

# High Precision for Light and Heavy Partons and $\alpha_s(M_Z^2)$ in DIS

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C. Schneider<sup>2</sup>, F. Wißbrock<sup>1</sup> [HQ corrections]  
S. Alekhin<sup>1,4</sup>, JB<sup>1</sup>, S.-O. Moch<sup>5</sup> [ABM PDFs]

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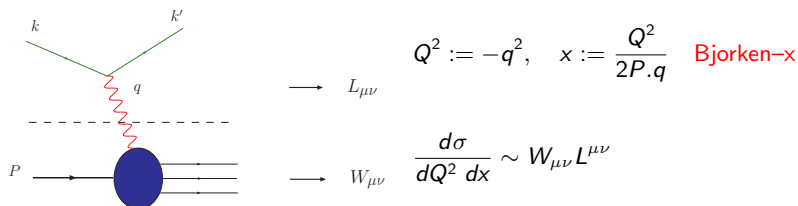


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# Introduction

Unpolarized Deep-Inelastic Scattering (DIS):



$$W_{\mu\nu}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s | [J_\mu^{em}(\xi), J_\nu^{em}(0)] | P, s \rangle =$$

$$\frac{1}{2x} \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left( P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2).$$

Structure Functions:  $F_{2,L}$

contain light and heavy quark contributions.

## I) Main Results of the PDF Analysis:

- Improved PDFs [Drell-Yan & 1st Jets from LHC]
- Improved Polarized PDFs
- Improved Standard Candles
- $\alpha_s(M_Z^2)$  remains stable

JB, B. Böttcher, A. Guffanti, Nucl.Phys. B774 (2007) 182

S. Alekhin, JB, S. Moch, S. Klein, Phys.Rev. D81 (2010) 014032

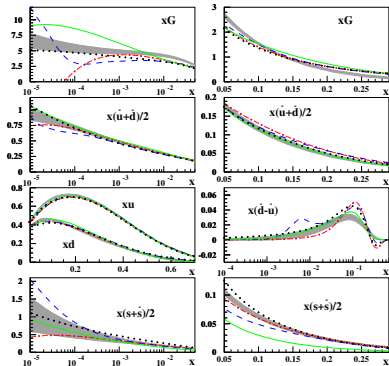
S. Alekhin, JB, S. Moch, Phys.Rev. D86 (2012) 054009,

Phys.Rev. D89 (2014) 054028

JB and H. Böttcher Nucl.Phys. B841 (2010) 205

S. Alekhin et al. arXiv:1404.6469

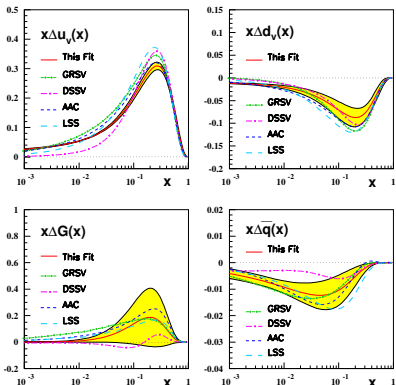
$\mu=2 \text{ GeV}, n_f=4$



unpolarized

NNLO

ABM13



polarized

NLO (NNLO in preparation)

BB10

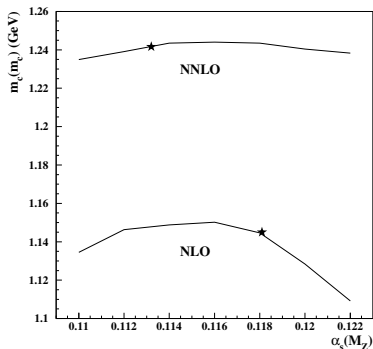
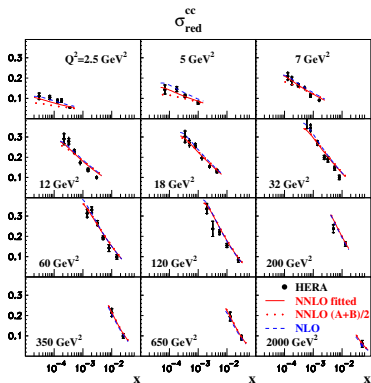
$\alpha_s(M_Z^2)$  from NNLO DIS(+) analyses

	$\alpha_s(M_Z^2)$	
BBG	$0.1134^{+0.0019}_{-0.0021}$	valence analysis, NNLO
GRS	0.112	valence analysis, NNLO
ABKM	$0.1135 \pm 0.0014$	HQ: FFNS $N_f = 3$
JR	$0.1128 \pm 0.0010$	dynamical approach
JR	$0.1162 \pm 0.0006$	including NLO-jets
MSTW	$0.1171 \pm 0.0014$	
Thorne	0.1136	[DIS+DY+HT*] (2014)
ABM11 <sub>J</sub>	$0.1134 - 0.1149 \pm 0.0012$	Tevatron jets (NLO) incl.
ABM13	$0.1133 \pm 0.0011$	
ABM13	$0.1132 \pm 0.0011$	(without jets)
CTEQ	$0.1159..0.1162$	
CTEQ	0.1140	(without jets)
NN21	$0.1174 \pm 0.0006 \pm 0.0001$	
Gehrmann et al.	$0.1131^{+0.0028}_{-0.0022}$	$e^+e^-$ thrust
Abbate et al.	$0.1140 \pm 0.0015$	$e^+e^-$ thrust
BBG	$0.1141^{+0.0020}_{-0.0022}$	valence analysis, $N^3\text{LO}$

$$\Delta_{\text{TH}}\alpha_s = \alpha_s(N^3\text{LO}) - \alpha_s(\text{NNLO}) + \Delta_{\text{HQ}} = +0.0009 \pm 0.0006_{\text{HQ}}$$

NNLO accuracy is needed to analyze the world data.  $\implies$  NNLO HQ corrections needed.

# Deep-Inelastic Scattering (DIS):



## NNLO:

S. Alekhin, J. Blümlein, K. Daum, K. Lipka, Phys.Lett. B720 (2013) 172 [1212.2355]

$$m_c(m_c) = 1.24 \pm 0.03(\text{exp}) \begin{matrix} +0.03 \\ -0.02 \end{matrix} (\text{scale}) \begin{matrix} +0.00 \\ -0.07 \end{matrix} (\text{thy}),$$

$$m_b(m_b) = 3.97 \pm 0.14(\text{exp}) \begin{matrix} +0.00 \\ -0.11 \end{matrix} (\text{thy}) \text{ (preliminary)},$$

Yet approximate NNLO treatment [Kawamura et al. [1205.5227]].

## II) Major Goals: NNLO HQ Corrections to DIS

- ▶ Complete the NNLO heavy flavor Wilson coefficients for twist-2 in the dynamical safe region  $Q^2 > 20\text{GeV}^2$  (no higher twist) for  $F_2(x, Q^2)$
- ▶ Measure  $m_c$  and  $\alpha_s$  as precisely as possible
- ▶ Provide precise charged current heavy flavor corrections
- ▶ **Consequences for LHC:**
  - ▶ NNLO VFNS will be provided
  - ▶ better constraint on sea quarks and the gluon
  - ▶ precise  $m_c, m_b$  and  $\alpha_s$  on input



## Publications: Physics

JB, A. De Freitas, S. Klein, W.L. van Neerven, Nucl. Phys. B755 (2006) 272  
I. Bierenbaum, JB, S. Klein, Nucl. Phys. B780 (2007) 40; Nucl.Phys. B820 (2009) 417; Phys.Lett. B672 (2009) 401  
JB, S. Klein, B. Tödtli, Phys. Rev. D80 (2009) 094010  
I. Bierenbaum, JB, S. Klein, C. Schneider, Nucl. Phys. B803 (2008) 1  
J. Ablinger, JB, S. Klein, C. Schneider, F. Wißbrock, Nucl. Phys. B844 (2011) 26  
JB, A. Hasselhuhn, S. Klein, C. Schneider, Nucl. Phys. B866 (2013) 196  
J. Ablinger et al., Nucl. Phys. B864 (2012) 52; Nucl. Phys. B882 (2014) 263; Nucl. Phys. B885 (2014) 409; Nucl. Phys. B885 (2014) 280; Nucl. Phys. B886 (2014) 733; 1409.1135 (Nucl.Phys. B (2014) in print).  
A. Behring et al., Eur.Phys.J. C74 (2014) 9, 3033

## Publications: Mathematics

JB, Comput. Phys. Commun. 159 (2004) 19  
JB, Comput. Phys. Commun. 180 (2009) 2143; 0901.0837  
JB, D. Broadhurst, J. Vermaseren, Comput. Phys. Commun. 181 (2010) 582  
JB, M. Kauers, C. Schneider, Comput. Phys. Commun. 180 (2009) 2143  
JB, S.Klein, C. Schneider, F. Stan. J. Symbolic Comput. 47 (2012) 1267  
J. Ablinger, JB, C. Schneider, J. Math. Phys. 52 (2011) 102301, J. Math. Phys. 54 (2013) 082301  
J. Ablinger, JB, 1304.7071 [Contr. to a Book: Springer, Wien]  
J. Ablinger, JB, C. Raab, C. Schneider, 1407.1822 (J. Math. Phys. in print)

# Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x, Q^2) = \sum_j \underbrace{C_{j,(2,L)} \left( x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right)}_{\text{perturbative}} \otimes \underbrace{f_j(x, \mu^2)}_{\text{nonpert.}}$$

into (pert.) **Wilson coefficients** and (nonpert.) **parton distribution functions (PDFs)**.

$\otimes$  denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \delta(x - yz) f(y) g(z) .$$

The subsequent calculations are performed in Mellin space, where  $\otimes$  reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx x^{N-1} f(x) .$$

Wilson coefficients:

$$C_{j,(2,L)} \left( N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = C_{j,(2,L)} \left( N, \frac{Q^2}{\mu^2} \right) + H_{j,(2,L)} \left( N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) .$$

At  $Q^2 \gg m^2$  the heavy flavor part

$$H_{j,(2,L)} \left( N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \sum_i C_{i,(2,L)} \left( N, \frac{Q^2}{\mu^2} \right) A_{ij} \left( \frac{m^2}{\mu^2}, N \right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients  $C$  and the massive operator matrix elements (OMEs) of local operators  $O_i$  between partonic states  $j$

$$A_{ij} \left( \frac{m^2}{\mu^2}, N \right) = \langle j | O_i | j \rangle .$$

→ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO

[Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For  $F_2(x, Q^2)$  : at  $Q^2 \gtrsim 10m^2$  the asymptotic representation holds at the 1% level.

# Status of OME calculations

Leading Order: [Witten 1976, Babcock, Sivers 1978, Shifman, Vainshtein, Zakharov 1978, Leveille, Weiler 1979, Glück, Reya 1979, Glück, Hoffmann, Reya 1982]

Next-to-Leading Order:

[Laenen, van Neerven, Riemersma, Smith 1993]

$Q^2 \gg m^2$ : via IBP [Buza, Matiounine, Smith, Migneron, van Neerven 1996]

Compact results via  ${}_pF_q$ 's [Bierenbaum, Blümlein, Klein, 2007]

$O(\alpha_s^2 \varepsilon)$  (for general  $N$ ) [Bierenbaum, Blümlein, Klein 2008, 2009]

Next-to-Next-to-Leading Order:  $Q^2 \gg m^2$

- ▶ Moments for  $F_2$ :  $N = 2 \dots 10(14)$  [Bierenbaum, Blümlein, Klein 2009]  
mapping large expressions to [MATAD, Steinhauser 2000]
- ▶ Contributions to transversity:  $N = 1 \dots 13$  [Blümlein, Klein, Tödtli 2009]
- ▶ Two masses  $m_1 \neq m_2 \rightarrow$  Moments  $N = 2, 4, 6$  [JB, Wißbrock 2011]

At 3-loop order for general values of  $N$ :

- ▶ All OMEs: terms  $O(n_f T_F^2 C_{A/F})$  to  $F_2$  [Ablinger et al. 2011, 2012]
- ▶ First contributions to  $O(T_F^2 C_{A/F})$   $A_{gg,Q}$  [Ablinger et al. 2014]

# The Wilson Coefficients at large $Q^2$

$$\begin{aligned}
 L_{q,(2,L)}^{\text{NS}}(N_F + 1) &= a_s^2 \left[ A_{qq,Q}^{(2),\text{NS}}(N_F + 1) \delta_2 + \hat{C}_{q,(2,L)}^{(2),\text{NS}}(N_F) \right] \\
 &+ a_s^3 \left[ A_{qq,Q}^{(3),\text{NS}}(N_F + 1) \delta_2 + A_{qq,Q}^{(2),\text{NS}}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + \hat{C}_{q,(2,L)}^{(3),\text{NS}}(N_F) \right] \\
 2010 \quad L_{q,(2,L)}^{\text{PS}}(N_F + 1) &= a_s^3 \left[ A_{qq,Q}^{(3),\text{PS}}(N_F + 1) \delta_2 + A_{gg,Q}^{(2)}(N_F) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + N_F \hat{C}_{q,(2,L)}^{(3),\text{PS}}(N_F) \right] \\
 L_{g,(2,L)}^{\text{S}}(N_F + 1) &= a_s^2 A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + a_s^3 \left[ A_{qq,Q}^{(3)}(N_F + 1) \delta_2 \right. \\
 &+ A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
 &+ A_{Qq}^{(1)}(N_F + 1) N_F \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) + N_F \hat{C}_{g,(2,L)}^{(3)}(N_F) \left. \right], \\
 H_{q,(2,L)}^{\text{PS}}(N_F + 1) &= a_s^2 \left[ A_{Qq}^{(2),\text{PS}}(N_F + 1) \delta_2 + \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) \right] + a_s^3 \left[ A_{Qq}^{(3),\text{PS}}(N_F + 1) \delta_2 \right. \\
 &+ \tilde{C}_{q,(2,L)}^{(3),\text{PS}}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
 &+ A_{Qq}^{(2),\text{PS}}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) \left. \right], \\
 H_{g,(2,L)}^{\text{S}}(N_F + 1) &= a_s \left[ A_{Qq}^{(1)}(N_F + 1) \delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \right] + a_s^2 \left[ A_{Qq}^{(2)}(N_F + 1) \delta_2 \right. \\
 &+ A_{Qq}^{(1)}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
 &+ \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) \left. \right] + a_s^3 \left[ A_{Qq}^{(3)}(N_F + 1) \delta_2 + A_{Qq}^{(2)}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) \right. \\
 &+ A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qq}^{(1)}(N_F + 1) \left\{ C_{q,(2,L)}^{(2),\text{NS}}(N_F + 1) \right. \\
 &+ \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) \left. \right\} + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F + 1) \left. \right]
 \end{aligned}$$

[Ablinger et al., 2010]

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 &+ a_s^3 \left[ A_{qq,Q}^{(3),\text{NS}}(N_F + 1) \delta_2 + A_{qq,Q}^{(2),\text{NS}}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + \hat{C}_{q,(2,L)}^{(3),\text{NS}}(N_F) \right] \\
 2010 \quad L_{q,(2,L)}^{\text{PS}}(N_F + 1) &= a_s^3 \left[ A_{qq,Q}^{(3),\text{PS}}(N_F + 1) \delta_2 + A_{gg,Q}^{(2)}(N_F) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + N_F \hat{C}_{q,(2,L)}^{(3),\text{PS}}(N_F) \right] \\
 2010 \quad L_{g,(2,L)}^{\text{S}}(N_F + 1) &= a_s^2 A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + a_s^3 \left[ A_{qq,Q}^{(3)}(N_F + 1) \delta_2 \right. \\
 &+ A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
 &+ A_{Qg}^{(1)}(N_F + 1) N_F \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) + N_F \hat{C}_{g,(2,L)}^{(3)}(N_F) \left. \right], \\
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 &+ \tilde{C}_{q,(2,L)}^{(3),\text{PS}}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
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 H_{g,(2,L)}^{\text{S}}(N_F + 1) &= a_s \left[ A_{Qg}^{(1)}(N_F + 1) \delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \right] + a_s^2 \left[ A_{Qg}^{(2)}(N_F + 1) \delta_2 \right. \\
 &+ A_{Qg}^{(1)}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
 &+ \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) \left. \right] + a_s^3 \left[ A_{Qg}^{(3)}(N_F + 1) \delta_2 + A_{Qg}^{(2)}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) \right. \\
 &+ A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qg}^{(1)}(N_F + 1) \left\{ C_{q,(2,L)}^{(2),\text{NS}}(N_F + 1) \right. \\
 &+ \left. \left. \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) \right\} + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F + 1) \left. \right]
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[Ablinger et al., 2010]

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 &+ a_s^3 \left[ A_{qq,Q}^{(3),\text{NS}}(N_F + 1) \delta_2 + A_{qq,Q}^{(2),\text{NS}}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + \hat{C}_{q,(2,L)}^{(3),\text{NS}}(N_F) \right] \\
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 &+ A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
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 &+ \tilde{C}_{q,(2,L)}^{(3)}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
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 &+ A_{Qg}^{(1)}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
 &+ \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) \left. \right] + a_s^3 \left[ A_{Qg}^{(3)}(N_F + 1) \delta_2 + A_{Qg}^{(2)}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) \right. \\
 &+ A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qg}^{(1)}(N_F + 1) \left\{ C_{q,(2,L)}^{(2),\text{NS}}(N_F + 1) \right. \\
 &+ \left. \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) \right\} + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F + 1) \left. \right]
 \end{aligned}$$

[Ablinger et al. 2010, Ablinger et al., 2014a]

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 &+ a_s^3 \left[ A_{qq,Q}^{(3),\text{NS}}(N_F + 1) \delta_2 + A_{qq,Q}^{(2),\text{NS}}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + \hat{C}_{q,(2,L)}^{(3),\text{NS}}(N_F) \right] \\
 2010 \quad L_{q,(2,L)}^{\text{PS}}(N_F + 1) &= a_s^3 \left[ A_{qq,Q}^{(3),\text{PS}}(N_F + 1) \delta_2 + A_{qq,Q}^{(2)}(N_F) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + N_F \hat{\tilde{C}}_{q,(2,L)}^{(3),\text{PS}}(N_F) \right. \\
 2010 \quad L_{g,(2,L)}^{\text{S}}(N_F + 1) &= a_s^2 A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + a_s^3 \left[ A_{qq,Q}^{(3)}(N_F + 1) \delta_2 \right. \\
 &+ A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
 &+ A_{Qg}^{(1)}(N_F + 1) N_F \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) + N_F \hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F) \left. \right], \\
 2014 \quad H_{q,(2,L)}^{\text{PS}}(N_F + 1) &= a_s^2 \left[ A_{Qq}^{(2),\text{PS}}(N_F + 1) \delta_2 + \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) \right] + a_s^3 \left[ A_{Qq}^{(3),\text{PS}}(N_F + 1) \delta_2 \right. \\
 &+ \tilde{C}_{q,(2,L)}^{(3),\text{PS}}(N_F + 1) + A_{qq,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
 &+ A_{Qq}^{(2),\text{PS}}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) \left. \right], \\
 H_{g,(2,L)}^{\text{S}}(N_F + 1) &= a_s \left[ A_{Qg}^{(1)}(N_F + 1) \delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \right] + a_s^2 \left[ A_{Qg}^{(2)}(N_F + 1) \delta_2 \right. \\
 &+ A_{Qg}^{(1)}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
 &+ \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) \left. \right] + a_s^3 \left[ A_{Qg}^{(3)}(N_F + 1) \delta_2 + A_{Qg}^{(2)}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) \right. \\
 &+ A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qg}^{(1)}(N_F + 1) \left\{ C_{q,(2,L)}^{(2),\text{NS}}(N_F + 1) \right. \\
 &+ \left. \left. \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) \right\} + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F + 1) \right]
 \end{aligned}$$

[Ablinger et al. 2010, Ablinger et al., 2014a, Ablinger et al., 2014b]



# Variable Flavor Number Scheme

$$f_k(n_f + 1, \mu^2) + \bar{f}_k(n_f + 1, \mu^2) = A_{qq,Q}^{\text{NS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes [f_k(n_f, \mu^2) + \bar{f}_k(n_f, \mu^2)] \\ + \tilde{A}_{qq,Q}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) + \tilde{A}_{qg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2)$$

$$f_{Q+\bar{Q}}(n_f + 1, \mu^2) = \tilde{A}_{Qq}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) + \tilde{A}_{Qg}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2).$$

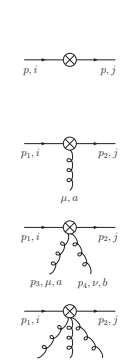
$$G(n_f + 1, \mu^2) = A_{gq,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) + A_{gg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2).$$

$$\Sigma(n_f + 1, \mu^2) = \sum_{k=1}^{n_f+1} [f_k(n_f + 1, \mu^2) + \bar{f}_k(n_f + 1, \mu^2)] \\ = \left[ A_{qq,Q}^{\text{NS}}\left(n_f, \frac{\mu^2}{m^2}\right) + n_f \tilde{A}_{qq,Q}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) + \tilde{A}_{Qq}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \right] \\ \otimes \Sigma(n_f, \mu^2) \\ + \left[ n_f \tilde{A}_{qg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) + \tilde{A}_{Qg}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \right] \otimes G(n_f, \mu^2)$$

The choice of matching scales is **not free** and varies with the process in case of precision observables. Blümlein, van Neerven [hep-ph/9811351]  $\implies$  More complicated for 2 masses J. Blümlein, Wißbrock, 2014.

# Calculation of the 3-loop operator matrix elements

The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



$$\overline{p, \nu, b} \otimes p, \mu, a$$

$$\frac{1+(-1)^N \delta^{ab} (\Delta \cdot p)^{N-2}}{2} \left[ g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu \right], \quad N \geq 2$$

$$\delta^{ij} \Delta_\mu \gamma_\pm (\Delta \cdot p)^{N-1}, \quad N \geq 1$$

$$g t_{ji}^a \Delta_\mu \Delta_\nu \Delta_\rho \gamma_\pm \sum_{j=0}^{N-2} \sum_{l=j+1}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}, \quad N \geq 2$$

$$g^2 \Delta^\mu \Delta^\nu \Delta^\rho \Delta_\sigma \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^l (\Delta p_1)^{N-l-2} \left[ (t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \right], \quad N \geq 3$$

$$g^3 \Delta_\mu \Delta_\nu \Delta_\rho \Delta_\sigma \Delta_\tau \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-m-2} \left[ (t^a t^b t^c)_{jil} (\Delta p_4 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_5 + \Delta p_1)^{m-l-1} + (t^a t^c t^b)_{jil} (\Delta p_4 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_4 + \Delta p_1)^{m-l-1} + (t^b t^a t^c)_{jil} (\Delta p_3 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_5 + \Delta p_1)^{m-l-1} + (t^b t^c t^a)_{jil} (\Delta p_3 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_3 + \Delta p_1)^{m-l-1} + (t^c t^a t^b)_{jil} (\Delta p_3 + \Delta p_4 + \Delta p_1)^{l-j-1} (\Delta p_4 + \Delta p_1)^{m-l-1} + (t^c t^b t^a)_{jil} (\Delta p_3 + \Delta p_4 + \Delta p_1)^{l-j-1} (\Delta p_3 + \Delta p_1)^{m-l-1} \right], \quad N \geq 4$$

$$\gamma_+ = 1, \quad \gamma_- = \gamma_5.$$

$$-ig \frac{1+(-1)^N}{2} f^{abc} \left( \left[ (\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu) \right] (\Delta \cdot p_1)^{N-2} + \Delta_\lambda \left[ \Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu \right] \times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j} + \left\{ \begin{matrix} p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow p_1 \\ \mu \rightarrow \nu \rightarrow \lambda \rightarrow \mu \end{matrix} \right\} + \left\{ \begin{matrix} p_1 \rightarrow p_3 \rightarrow p_2 \rightarrow p_1 \\ \mu \rightarrow \lambda \rightarrow \nu \rightarrow \mu \end{matrix} \right\} \right), \quad N \geq 2$$

$$g^2 \frac{1+(-1)^N}{2} \left( f^{abc} f^{cde} O_{\mu\nu\lambda\sigma} (p_1, p_2, p_3, p_4) + f^{a\alpha\epsilon} f^{b\delta\epsilon} O_{\mu\lambda\nu\sigma} (p_1, p_3, p_2, p_4) + f^{a\delta\epsilon} f^{b\alpha\epsilon} O_{\mu\nu\sigma\lambda} (p_1, p_4, p_2, p_3) \right),$$

$$O_{\mu\nu\lambda\sigma} (p_1, p_2, p_3, p_4) = \Delta_\nu \lambda \left\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2} + [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i} - [p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i} + [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma] \times \sum_{i=0}^{N-4} \sum_{j=0}^i (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \right\} - \left\{ \begin{matrix} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu \end{matrix} \right\} - \left\{ \begin{matrix} p_3 \leftrightarrow p_4 \\ \lambda \leftrightarrow \sigma \end{matrix} \right\} + \left\{ \begin{matrix} p_1 \leftrightarrow p_2, p_3 \leftrightarrow p_4 \\ \mu \leftrightarrow \nu, \lambda \leftrightarrow \sigma \end{matrix} \right\}, \quad N \geq 2$$

The diagrams are generated using **QGRAF** [Nogueira 1993 J. Comput. Phys].

	$A_{qq,Q}^{(3),NS}$	$A_{gq,Q}^{(3)}$	$A_{Qq}^{(3),PS}$	$A_{gg,Q}^{(3)}$	$A_{Qg}^{(3)}$
No. diagrams	110	86	125	642	1233

A **FORM** [Vermaseren 2000] program was written in order to perform the  $\gamma$ -matrix algebra in the numerator of all diagrams, which are then expressed as a linear combination of scalar integrals.

$$A_{qq,Q}^{(3),NS} \rightarrow 7426 \text{ scalar integrals.}$$

$$A_{gq,Q}^{(3)} \rightarrow 12529 \text{ scalar integrals.}$$

$$A_{Qq}^{(3),PS} \rightarrow 5470 \text{ scalar integrals.}$$

$\Rightarrow$  Need to use integration by parts identities.

$\Rightarrow$  The reduction for all OMEs has been completed.

$\Rightarrow$  Use special computers: 12 units with overall **3.2 TB** RAM, **97 TB** fast disc, **hundreds** of mathematica lic. ; IBP: **several TB** of final relations.

# Integration by parts

We use **Reduze** [A. von Manteuffel, C. Studerus, 2012] to express all scalar integrals required in the calculation in terms of a small(er) set of master integrals.

Reduze is a **C++** program based on **Laporta's algorithm**.

$$(\Delta \cdot k)^N \rightarrow \sum_{N=0}^{\infty} x^N (\Delta \cdot k)^N = \frac{1}{1 - x \Delta \cdot k}$$

$\Rightarrow$  additional propagator.

Number of master integrals:

$$A_{qq,Q}^{(3),NS} \rightarrow 35 \text{ master integrals.}$$

$$A_{gq,Q}^{(3)} \rightarrow 41 \text{ master integrals.}$$

$$A_{Qq}^{(3),PS} \rightarrow 66 \text{ master integrals.}$$

If we also include  $A_{gg,Q}^{(3)}$  and  $A_{Qg}^{(3)}$ , there is a total of more than 600 master integrals for the entire project.

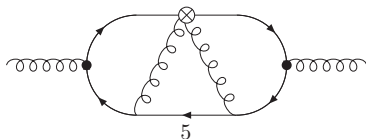
24 integral families are required and implemented in Reduze.

# Calculation of the master integrals

For the calculation of the master integrals we use a wide variety of tools:

- ▶ Hypergeometric functions.
- ▶ Summation methods based on difference fields, implemented in the Mathematica program **Sigma** [C. Schneider, 2005–].
  - ▶ Reduction of the sums to a small number of key sums.
  - ▶ Expansion the summands in  $\varepsilon$ .
  - ▶ Simplification by symbolic summation algorithms based on  $\Pi\Sigma$ -fields [Karr 1981 J. ACM, Schneider 2005–].
  - ▶ Harmonic sums, polylogarithms and their various generalizations are algebraically reduced using the package HarmonicSums [Ablinger 2010, 2013, Ablinger, Blümlein, Schneider 2011,2013].
- ▶ Mellin-Barnes representations.
- ▶ In the case of **convergent** massive 3-loop Feynman integrals, they can be performed in terms of **Hyperlogarithms** [Generalization of a method by F. Brown, 2008, to non-vanishing masses and local operators].
- ▶ Systems of Differential Equations.
- ▶ Almquist-Zeilberger Theorem as Integration Method.

# V-Topology



- ▶ Emergence of a new function class : **nested generalized cyclotomic sums, weighted with binomials and inverse binomials** of the type  $\binom{2i}{i}$ .
- ▶ At the side of the iterated integrals **many root-valued letters** appear (around 30).
- ▶ The scalar diagram exhibits terms growing like  $8^N, 4^N, 2^N, N \rightarrow \infty$ . The growth  $2^N$  survives in the scalar case. **Physics case finishes during these days.**
- ▶ Asymptotic representations can be constructed analytically to arbitrary precision.
- ▶ Various special **new numbers** appear, the simplest of which is  $\pi$ , through which  $\zeta_2$  is no longer and elementary constant here.

## Emergence of new nested sums :

$$\begin{aligned} & \sum_{i=1}^N \binom{2i}{i} (-2)^i \sum_{j=1}^i \frac{1}{j \binom{2j}{j}} S_{1,2} \left( \frac{1}{2}, -1; j \right) \\ &= \int_0^1 dx \frac{x^N - 1}{x - 1} \sqrt{\frac{x}{8+x}} \left[ H_{w_{17}, -1, 0}^*(x) - 2H_{w_{18}, -1, 0}^*(x) \right] \\ &+ \frac{\zeta_2}{2} \int_0^1 dx \frac{(-x)^N - 1}{x + 1} \sqrt{\frac{x}{8+x}} \left[ H_{12}^*(x) - 2H_{13}^*(x) \right] \\ &+ c_3 \int_0^1 dx \frac{(-8x)^N - 1}{x + \frac{1}{8}} \sqrt{\frac{x}{1-x}}, \end{aligned}$$

---

$$w_{12} = \frac{1}{\sqrt{x(8-x)}},$$

$$w_{13} = \frac{1}{(2-x)\sqrt{x(8-x)}},$$

$$w_{17} = \frac{1}{\sqrt{x(8+x)}},$$

$$w_{18} = \frac{1}{(2+x)\sqrt{x(8+x)}}.$$

~ 100 associated independent nested sums. The associated iterated integrals request root-valued alphabets with about 30 new letters.

[J. Ablinger, J. Bümlein, J. Raab, C. Schneider, F. Wißbrock 2014.]

[J. Ablinger, J. Bümlein, J. Raab, C. Schneider 2014.]

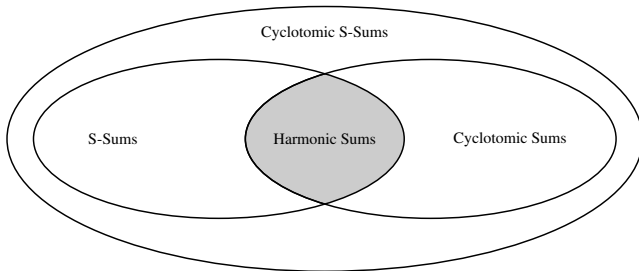
# Spill-Off:

## New Mathematical Function Classes and Algebras

- ▶ 1998: Harmonic Sums [Vermaseren; JB]
- ▶ 1999: Harmonic Polylogarithms [Remiddi, Vermaseren]
- ▶ 2001: Generalized Harmonic Sums [Moch, Uwer, Weinzierl]
- ▶ 2004: Infinite harmonic (inverse) binomial sums [Davydychev, Kalmykov; Weinzierl]
- ▶ 2011: (generalized) Cyclotomic Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- ▶ 2013: Systematic Theory of Generalized Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- ▶ 2014: Finite nested Generalized Cyclotomic Harmonic Sums with (inverse) Binomial Weights [Ablinger, JB, Raab, Schneider]

Particle Physics Generates **NEW** Mathematics.





Nested (inverse) binomial sums

.....

More and more onion skins to be added during these calculations.

# $T_F^2 C_{F,A}$ Contributions to $A_{gg}$ for two massive lines ( $m_1 = m_2$ )

$$\begin{aligned}
 a_{gg,Q;T_F^2}^{(3)}(N) = & C_F T_F^2 \left\{ \frac{16}{27} FS_1^3 + \frac{16P_4}{27(N-1)N^3(N+1)^3(N+2)} S_1^2 + \left[ -\frac{16}{3} FS_2 \right. \right. \\
 & - \left. \frac{32P_{10}}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \right] S_1 - \frac{16P_4}{9(N-1)N^3(N+1)^3(N+2)} S_2 \\
 & - \frac{2P_{13}}{243(N-1)N^5(N+1)^5(N+2)(2N-3)(2N-1)} - F \left[ \frac{352}{27} S_3 - \frac{64}{3} S_{2,1} \right] \\
 & + \left[ \frac{16}{3} FS_1 - \frac{8P_8}{9(N-1)N^3(N+1)^3(N+2)} \right] \zeta_2 + \frac{P_3}{9(N-1)N^2(N+1)^2(N+2)} \zeta_3 \\
 & - \left( \frac{2N}{N} \right) \frac{16P_5}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} \frac{1}{4N} \left( \sum_{i=1}^N \frac{4^i S_1(i-1)}{i^2 \binom{2i}{i}} - 7\zeta_3 \right) \left. \right\} \\
 & + C_A T_F^2 \left\{ -\frac{4P_2}{135(N-1)N^2(N+1)^2(N+2)} S_1^2 + \frac{16(4N^3 + 4N^2 - 7N + 1)}{15(N-1)N(N+1)} [S_{2,1} - S_3] \right. \\
 & + \frac{P_{12}}{3645(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \\
 & - \frac{8P_{11}}{3645(N-1)N^3(N+1)^3(N+2)(2N-3)(2N-1)} S_1 + \frac{4P_7}{135(N-1)N^2(N+1)^2(N+2)} S_2 \\
 & - \left( \frac{2N}{N} \right) \frac{4P_9}{45(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} \frac{1}{4N} \left( \sum_{i=1}^N \frac{4^i S_1(i-1)}{i^2 \binom{2i}{i}} - 7\zeta_3 \right) \\
 & \left. + \left[ \frac{4P_6}{27(N-1)N^2(N+1)^2(N+2)} - \frac{560}{27} S_1 \right] \zeta_2 + \left[ -\frac{7P_1}{270(N-1)N(N+1)(N+2)} - \frac{1120}{27} S_1 \right] \zeta_3 \right\},
 \end{aligned}$$

# Moments for graphs with two massive lines ( $m_1 \neq m_2$ )

$$\begin{aligned}
 a_{Q_9}^{(3)}(N=6) = & \frac{1}{2} \left\{ T_F^2 C_A \left[ \frac{69882273800453}{367569090000} - \frac{395296}{19845} \zeta_3 + \frac{1316809}{39690} \zeta_2 + \frac{832369820129}{14586075000} x + \frac{1511074426112}{624023544375} x^2 - \frac{84840004938801319}{690973782403905000} x^3 \right] \right. \\
 & + \ln\left(\frac{m_2^2}{\mu^2}\right) \left[ \frac{11771644229}{194481000} + \frac{78496}{2205} \zeta_2 - \frac{1406143531}{69457500} x - \frac{105157957}{180093375} x^2 + \frac{2287164970759}{7669816654500} x^3 \right] \\
 & + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \left[ \frac{2668087}{79380} + \frac{112669}{661500} x - \frac{49373}{51975} x^2 - \frac{31340489}{34054020} x^3 \right] + \ln^3\left(\frac{m_2^2}{\mu^2}\right) \frac{324148}{19845} + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \frac{156992}{6615} \\
 & + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \left[ \frac{128234}{3969} - \frac{112669}{330750} x + \frac{98746}{51975} x^2 + \frac{31340489}{17027010} x^3 \right] + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln^2\left(\frac{m_1^2}{\mu^2}\right) \frac{68332}{6615} \\
 & + \ln\left(\frac{m_1^2}{\mu^2}\right) \left[ \frac{83755534727}{583443000} + \frac{78496}{2205} \zeta_2 + \frac{1406143531}{69457500} x + \frac{105157957}{180093375} x^2 - \frac{2287164970759}{7669816654500} x^3 \right] \\
 & + \ln^2\left(\frac{m_1^2}{\mu^2}\right) \left[ \frac{2668087}{79380} + \frac{112669}{661500} x - \frac{49373}{51975} x^2 - \frac{31340489}{34054020} x^3 \right] + \ln^3\left(\frac{m_1^2}{\mu^2}\right) \frac{412808}{19845} \left. \right\} \\
 & + T_F^2 C_F \left\{ -\frac{3161811182177}{71471767500} + \frac{447392}{19845} \zeta_3 + \frac{9568018}{4862025} \zeta_2 - \frac{64855635472}{2552563125} x + \frac{1048702178522}{97070329125} x^2 + \frac{1980566069882672}{2467763508585375} x^3 \right. \\
 & + \ln\left(\frac{m_2^2}{\mu^2}\right) \left[ \frac{1786067629}{204205050} - \frac{111848}{15435} \zeta_2 - \frac{128543024}{24310125} x - \frac{22957168}{3361743} x^2 - \frac{2511536080}{2191376187} x^3 \right] \\
 & + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \left[ \frac{3232799}{4862025} + \frac{752432}{231525} x + \frac{177944}{40425} x^2 + \frac{127858928}{42567525} x^3 \right] - \ln^3\left(\frac{m_2^2}{\mu^2}\right) \frac{111848}{19845} - \ln^2\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \frac{223696}{46305} \\
 & + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \left[ \frac{22238456}{4862025} - \frac{1504864}{231525} x - \frac{355888}{40425} x^2 - \frac{255717856}{42567525} x^3 \right] + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln^2\left(\frac{m_1^2}{\mu^2}\right) \frac{223696}{46305} \\
 & + \ln\left(\frac{m_1^2}{\mu^2}\right) \left[ -\frac{24797875607}{1021025250} - \frac{111848}{15435} \zeta_2 + \frac{128543024}{24310125} x + \frac{22957168}{3361743} x^2 + \frac{2511536080}{2191376187} x^3 \right] \\
 & + \ln^2\left(\frac{m_1^2}{\mu^2}\right) \left[ \frac{3232799}{4862025} + \frac{752432}{231525} x + \frac{177944}{40425} x^2 + \frac{127858928}{42567525} x^3 \right] - \ln^3\left(\frac{m_1^2}{\mu^2}\right) \frac{1230328}{138915} \left. \right\} + O(x^4 \ln^3(x))
 \end{aligned}$$

## Complete OMEs and Wilson Coefficients:

- 3 Loop Anomalous Dimensions
- Wilson Coefficients and OMEs

6 of 8 OMEs have been calculated

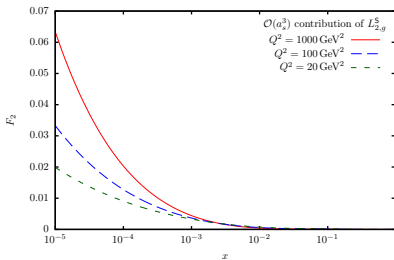
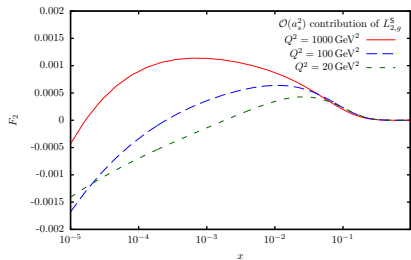
# 3-Loop Anomalous Dimensions $\propto T_F$

## Transversity:

$$\begin{aligned}
 \gamma_{qq,NS,TR}^{(1)} &= -C_F \left( \frac{C_A}{2} - C_F \right) \left[ 64S_{-2,1} + \frac{2(17N^2 + 17N - 12)}{3N(N+1)} - 64S_{-2}S_1 - \frac{1072}{9}S_1 \right. \\
 &\quad \left. + \frac{176}{3}S_2 - 32S_3 - 32S_{-3} \right] + C_F^2 \left[ S_1 \left[ \frac{1072}{9} - 32S_2 \right] - \frac{104}{3}S_2 - \frac{43}{3} \right] \\
 &\quad + C_F T_F N_F \left[ -\frac{160}{9}S_1 + \frac{32}{3}S_2 + \frac{4}{3} \right], \\
 \gamma_{qq,NS,TR}^{(1)} &= -C_F \left( \frac{C_A}{2} - C_F \right) \frac{8}{N(N+1)}, \\
 \gamma_{qq,NS,TR}^{(2)} &= C_F^2 T_F N_F \left\{ \frac{256}{3}S_{3,1} + \left[ -\frac{512}{3}S_{-2,1} + \frac{1280}{9}S_2 - \frac{512}{3}S_3 - \frac{440}{3} \right] S_1 - \frac{2560}{9}S_{-2,1} \right. \\
 &\quad \left. - \frac{256}{3}S_{-2,2} + \frac{1024}{3}S_{-2,1,1} + \frac{4(207N^3 + 414N^2 + 311N + 56)}{9N(N+1)^2} - \frac{128}{3}S_2^2 - \frac{80}{3}S_2 \right. \\
 &\quad \left. + \left[ \frac{1280}{9} - \frac{256}{3}S_1 \right] S_{-3} + \left[ \frac{2560}{9}S_1 - \frac{256}{3}S_2 \right] S_{-2} + \frac{1856}{9}S_3 - \frac{512}{3}S_4 \right. \\
 &\quad \left. - \frac{256}{3}S_{-4} + \left[ 128S_1 - 96 \right] \zeta_3 \right\} \\
 &\quad + C_F C_A T_F N_F \left\{ \left[ \frac{256}{3}S_{-2,1} - \frac{16(209N^2 + 209N - 9)}{27N(N+1)} + 64S_3 \right] S_1 - \frac{256}{3}S_{3,1} \right. \\
 &\quad \left. + \frac{1280}{9}S_{-2,1} + \frac{128}{3}S_{-2,2} - \frac{512}{3}S_{-2,1,1} - \frac{16(15N^3 + 30N^2 + 12N - 5)}{3N(N+1)^2} + \left[ \frac{128}{3}S_1 \right. \right. \\
 &\quad \left. \left. - \frac{640}{9} \right] S_{-3} + \frac{5344}{27}S_2 + \left[ \frac{128}{3}S_2 - \frac{1280}{9}S_1 \right] S_{-2} - \frac{448}{3}S_3 + \frac{320}{3}S_4 \right. \\
 &\quad \left. + \frac{128}{3}S_{-4} + \left[ 96 - 128S_1 \right] \zeta_3 \right\} \\
 &\quad + C_F T_F^2 N_F^2 \left\{ \frac{8(17N^2 + 17N - 8)}{9N(N+1)} - \frac{128}{27}S_1 - \frac{640}{27}S_2 + \frac{128}{9}S_3 \right\}, \\
 \gamma_{qq,NS,TR}^{(2)} &= \frac{32}{3} C_F T_F N_F \left( \frac{C_A}{2} - C_F \right) \left[ \frac{13N+7}{3N(N+1)^2} - \frac{2}{N(N+1)} S_1 \right].
 \end{aligned}$$

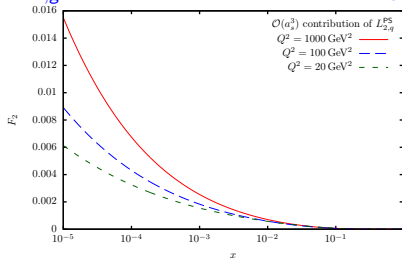
- ▶ Independent confirmation of full two-loop results.
- ▶ 1st ab initio calculation of the contribution  $\propto T_F$  at 3 loops.
- ▶ Note a typo in the 15th moment in 1203.1022.
- ▶ Independent calculation of the anomalous dimensions ( $\propto T_F$ )  $\gamma_{qq}^{NS\pm}$  and  $\gamma_{gq}$  at 3 loops.

# $L_{g,2}^S$ and $L_{q,2}^{PS}$



$\mathcal{O}(a_s^2)$   $L_{2,g}^S$

$\mathcal{O}(a_s^3)$   $L_{2,g}^S$



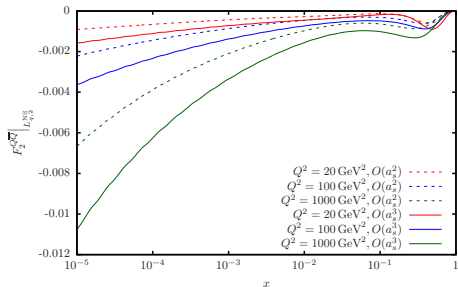
$L_{q,2}^{PS}$



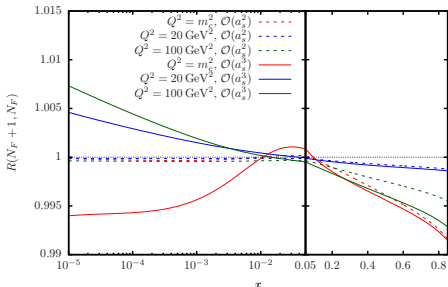
# 3-Loop OME: Non-Singlet

$$\begin{aligned}
 a_{NS}^{(3)}(N) = & C_F^2 T_F \left\{ \left[ \frac{128}{27} S_2 - \frac{16(2N+1)}{27N^2(N+1)^2} \right] S_1^2 + \left[ (-1)^N \frac{64(2N^2+2N+1)}{9N^3(N+1)^3} \right. \right. \\
 & - \left. \frac{8(2N^3+7N^4+3N^5-9N^2-7N+2)}{9N^3(N+1)^3} - \frac{64}{9N(N+1)} S_2 + \frac{64}{3} S_{-1} - \frac{128}{9} S_{-2,1} - \frac{256}{9} S_{-2,1} \right] S_1^2 \\
 & + \left[ \frac{64}{9} S_2^2 + \frac{16(448N^4+896N^5+484N^2+54N+45)}{81N^2(N+1)^2} S_2 - (-1)^N \frac{32P_1}{27N^3(N+1)^4} \right. \\
 & + \frac{8P_3}{81N^4(N+1)^4} - \frac{64(40N^2+40N+9)}{27N(N+1)} S_3 + \frac{704}{9} S_4 + \frac{128}{9} S_{-1,1} S_2 + \frac{320}{9} S_{-1,1} \\
 & \left. - \frac{256(10N^2+10N-3)}{27N(N+1)} S_{-2,1} - \frac{256}{9} S_{-2,2} + \frac{64}{3} S_{-2,1,1} + \frac{1024}{9} S_{-2,1,1} \right] S_1 \\
 & - \frac{16(31N^2+31N-6)}{27N(N+1)} S_2^2 + (-1)^N \frac{16P_2}{81N^3(N+1)^3} + \frac{P_4}{162N^3(N+1)^3} + \left[ \frac{16(3N^2+3N+2)}{3N(N+1)} \right. \\
 & - \frac{64}{3} S_1 \left] S_{-4} + \left[ \frac{256S_1(N)}{9} - \frac{128(10N^2+10N+3)}{27N(N+1)} \right] B_4 + \left[ 96S_1(N) \right. \\
 & \left. - \frac{24(3N^2+3N+2)}{N(N+1)} \right] \zeta_4 + \left[ \frac{128}{9} |S_1^2 + S_2| - \frac{128(10N^2+10N+3)}{27N(N+1)} S_1 \right. \\
 & \left. + \frac{64(112N^3+224N^2+169N+39)}{81N(N+1)^2} \right] S_{-3} - \frac{176(17N^2+17N+6)}{27N(N+1)} S_4 \\
 & + \frac{8(1301N^4+2602N^3+2177N^2+492N-84)}{81N^2(N+1)^2} S_5 + \frac{512}{9} S_5 + \frac{256}{9} S_{-3} + \left[ \frac{256}{27} S_1^2 \right. \\
 & - \frac{128}{9N(N+1)} S_1^2 + \frac{128(112N^4+224N^3+121N^2+9N+9)}{81N^3(N+1)^2} S_1 \\
 & \left. - \frac{64(181N^4+266N^3+82N^2-3N+18)}{81N^3(N+1)^3} - \frac{512}{9} S_{-2,1} - \frac{1280}{27} S_2 + \frac{512}{27} S_2 \right] S_{-2} \\
 & + \frac{16(7N^4+14N^3+3N^2-4N-4)}{9N^2(N+1)^2} S_{-4} + \frac{256}{9} S_{-2,3} - \frac{512}{9} S_{-2,3} + \frac{16(89N^2+89N+30)}{27N(N+1)} S_{-3,1} \\
 & - \frac{512}{9} S_{-4,1} - \frac{128(112N^3+112N^2-39N+18)}{81N^2(N+1)} S_{-2,1} + \left[ \frac{64(-1)^N(2N^2+2N+1)}{9N^3(N+1)^3} \right. \\
 & - \frac{8P_3}{81N^3(N+1)^3} + \frac{256}{27} S_3 + \frac{256}{3} S_{-2,1} \left] S_{-2} - \frac{128(10N^2+10N-3)}{27N(N+1)} S_{-2,2} + \frac{512}{9} S_{-2,3} \\
 & - \frac{16(3N^2+3N+2)}{3N(N+1)} S_{-2,1,1} + \frac{512}{9} S_{-2,1,1} - \frac{256}{9} S_{-2,1,1} + \frac{512(10N^2+10N-3)}{27N(N+1)} S_{-2,1,1} \\
 & + \frac{512}{9} S_{-2,2,1} - \frac{2048}{9} S_{-2,1,1,1} + \left[ (-1)^N \frac{16(2N^2+2N+1)}{3N^3(N+1)^3} + \frac{P_6}{3N^3(N+1)^3} \right] + \left[ \frac{64}{3} S_1 \right. \\
 & - \frac{32}{3N(N+1)} S_{-2} - \frac{8(3N^2+3N+2)}{3N(N+1)} S_2 + \left[ \frac{8(15N^4+30N^3+15N^2-4N-2)}{3N^2(N+1)^2} - \frac{4N+2}{3} \right] S_1 \\
 & \left. + \frac{32}{3} S_3 + \frac{32}{3} S_{-3} - \frac{64}{3} S_{-2,1} \right] \zeta_4 + \left[ \frac{2(561N^4+1122N^3+767N^2+302N+48)}{9N^2(N+1)^2} - \frac{1208}{9} S_1 \right. \\
 & \left. + C_F T_F^2 \left\{ \frac{4P_5}{729N^4(N+1)^4} - \frac{1924S_1(N)}{729} - \frac{1856S_2(N)}{81} - \frac{608S_3(N)}{81} - \frac{128S_4(N)}{27} \right\} \right. \\
 & + N_F \left[ \frac{2P_5}{729N^4(N+1)^4} - \frac{5552}{729} S_1 + \frac{640}{27} S_2 - \frac{320}{81} S_3 + \frac{64}{27} S_4 \right] \\
 & + \left[ \frac{4(3N^4+6N^3+47N^2+20N-12)}{27N^3(N+1)^3} - \frac{160}{27} S_1 + \frac{32}{9} S_2 \right] (2 + N_F) \zeta_4 \\
 & \left. + \left[ \frac{256(3N^2+3N+2)}{27N(N+1)} - \frac{1024}{27} S_1 + N_F \left[ \frac{448}{27} S_1 - \frac{112(3N^2+3N+2)}{27N(N+1)} \right] \right] \zeta_4 \right\} \\
 & - \left. \frac{64}{3} S_2 \right] \zeta_4 \left. \right\} \\
 & + C_F C_A T_F \left\{ \frac{64}{27} S_2 S_1^2 + \left[ (-1)^N \frac{32(2N^2+2N+1)}{9N^3(N+1)^3} + \frac{4P_5}{9N^3(N+1)^3} + \frac{32}{9N(N+1)} S_2 - \frac{80}{9} S_{-3} \right. \right. \\
 & + \frac{128}{9} S_{-2,1} + \frac{128}{9} S_{-2,1} \left] S_1^2 + \left[ \frac{80(2N+1)^2}{9N(N+1)^2} S_3 + \frac{112}{9} S_2^2 + (-1)^N \frac{16P_1}{27N^4(N+1)^4} + \frac{4P_{10}}{729N^4(N+1)^4} \right. \\
 & - \frac{16(N-1)(2N^3-N^2-N-2)}{9N^2(N+1)^2} S_2 - \frac{208}{9} S_4 - \frac{8(9N^2+9N+16)}{9N(N+1)} S_{-2,1} + \frac{64}{3} S_{-3,1} \\
 & \left. + \frac{128(10N^2+10N-3)}{27N(N+1)} S_{-2,1} + \frac{128}{9} S_{-2,2} - 32S_{-2,1,1} - \frac{512}{9} S_{-2,1,1} \right] S_1 \\
 & - \frac{4(15N^2+15N+14)}{9N(N+1)} S_2^2 + \frac{24(N-1)(N^2+2)}{5N(N+1)^2} \zeta_2 - (-1)^N \frac{8P_3}{81N^3(N+1)^3} + \frac{P_{11}}{1458N^3(N+1)^3} \\
 & + \left[ \frac{12(5N^3+13N^2+8N+6)}{N(N+1)^2} - 96S_1 \right] \zeta_4 + \left[ \frac{64(10N^2+10N+3)}{27N(N+1)} - \frac{128}{9} S_1 \right] S_{-4} \\
 & + \left[ \frac{32}{3} S_1 - 8(3N^2+3N+2) \right] B_4 + \left[ \frac{64}{9} |S_1^2 + S_2| + \frac{64(10N^2+10N+3)}{27N(N+1)} S_1 \right. \\
 & \left. - \frac{32(112N^3+224N^2+169N+39)}{81N(N+1)^2} \right] S_{-3} - \frac{8P_{12}}{81N^2(N+1)^2} S_3 \\
 & + \frac{4(311N^2+311N+78)}{27N(N+1)} S_4 - \frac{224}{9} S_5 - \frac{128}{9} S_{-5} - \frac{4(2N^3-35N^2-37N-24)}{9N^3(N+1)^3} |S_1^2 + S_2| \\
 & - \frac{8P_{13}}{9N^2(N+1)^2} S_{-2,1} + \left[ -\frac{64(112N^4+224N^3+121N^2+9N+9)}{81N^2(N+1)^2} S_1 - \frac{128}{27} S_1^2 + \frac{64}{9N(N+1)} S_1^2 \right. \\
 & \left. - \frac{640}{27} S_2 - \frac{256}{27} S_3 + \frac{256}{9} S_{-2,1} + \frac{32(181N^4+266N^3+82N^2-3N+18)}{81N^3(N+1)^3} \right] S_{-2} \\
 & - \frac{128}{3} S_{-2,3} + \frac{256}{9} S_{-2,3} - \frac{8(13N+4)(13N+9)}{27N(N+1)} S_{-2,1} + \frac{256}{9} S_{-4,1} + \left[ (-1)^N \frac{32(2N^2+2N+1)}{9N^3(N+1)^3} \right. \\
 & - \frac{4P_{14}}{81N^3(N+1)^3} + \frac{496}{27} S_3 - \frac{64}{3} S_{-2,1} - \frac{128}{3} S_{-2,1} \left] S_2 + \frac{64(10N^2+10N-3)}{27N(N+1)} S_{-2,2} \\
 & + \frac{64(112N^3+112N^2-39N+18)}{81N^2(N+1)} S_{-2,1} - \frac{256}{9} S_{-2,3} + \frac{8(3N^2+3N+2)}{N(N+1)} S_{-2,1,1} - \frac{256}{9} S_{-2,1,1} \\
 & + \frac{64}{3} S_{-2,2,1} - \frac{256}{9} S_{-2,1,1,1} - \frac{256(10N^2+10N-3)}{27N(N+1)} S_{-2,1,1} - \frac{256}{9} S_{-2,2,1} + \frac{224}{9} S_{-2,1,1,1} \\
 & + \frac{1024}{9} S_{-2,1,1,1} + \left[ (-1)^N \frac{8(2N^2+2N+1)}{3N^3(N+1)^3} + \frac{P_{15}}{27N^3(N+1)^3} \right] + \left[ \frac{16}{3N(N+1)} - \frac{32}{3} S_1 \right] S_{-2} \\
 & - \frac{16}{27} S_1 - \frac{88}{9} S_2 - \frac{16}{3} S_3 - \frac{16}{3} S_{-3} + \frac{32}{3} S_{-2,1} \left. \right] \zeta_4 + \left[ -16S_1^2 + \frac{4(637N^2+637N+108)}{27N(N+1)} S_1 \right. \\
 & \left. + \frac{P_{16}}{27N^2(N+1)^2} + 16S_2 \right] \zeta_4 \left. \right\}
 \end{aligned}$$

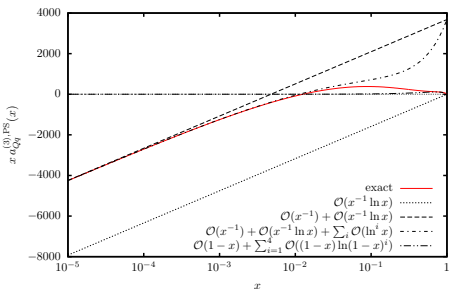
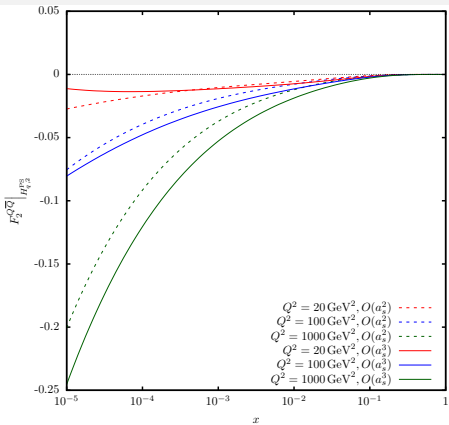




Contribution to  $F_2(x, Q^2)$



VFNS matching


 $a_{Qq}^{(3),\text{PS}}$ 

 Contribution to  $F_2(x, Q^2)$ 

The leading small  $x$  approximation corresponding to CCH, 1991, departs from the physical result everywhere except for  $x = 1$ .

### 3. Conclusions

- ▶ 2009: 10-14 Mellin Moments for all massive 3-loop OMEs, WC.  
2010: Wilson Coefficients  $L_q^{(3),PS}(N)$ ,  $L_g^{(3),S}(N)$ .
- ▶ 2013: Ladder, V-Graph and Benz-topologies for graphs, with no singularities in  $\varepsilon$  can be systematically calculated for **general  $N$** .
- ▶ Here **new functions** occur (including a larger number of root-letters in iterated integrals).
- ▶ A method for the calculation of **graphs with two massive lines** of equal masses and operator insertions has been developed and applied  $A_{gg,Q}^{(3)}$ .
- ▶ The method can be generalized to the case of unequal masses. Here the moments for  $N = 2, 4, 6$  for all graphs with two quark lines of unequal masses are now known [ $\rightarrow$  **extended renormalization**].
- ▶ The  $O(\alpha_s^2)$  charged current Wilson coefficients have been completed.
- ▶ 2014  $L_q^{NS,(3)}$ ,  $A_{gg,Q}^{S,(3)}$ ,  $A_{qq,Q}^{NS,TR(3)}$ ,  $H_{2,q}^{PS(3)}$  and  $A_{Qq}^{PS(3)}$  were completed.

### 3. Conclusions

- ▶ The corresponding 3-loop anomalous dimensions were computed, those for **transversity** for the first time ab initio.
- ▶  $A_{gg,Q}^{(3)}$  and  $A_{Qg}^{(3)}$  are underway. Here all the topologies, including the ladder- and V-topologies have been solved.
- ▶ Different new computer-algebra and mathematical technologies were developed.
- ▶ A recent update of the **ABM** parton distribution functions has been obtained including the **LHC Drell-Yan** (ATLAS, CMS, LHCb) and first **Jet** data (ATLAS) improving the PDFs.
- ▶ The benchmarks for LHC were updated.
- ▶ The DIS  $\alpha_s(M_Z^2)$  remains low  $\sim 0.1140$ . The inclusion of the LHC jet-data lead to lower values of  $\alpha_s(M_Z^2)$  also in other analyses. [Final NNLO jet analysis after arrival of the complete corrections  $\sim 6$  months.]
- ▶ Further **highly precise data** from HERA and the LHC are hoped to come to be used in the **upcoming higher loop professional QCD analyses**.