Integrability and scattering amplitudes in $\mathcal{N} = 4$ sYM

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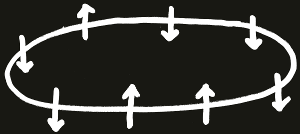
- what is integrability? (example)

- susy Yang-Mills as an integrable model
    - spectrum
    - scattering amplitudes

The Heisenberg model



$$H \sim \sum_{i=1}^{N} \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} \sim \sum_{i=1}^{N} \mathbb{P}_{ii+1}$$

$\mathbb{P}\, v \otimes w = w \otimes v$

diagonalise $2^N \times 2^N$ matrix

[Bethe 30's] [Faddeev et al. 70's]

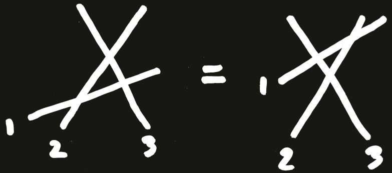
Hilbert space: $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ (N times)

R matrix

$$R_{ij}(u) = u\mathbb{I}_{ij} + i\mathbb{P}_{ij}$$

Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$



monodromy matrix

$$M_0(u) = R_{01}(u) \cdots R_{0N}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_0$$

transfer maxtrix

$$T(u) = \mathrm{tr}_0\, M_0(u) = A(u) + D(u)$$

Yang-Baxter

$$\implies R_{ab} M_a M_b = M_b M_a R_{ab}$$

$$\implies [T(u), T(v)] = 0$$

$$H_{\text{Heisenberg}} = \frac{\mathrm{d}}{\mathrm{d}u} \log T(u)|_{u=0}$$

diagonalise $T(u)$

Ansatz: $|\psi\rangle = B(u_1)\cdots B(u_M) \quad |\Omega\rangle$

"creation operators" "vacuum"

$|\uparrow\uparrow\cdots\uparrow\uparrow\rangle$

determine $u_1, \cdots, u_M$, s.t.

$$T(u)\,|\psi\rangle = (A(u) + D(u))\,|\psi\rangle = f(u)\,|\psi\rangle$$

$$R_{ab} M_a M_b = M_b M_a R_{ab}$$

$$\Longrightarrow \begin{cases} [A(u), B(v)] = \cdots \\ [D(u), B(v)] = \cdots \end{cases}$$

condition for eigenvectors

by commuting A and D past the B's

Bethe equations

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^N = -\prod_{k=1}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}$$

energy: $$E = \sum_{j=1}^{M} \frac{1}{u_j^2 + \frac{1}{4}}$$

extreme simplification of the problem
solve M algebraic equations

$$R_{ab} M_a M_b = M_b M_a R_{ab}$$

$$\simeq \text{ algebra for } \begin{cases} A(u) = A_0 + uA_1 + \cdots \\ B(u) = B_0 + uB_1 + \cdots \\ C(u) = C_0 + uC_1 + \cdots \\ D(u) = D_0 + uD_1 + \cdots \end{cases}$$

∞-dimensional algebra called the
Yangian (of $g\ell(2)$)

"integrability $\sim$ hidden infinite symmetry + tools"

$\mathcal{N} = 4$ super Yang-Mills

unique maximally supersymmetric gauge theory in 4d

fields: $A_\mu \quad \psi^i_\alpha \quad \bar{\psi}^i_{\dot{\alpha}} \quad \phi^{ij}$

all in the adjoint of the gauge group $\mathrm{SU}(N)$ $\quad (N \to \infty)$

- $\beta = 0$, superconformal, $\mathrm{psu}(2,2|4)$
- AdS/CFT: $\mathcal{N} = 4$ sYM= Type IIb string on $\mathrm{AdS}_5 \times \mathrm{S}^5$
- testing ground for new techniques
- integrablity

The spectral problem

gauge invariant operators

$$\mathcal{O}(x) = \mathrm{tr}(\phi(x)\psi(x)\phi(x)\cdots)$$

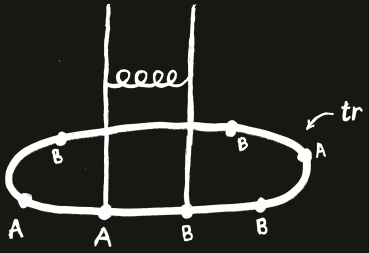
they renormalise and mix: $\mathcal{O}_{\mathrm{ren},\,a} = Z_{ab}\mathcal{O}_b$

observable: anomalous dimension $\gamma = \frac{\mathrm{d}\log Z}{\mathrm{d}\log \Lambda}$

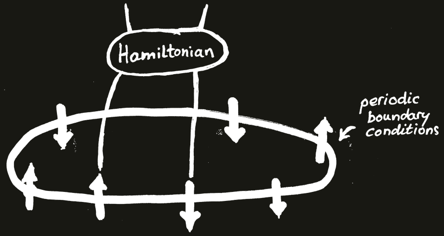
scaling dimension / energy spectrum

Z_{ab} at one-loop

for two scalars A, B



Heisenberg model



$$\gamma_{\text{one-loop}} = H_{\text{Heisenberg}}$$

can be lifted to all operators / Yangian(su(2)) $\to$ Yangian(psu(2, 2|4))

all-loop / finite coupling

Scattering amplitudes

incredible progress

(for gauge theories in general)

- spinor helicity variables, color decomposition
- unitarity methods:

  BCFW recursion, generalised unitarity,. . .

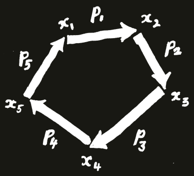
in $\mathcal{N} = 4$ sYM: connection to integrability

Symmetries in $\mathcal{N} = 4$ sYM

superconformal invariance

$$J = \left\{ \begin{array}{ll} L = \sum_i \lambda_i \tilde{\partial}_i & P = \sum_i \lambda_i \tilde{\lambda}_i \\ K = \sum_i \partial_i \tilde{\partial}_i & \tilde{L} = \sum_i \tilde{\lambda}_i \partial_i \end{array} \right\} \qquad J\,\mathcal{A} = 0$$

dual superconformal invariance



x_i: "region momenta"

superconformal generators in x-space

$J^{\text{dual}}\,\mathcal{A} = 0!$

now we can look at

$$[J, J^{\text{dual}}] = \text{new generators}$$

$$\begin{array}{r} \text{superconformal psu}(2,2|4) \\ +\text{dual superconformal psu}(2,2|4) \\ \hline \underline{\text{Yangian}} \text{ of psu}(2,2|4) \end{array}$$

Amplitudes are Yangian invariant

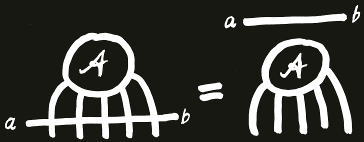
– hint at integrability

remember: we can pack all Yangian generators into a "monodromy matrix"

$$M(u) =$$

$$= \mathbb{I} + uJ + \cdots$$

Yangian invariance $\simeq M(u)\mathcal{A} = \mathcal{A}$

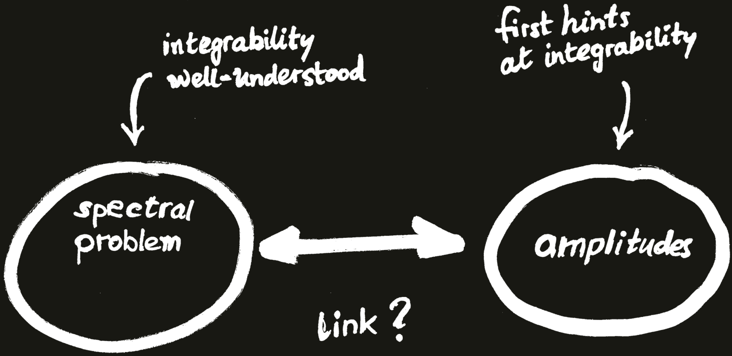


$$\implies T(u)\mathcal{A} = \text{const} \times \mathcal{A}$$

"amplitude = special Bethe vector"

Summary

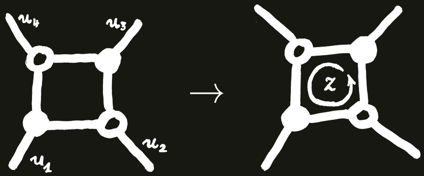
integrability in $\mathcal{N} = 4$ sYM



References

Integrability	Bethe, "Zur Theorie der Metalle" (1931)
	Franchini, "Notes on Bethe Ansatz Techniques"
	Faddeev, hep-th/9605187
	Dorey, hep-th/9810026
Spectral problem	Beisert et al., 1012.3983
	Gromov et al., 1405.4857
Amplitudes	Elvang & Huang, 1308.1697
	Drummond, Henn & Plefka, 0902.2987
	Arkani-Hamed et al., 1212.5605
	Frassek et al., 1312.1693
	Ferro et al., 1308.3494

- introduce deformations / spectral parameters

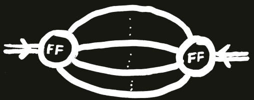


$\sim$ R matrix!

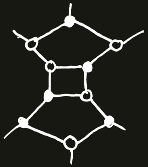
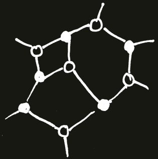
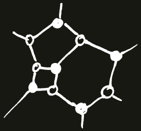
- form factors $\langle 0|\,\mathcal{O}(x)\,|\Phi_1 \cdots \Phi_n\rangle$

off-shell on-shell

very similar to amplitudes,
generalised unitarity, BCFW, ...

$\langle \mathcal{O}\mathcal{O}\rangle \sim$ 

example for on-shell diagrams

$\mathcal{A}_{6,3} =$  $+$  $+$ 

basic building blocks:

$\mathcal{A}_{3,2} =$  $\mathcal{A}_{3,1} =$ 

only on-shell gauge-invariant quantities!

The Grassmannian integral

"generating function" for amplitudes:

$$\mathcal{A}_{n,k} = \int \frac{d^{k\times n} c_{ij} / \text{Vol}[GL(k)]}{M_1 \cdots M_n} \delta^{k\times 2}(C\cdot\tilde{\lambda})\delta^{2\times(n-k)}(\lambda\cdot C^{\perp})\delta^{k\times 4}(C\cdot\tilde{\eta})$$

C: $k \times n$ matrix (c_{ij})
M_i: i'th $k \times k$ minor of C

BCFW terms as residues