.4811 + 496.30 \frac{1}{\beta} \right) \ln^2 \beta + \left(3144.4 + 321.17 \frac{1}{\beta} \right) \ln \beta + 68.547 \frac{1}{\beta^2} - 196.93 \frac{1}{\beta} + C_{gg}^{(2)} \right\}$$

- Add all scale dependent terms
 - $\ln(\mu/m)$ -terms exactly known from renormalization group methods

Upshot

- Best approximation to complete NNLO
- Similar results for new massive colored particles (4th generation quarks, squarks, gluinos, ...)
 S.M., Uwer '08; S.M., Langenfeld '08

Top-quark pair-production at NNLO

- NLO (with MRST2006 PDF set)
 - scale uncertainty $\mathcal{O}(10\%) \oplus \mathsf{PDF}$ uncertainty $\mathcal{O}(5\%)$
- NNLO_{approx} (with MRST2006 PDF set)
 - Scale uncertainty O(3%) ⊕ PDF uncertainty O(2%)



Theory at NNLO matches anticipated experimental precision $\mathcal{O}(10\%)$

Tevatron analyses

- Total cross section and different channels of Tevatron analyses (theory uncertainty band from scale variation)
- NNLO allows for precision determinations of m_t from total cross section (slope $d\sigma/dm_t$)





Single top-quark production

- Single-top production allows study of charged-current weak interaction of top quark
 - direct extraction of the CKM-matrix element V_{tb}
 - flagship measurement of Tevatron run II (control QCD bckgrd !)



- Large corrections from extensions of Standard Model
 - t-channel: anomalous couplings or flavor changing neutral currents
 - s-channel: charged "top-pion", Kaluza-Klein modes of W or W'-boson

Higgs production at LHC



Standard model Higgs

branching ratios for decay (left) and dominant production modes (right) Djouadi '05 Sven-Olaf Moch

Hard scattering cross sections in QCD - p.37

Gluon fusion

• Largest rate for all values of Higgs mass M_H (top-Yukawa coupling)





heavy top limit $m_t \rightarrow \infty$: effective gg Higgs vertex

- Total cross section with QCD corrections
- Variation of renormalization scale for Higgs mass $M_H = 120 \text{GeV}$
 - NNLO corrections
 Harlander, Kilgore '02; Anastasiou, Melnikov '02;
 Ravindran, Smith, van Neerven '03
 - complete soft N³LO corrections
 S.M., Vogt '05

Differential distributions in gluon fusion

- Bin-integrated Higgs rapidity distribution including decay $H \rightarrow \gamma \gamma$
 - QCD corrections up to NNLO Anastasiou, Melnikov, Petriello '05
 - fast parton level Monte Carlo HNNLO Catani, Grazzini '07



- Impact of kinematical cuts on higher order corrections
 - left: Higgs mass $M_h = 125$ GeV, no cuts on p_t of jets
 - right: Higgs mass $M_h = 165 \text{ GeV}$ and veto on jets with $p_t > 40 \text{ GeV}$ (k_t algorithm for jet reconstruction with jet size D = 0.4)

Weak vector-boson fusion

- Channel $qq \rightarrow qqH$ (with cuts on jets energies)
- Second largest rate (WWH coupling)
 - mostly dominated by u, d-quarks





Higgs-strahlung

- Channel $q\bar{q} \rightarrow W(Z)H$
- Third largest rate (same couplings as vector boson fusion)

tĪ⊢

- Channel $pp \rightarrow t\bar{t}H$
 - discovery channel in low mass region $M_H \lesssim 130 \text{ GeV}$
 - driven by gluon luminosity, but large SM background $pp \rightarrow t\bar{t}H \rightarrow t\bar{t}b\bar{b}$



- Main backgrounds for $pp \rightarrow t\bar{t}H$
 - combinatorial background from signal (4 b-quarks in final state)
 - $t\bar{t}+2$ jets, $t\bar{t}b\bar{b}, t\bar{t}Z$
 - complex final states
- New: NLO QCD corrections to $q\bar{q} \rightarrow t\bar{t}b\bar{b}$ Denner, Dittmaier, Pozzorini '08
 - extremely diffcult hexagon integrals with masses



Progress in theory

- Sensitivity for Higgs production at LHC
 - inclusion of higher order theory predictions in new studies
 - e.g. $pp \rightarrow t\bar{t}H$ absent in CMS plot



Summary

Hard QCD

- Hard parton cross section
 - Structure functions in DIS
 - W^{\pm}/Z -boson production
 - hadro-production of top quarks
 - Higgs total cross section
- Hadronic final state
 - (multi) jet cross sections
 - jet algorithms and fragmentation of (heavy) quarks
 - parton shower Monte Carlo simulation

Outlook

- QCD tool box ready for LHC challenges
 - however, still much dedicated work to do

Literature

Review

Expectations at LHC from hard QCD
 J. Phys. G: Nucl. Part. Phys. 35, 073001 (2008) [arXiv:0803.0457] [hep-ph].