Computing correlation functions

LAP08, DESY Zeuthen

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The idea of these lectures

Assume you were given a set of gauge configurations

- you want to compute basic observables like
  - pion mass
  - pion decay constant
  - pion electromagnetic form factor
  - ...
- what has to be done to compute these observables?
- these lectures should provide and explain some tools that are necessary to complete this task
- by no means comprehensive!

Content

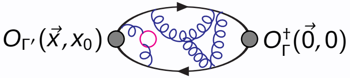
1st lecture

- correlation functions
- quark propagators
  - types of quark sources
  - inversion algorithm
  - sequential sources
  - smearing
  - boundary conditions
- ratios of correlation functions
- the Dublin approach
- deflation

2nd lecture

- implementation of these ideas in chroma
- do it yourself

Meson correlation functions

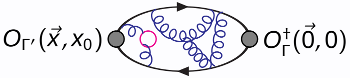


Assume generic interpolating operator $O^{12}_\Gamma(x) = \bar{\psi}_a(x)\Gamma\psi_b(x)$:

- 2pt-function

$$C_2(\vec{p},x_0) \quad = \quad \sum_{\vec{x}} e^{-i\vec{p}\vec{x}} \left\langle O^{12}_{\Gamma'}(\vec{x},x_0)\left(O^{12}_\Gamma(\vec{0},0)\right)^\dagger\right\rangle_{\rm QCD}$$

Meson correlation functions



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\begin{aligned}
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&= \sum_{\vec{x}} e^{-i\vec{p}\vec{x}} \left\langle {\rm Tr}\left\{S_1^\dagger(\vec{x},x_0;\vec{0},0)\gamma_5\Gamma' S_2(\vec{x},x_0;\vec{0},0)\gamma_0\Gamma^\dagger\gamma_0\gamma_5\right\}\right\rangle_{\rm QCD}
\end{aligned}
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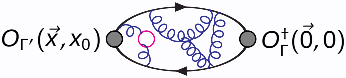
Meson correlation functions



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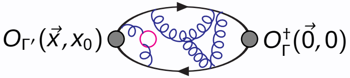
&\overset{t\to\infty}{=} \frac{1}{E(\vec{p})} |\langle 0|P|\pi\rangle|^2 e^{-E(\vec{p})T/2} \cosh\left((t-T/2)E(\vec{p})\right)
\end{aligned}
$$

Meson correlation functions



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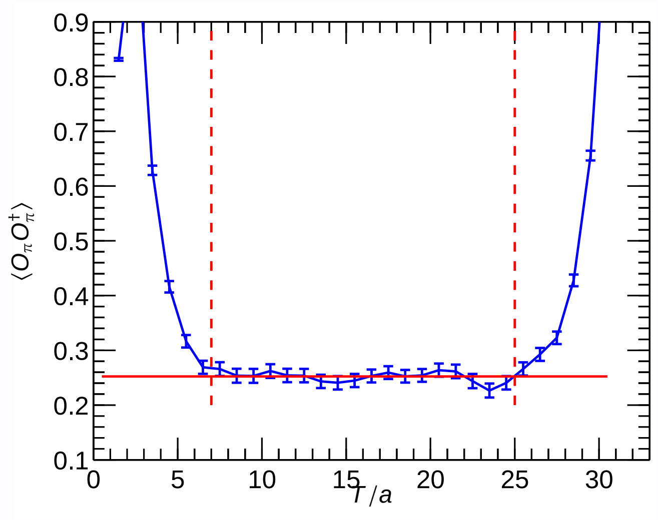
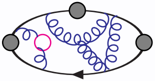
$$

- more complicated if disconnected contribution

Meson correlation functions



Meson correlation functions



3pt-function

$$
\begin{aligned}
C_3(\dots) &= \sum_{\vec{x},\vec{y}} e^{i\vec{p}_f\cdot(\vec{x}-\vec{y})} e^{i\vec{p}_i\cdot\vec{x}} \langle\, O_{\Gamma'}(\vec{x}, x_0)\, j_{\Gamma_j}(\vec{y}, y_0)\, O_\Gamma^\dagger(\vec{0}, 0)\,\rangle_{\rm QCD} \\
&= \sum_{\vec{x},\vec{y}} e^{i\vec{p}_f(\vec{x}-\vec{y})} e^{i\vec{p}_i\vec{y}} \langle {\rm Tr}\left\{ S_1(\vec{0},0;\vec{x},x_0)\Gamma' S_3(\vec{x},x_0;\vec{y},y_0)\Gamma_j S_2(\vec{y},y_0;\vec{0},0)\gamma_0\Gamma^\dagger\gamma_0)\right\}\rangle_{\rm QCD} \\
&= \frac{Z_i Z_f}{4E_i E_f} \langle\, \Gamma'(\vec{p}_f)\,|\, j_{\Gamma_j}(0)\,|\,\Gamma_\Gamma(\vec{p}_i)\,\rangle_{\rm QCD} \\
&\quad \times \left\{ \theta(x_0 - y_0)\, e^{-E_i y_0 - E_f(x_0-y_0)} \;-\; \theta(y_0 - x_0)\, e^{-E_i(T-y_0)-E_f(y_0-x_0)} \right\}
\end{aligned}
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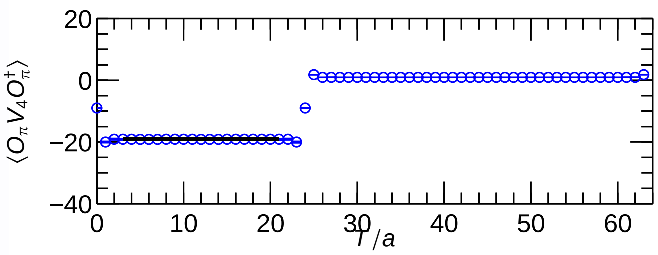

Gamma matrices and symmetries

Gupta, hep-lat/9807028

State	$I^G(J^{PC})$	Operator
Scalar(σ)	$1^-(0^{++})$	$\overline{u}(x)d(x)$
	$1^-(0^{++})$	$\overline{u}(x)\gamma_4 d(x)$
Pseudoscalar	$1^-(0^{-+})$	$\overline{u}(x)\gamma_5 d(x)$
	$1^-(0^{-+})$	$\overline{u}(x)\gamma_4\gamma_5 d(x)$
Vector	$1^+(1^{--})$	$\overline{u}(x)\gamma_i d(x)$
	$1^+(1^{--})$	$\overline{u}(x)\gamma_i\gamma_4 d(x)$
Axial (a_1)	$1^-(1^{++})$	$\overline{u}(x)\gamma_i\gamma_5 d(x)$
Tensor(b_1)	$1^+(1^{+-})$	$\overline{u}(x)\gamma_i\gamma_j d(x)$

Also very helpful: symmetry transformation of lattice propagators

Bernard, Lectures given at TASI '89

$$
\begin{aligned}
P: &\quad Q(x,y,[U]) = \gamma^0 Q(x^P, y^P, [U]^P)\gamma^0 \\
T: &\quad Q(x,y,[U]) = \gamma^0\gamma^5 Q(x^\tau, y^\tau, [U]^\tau)\gamma^5\gamma^0 \\
C: &\quad Q(x,y,[U]) = C\widetilde{Q}(y,x,[U]^c)C^{-1} \\
H: &\quad Q(x,y,[U]) = \gamma_5 Q^\dagger(y,x,[U])\gamma_5 \\
CH: &\quad Q(x,y,[U]) = C\gamma_5 Q^*(x,y,[U]^c)\gamma_5 C^{-1}\ ,
\end{aligned}
$$

where $C = \gamma_0\gamma_2$ and $C\gamma_\mu C = -\gamma_\mu^T$

Definition of the propagator $S(\vec{y}, y_0; \vec{x}, x_0)$ - the inverse of the Dirac operator:

$$\underbrace{D^{ab}_{\alpha\beta}(z,y)}_{(V\cdot 4\cdot 3\cdot \mathbb{C})^2} S^{bc}_{\beta\gamma}(y,x) = \delta(z-x)\delta^{ac}\delta_{\alpha\gamma}$$

e.g. $24^3 \times 64 \times 4 \times 3 = 10616832$
$\rightarrow$ huge!!!! ”naive” numerical inversion is impossible

Let's phrase the problem in more general terms: we want $S = D^{-1}$ where S is a $N \times N$-matrix

- exact solution: solve N linear problems

$$Dz_1 = e_1, \ldots, Dz_N = e_N$$

  where e_i is ith column of id_N
  result: $N \times N$-matrix $S = [z_1, \ldots, z_N]$
  example: I think nobody has ever done it for LQCD

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  example: point source propagator (similarly for smeared source props):
  $N \times 12$-matrix with e.g. $V = [e_1, \ldots, e_{12}]$
  this corresponds to a propagator from one space-time point to all others (point-to-all propagator)

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- this discards valuable information encoded in the gauge configurations
- idea: look for W such that $(SV)W = S$; i.e. $VW = 1$ (impossible since rank of V, W would be smaller than N in practical implementations)
- but: $X = VW \approx 1$ through a Monte Carlo method for choosing V and W

before studying different choices for the quark source let's look at how to compute the propagator given a source b

- the generic problem: solve

$$Ax = b$$

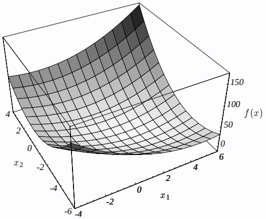
  for the vector x
  this is a minimisation problem:
  consider

$$f(x) = \frac{1}{2}x^T Ax - b^T x + c$$

  the minimum of this function is

$$f'(x) = 0 = \frac{1}{2}(Ax + x^T A) - b$$

  so for positive definite and symmetric A the minimum of $f(x)$ is what we are looking for $Ax = b$



entertaining reading: An Introduction to the Conjugate Gradient Method Without the Agonizing Pain Edition $1\frac{1}{4}$ by Jonathan Richard Shewchuk

- the problem of finding x is equivalent to find the extremum of a quadratic form

- correlation functions in terms of propagators ✓
- algorithm for computation of correlators ✓
- choosing a clever quark source

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- algorithm for computation of correlators ✓
- choosing a clever quark source
    - point source (nothing to say here)
    - noise source
    - smeared source
    - sequential sources

Noise source

Comp. Phys. Comm. 78 (1994) 256-264

- remember: look for W such that $(SV)W = S$; i.e. $VW \approx 1$
- the approximation $X = VW \approx 1$ is reached through e.g. a set of individually, identically distributed random numbers $\{\mu_{nl}\}$ where $n = 1, \ldots, N$, $l = 1, \ldots, L$ and which satisfy

$$E[\mu_{mk}(\mu_{nl})^*] = \delta_{mn}\delta_{kl}$$

(note: $E[\cdot]$ is expectation value over a series of experiments)

- we construct V as $V = \frac{1}{\sqrt{L}}[\mu_{nl}]$
- and $W \equiv (V^*)^T$ to construct

$$X \equiv VW$$

with elements $x_{mn} = \frac{1}{L}\sum_{l=1}^{L} \mu_{ml}\mu_{nl}^*$

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- The noise-average has the following properties

$$\begin{aligned}
E[x_{mn}] &= \delta_{mn} \\
E[(x_{mn} - \delta_{mn})(x^*_{pq} - \delta_{pq})] &= 0 \qquad \text{if } (mn) \neq (pq) \text{ or } (mn) \neq (qp) \\
E[|x_{mn} - \delta_{mn}|^2] &= \frac{1}{L} \qquad \text{if } m \neq n
\end{aligned}$$

Convergence of the noise source propagator

- let's look at the trace of the propagator (we always look at traces of objects constructed from propagators...)

$$\begin{aligned} E\,[\mathrm{Tr}(SX)] &= \\ var\,[\mathrm{Tr}(SX)] &= \end{aligned}$$

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- choices for the noise source which minimise the variance:
  - $Z(2)$-noise: $\mathcal{P}[\eta_{nl}=1]=\frac{1}{2}$ and $\mathcal{P}[\eta_{nl}=-1]=\frac{1}{2}$ since $var[Tr(SX^{Z(2)})]=\sum_{m\neq n}\frac{|s_{mn}|^2+s_{mn}s_{nm^*}}{L}$

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  - $Z(J)$-noise $\mathcal{P}[\mu^J_{nl}=e^{2\pi lj/J}]=\frac{1}{J}$
    since $var[Tr(SX^{Z(J)})]=\sum_{m\neq n}\frac{|s_{mn}|^2}{L}$
  - difficult to say which one is better - depends on term $\sum_{m\neq n} s_{mn}s^*_{nm}$

Constructing meson correlation functions from noise source props

- 2pt function from point source:

$$C_2(\vec{0},t) = \sum_{\vec{x}} \left\langle \mathrm{Tr}\left\{ S_1^\dagger(\vec{x},x_0;\vec{0},0)\gamma_5\Gamma' S_2(\vec{x},x_0;\vec{0},0)\gamma_0\Gamma^\dagger\gamma_0\gamma_5 \right\}\right\rangle_{\mathrm{QCD}}$$

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- 2pt-function from noise source (one-end trick Phys. Rev. D59 074503 (1999)):

$$\begin{aligned} C_2(\vec{0},t) &= \sum_{\vec{x},\vec{y}} \left\langle \mathrm{Tr}\left\{ S_1(\vec{y},y_0;\vec{x},x_0)\Gamma' S_2(\vec{x},x_0;\vec{y},y_0)\gamma_0\Gamma^\dagger\gamma_0 \right\} \right\rangle_{\mathrm{QCD}} \\ &\approx \sum_{\vec{y}} \frac{1}{L}\sum_{l} \left\langle \mathrm{Tr}\left\{ \Psi_{1,l}(\vec{y},y_0;x_0)\left(\Psi_{2,l}^{\gamma_5\Gamma'}(\vec{y},y_0;x_0)\right)^\dagger \gamma_5\gamma_0\Gamma^\dagger\gamma_0 \right\} \right\rangle_{\mathrm{QCD}} \end{aligned}$$

where

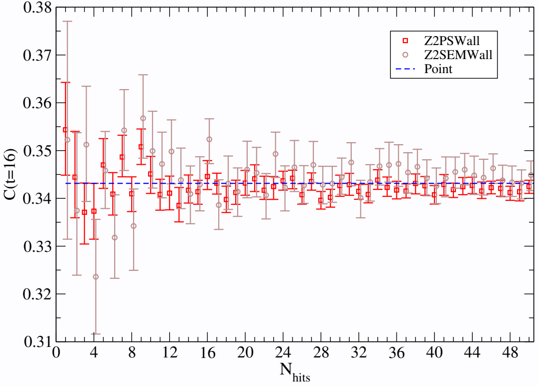
$$\begin{aligned} \Psi_{1,l}(\vec{y},y_0;x_0) &= \sum_{\vec{x}} [S_1(\vec{y},y_0;\vec{x},x_0)]_{a\alpha b\beta} \underbrace{[\eta_l(\vec{x},x_0)]_{b,\beta}}_{\text{noise source}} \\ \Psi_{2,l}^{\gamma_5\Gamma}(\vec{y},y_0;x_0) &= \sum_{\vec{x}} S_2(\vec{y},y_0;\vec{x},x_0)\underbrace{\gamma_5\Gamma\eta(\vec{x},x_0)}_{\text{noise source}} \end{aligned}$$

- $$
\begin{array}{llcl}
i) & \Psi_l(\vec{y},y_0;x_0) & = & \sum_{\vec{x}} S(\vec{y},y_0;\vec{x},x_0)\eta_l(\vec{x},x_0) \\
ii) & \Psi_l^{\gamma_5\Gamma}(\vec{y},y_0;x_0) & = & \sum_{\vec{x}} S(\vec{y},y_0;\vec{x},x_0)\gamma_5\Gamma\eta_l(\vec{x},x_0)
\end{array}
$$
- case $\Gamma = \gamma_5 \rightarrow \Psi_1^{\gamma_5\gamma_5} = \Psi_1$:
  need $\delta_{\kappa\lambda}\delta_{c,d}\delta(\vec{x}-\vec{z}) = \frac{1}{L}\sum_l \eta_l(\vec{x})_{\kappa,c}\eta_l^\dagger(\vec{z})_{\lambda,d}$
- general case: $\delta_{c,d}\delta(\vec{x}-\vec{z}) = \frac{1}{L}\sum_l \eta_l(\vec{x})_c\eta_l^\dagger(\vec{z})_d$

  in this case we need four noise vectors (spin explicit) in order to properly represent the gamma-structure
- remark for implementation: use your old inverter for noise source
  most codes expect 3×4 inversions. Here only 1 or 4 inversions, respectively, are necessary. Use standard inverter but skip inversion if norm of the source for a given *spin – color*-index vanishes
- I have secretly assumed that the noise source is limited to one time slice. This is called "dilution". The noise can be diluted more (spin-explicit, color-explicit) or e.g. onto the odd and even site on the lattice.
  Dilution in time is known to increase the signal/noise in correlation functions

Costs:

source type	# of inversions	
point source	12	-
single noise vector:	1	times the number of hits
spin explicit noise vector:	4	times the number of hits



study on a unit gauge configuration JHEP 0808:086,2008

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some experience:

- hit-average and gauge-average are often commutative
  in practice it may suffice to generate only few noise vectors per configuration
  (since the gauge noise is larger than the noise from the stochastic source)

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- when 'hitting' various times on a gauge configuration - move the source along the 0-direction; this reduces correlation effects originating in the gauge configuration
- how well the noise source works compared to the point source seems to be a function of the volume, the quark masses and the observable under investigation.
  example (JHEP 0807:112,2008):

$16^3 \times 32 \times 16$ DWF, $a^{-1} \approx 1.7$GeV, $m_\pi \approx 700$MeV noise source achieves same stat. error on m_π at half of the price	$24^3 \times 64 \times 16$ DWF, $a^{-1} \approx 1.7$GeV, $m_\pi \approx 330$MeV noise source achieves same stat. error on m_π at 1/12th of the price

Yet another source type: Gaussian smearing

Güsken et al.; NPB (Proc. Suppl.) 17 (1990) 361-364

- motivation: phenomenologically immitate what may or may not be the hadron wave function
- thus increase overlap with desired hadronic state
- similar to point source, but smearing out the delta function
- this corresponds to smear out the fermion fields

$$\tilde{\psi}(\vec{x},t) = \sum_{x'} F(\vec{x},\vec{x}')\psi(\vec{x}',t)$$

- Gaussian smearing:

$$F(x,x') = \delta_{x,x'} + \alpha H(x,x')$$

with the hopping matrix

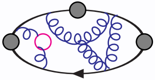
$$H(x,x') = \sum_{\mu=1}^{3} \left\{ U_\mu(x)\delta_{x',\vec{x}+\hat{\mu}} + U_\mu^\dagger(x-\hat{\mu})\delta_{x',x-\hat{\mu}} \right\}$$

with coupling α between nearest neighbours

- iterate this starting from a point source
- in the free case approximately gaussian shape and rotationally invariant

Still not enough sources: sequential source propagators

how to construct 3pt-functions?



- problem here: naively one would need an all-to-all propagator
- way out: sequential source propagator Nucl. Phys. B316 (1989) 355

$$\tilde{S}_{13}(\vec{y},y_0;\vec{p},x_0;\vec{0},0) = \sum_{\vec{x}} S_3(\vec{y},y_0;\vec{x},x_0)\underbrace{\gamma_5\Gamma'\gamma_5 S_1(\vec{x},x_0,\vec{0},0)e^{i\vec{p}\vec{x}}}_{\text{source}}$$

Still not enough sources: sequential source propagators

$$\tilde{S}_{13}(\vec{y},y_0;\vec{p}_f,x_0;\vec{0},0) = \sum_{\vec{x}} S_3(\vec{y},y_0;\vec{x},x_0)\underbrace{\gamma_5\Gamma'\gamma_5 S_1(\vec{x},x_0,\vec{0},0)e^{i\vec{p}_f\vec{x}}}_{\text{source}}$$

can be solved by

$$\sum_{z} D(\vec{z},z_0;\vec{z}',z_0')\tilde{S}(\vec{z}',z_0',\vec{0},0) = \sum_{\vec{y},y_0} e^{i\vec{p}_f\vec{y}}\delta_{z,y}\gamma_5\Gamma^\dagger\gamma_5 S(y,0)$$

with the source term

$$\sum_{\vec{y},y_0} e^{i\vec{p}_f\vec{y}}\delta_{z,y}\gamma_5\Gamma^\dagger\gamma_5 S(y,0) = \begin{cases} 0 & z_4 \neq \text{sink time slice} \\ \gamma_5\Gamma^\dagger\gamma_5 G(\vec{z},z_0,\vec{0},0)e^{i\vec{p}\vec{z}} & z_4 = \text{sink time slice} \end{cases}$$

- new inversions for each Γ and Fourier mode $e^{i\vec{p}\vec{x}}$
- straight forward to implement with noise source
- once $\tilde{S}$ has been constructed the three point function can be contracted like a two-point function

Comments on 3pt functions

- $C_3 = \sum_{\vec{x}_f,\vec{x}} e^{i\vec{p}_f\cdot(\vec{x}_f-\vec{x})} e^{i\vec{p}_i\cdot\vec{x}} \langle\, O_f(t_f,\vec{x}_f)\, j_{\Gamma_j}(t,\vec{x})\, O_i^\dagger(t_i,\vec{0})\,\rangle$
$= \sum_{\vec{x},\vec{y}} e^{i\vec{p}_f(\vec{x}-\vec{y})} e^{i\vec{p}_i\vec{y}} \langle \mathrm{Tr}\left\{S_1(\vec{0},0;\vec{x},x_0)\Gamma' S_3(\vec{x},x_0;\vec{y},y_0)\Gamma_j S_2(\vec{y},y_0;\vec{0},0)\gamma_0\Gamma^\dagger\gamma_0)\right\}\rangle$
$= \sum_{\vec{x},\vec{y}} e^{-i(\vec{p}_f-p_i)\vec{y}} \langle \mathrm{Tr}\left\{\left(\tilde{S}_{31}(\vec{y},y_0;\vec{p}_i,x_0;\vec{0},0)\right)^\dagger \gamma_5\Gamma_j S_2(\vec{y},y_0;\vec{0},0)\gamma_0\Gamma^\dagger\gamma_0\gamma_5)\right\}\rangle$

  chroma does not use the γ_0 - so there might be sign issues

Comments on 3pt functions

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  chroma does not use the γ_0 - so there might be sign issues
- we extract ME of type $\langle\, P_f(\vec{p}_f)\,|\,V_4(0)\,|\,P_i(\vec{p}_i)\,\rangle$
  It turns out to be advantageous to extract them from ratios like e.g.

$$\begin{aligned}
R_{1,P_iP_f}(\vec{p}_i,\vec{p}_f) &= 4\sqrt{E_iE_f}\,\sqrt{\frac{C_{P_iP_f}(t,\vec{p}_i,\vec{p}_f)\,C_{P_fP_i}(t,\vec{p}_f,\vec{p}_i)}{C_{P_i}(T/2,\vec{p}_i)\,C_{P_f}(T/2,\vec{p}_f)}},\\
R_{2,P_iP_f}(\vec{p}_i,\vec{p}_f) &= 2\sqrt{E_iE_f}\,\sqrt{\frac{C_{P_iP_f}(t,\vec{p}_i,\vec{p}_f)\,C_{P_fP_i}(t,\vec{p}_f,\vec{p}_i)}{C_{P_iP_i}(t,\vec{p}_i,\vec{p}_i)\,C_{P_fP_f}(t,\vec{p}_f,\vec{p}_f)}},\\
R_{3,P_iP_f}(\vec{p}_i,\vec{p}_f) &= 4\sqrt{E_iE_f}\,\frac{C_{P_iP_f}(t,\vec{p}_i,\vec{p}_f)}{C_{P_f}(T/2,\vec{p}_f)}\,\sqrt{\frac{C_{P_i}(T/2-t,\vec{p}_i)\,C_{P_f}(t,\vec{p}_f)\,C_{P_f}(T/2,\vec{p}_f)}{C_{P_f}(T/2-t,\vec{p}_f)\,C_{P_i}(t,\vec{p}_i)\,C_{P_i}(T/2,\vec{p}_i)}}.
\end{aligned} \tag{1}$$

  - sometimes cancellation of renormalisation factor
  - plateaus look different - optimise!
  - cancellation of correlations → better signal
  - time-dependence cancels - these ratios should be constant for large euclidean separations

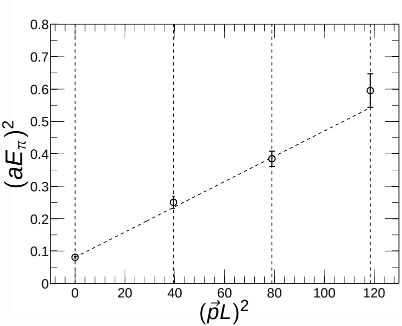
Twisted boundary conditions

periodic bc's

$$\psi(x_i + L) = \psi(x_i)$$

$$\vec{p}_{quark} = \vec{n}\frac{2\pi}{L}$$

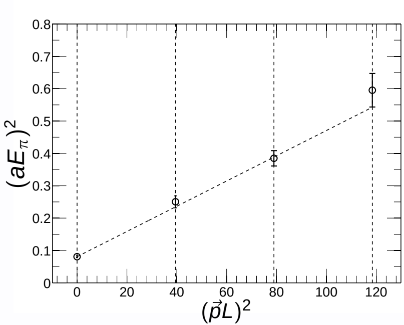
$$E_\pi = \sqrt{m_\pi^2 + (\vec{n}\frac{2\pi}{L})^2}$$

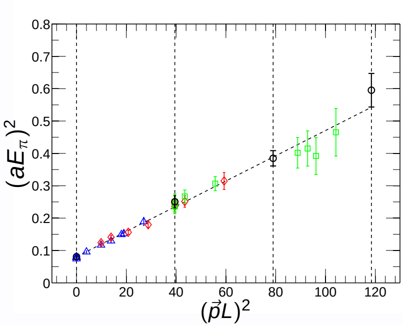


PLB 595 (2004) 408, PLB 593 (2004) 82,
PLB 609 (2005) 73, PLB 632 (2006) 313

Twisted boundary conditions

periodic bc's	twisted bc's
$\psi(x_i + L) = \psi(x_i)$	$\psi(x_i + L) = e^{i\theta_i}\psi(x_i)$
$\vec{p}_{quark} = \vec{n}\frac{2\pi}{L}$	$\vec{p}_{quark} = \vec{n}\frac{2\pi}{L} + \frac{\vec{\theta}}{L}$
$E_\pi = \sqrt{m_\pi^2 + (\vec{n}\frac{2\pi}{L})^2}$	$E_\pi = \sqrt{m_\pi^2 + (\vec{n}\frac{2\pi}{L} + \frac{\vec{\theta}_u - \vec{\theta}_d}{L})^2}$





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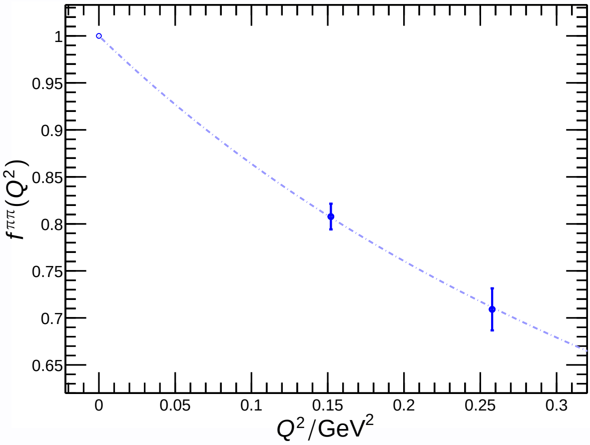
Twisted boundary conditions - applications

- pion form factor $\langle \pi(p') | V_\mu^{\mathrm{elmag}} | \pi(p) \rangle$



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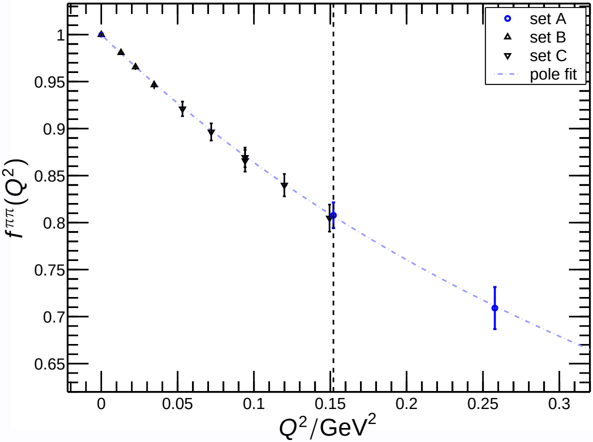

JHEP 0807:112,2008

Implementation of partially twisted BC

- twisted bc's $\quad \psi(x) = e^{i\frac{\vec{\theta}\cdot\vec{x}}{L}}\tilde{\psi}(x)$
- Wilson's hopping term:

$$\bar{\tilde{\psi}}(x)\left[e^{i\frac{a\theta_i}{L}}U_i(x)(1-\gamma_i)\tilde{\psi}(x+\hat{i}) + e^{-i\frac{a\theta_i}{L}}U_i^\dagger(x-\hat{i})(1+\gamma_i)\tilde{\psi}(x-\hat{i})\right]$$

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- equivalent: replace the link variables

$$\{U_i(x)\} \rightarrow \{e^{i\frac{a\theta_i}{L}}U_i(x)\}$$

  inverting on phase-shifted gauge field encodes the momentum shift for the valence quarks
- Applicable to other discretizations (e.g. DWF)

Kinematics with twisted bc's

- 2pt-function

$$E_\pi = \sqrt{m_\pi^2 + (\vec{n}\frac{2\pi}{L} + \frac{\vec{\theta}_u - \vec{\theta}_d}{L})^2}$$

- 3pt-function

$$q^2 = (p_i - p_f)^2 = \left\{[E_i(\vec{p}_i) - E_f(\vec{p}_f)]^2 - \left[(\vec{p}_{\mathrm{FT},i} + \vec{\theta}_i/L) - (\vec{p}_{\mathrm{FT},f} + \vec{\theta}_f/L)\right]^2\right\}$$

  - watch out for relative signs of twist angles when you construct the 3pt function
  - changing directions of twists can decrease correlation effects
  - keep track of direction of twist - correlators involving currents in spacial directions depend on the momentum

Some newer developments for computing propagators

- light quark physics is dominated by the low lying eigen modes of the Dirac operator
- noise from noise source propagators often still too large
- idea: separate treatment of low modes and high modes
  let $Q = \gamma_5 D$ be the herm. dirac operator

$$Q = Q_1 + Q_2 = \underbrace{\sum_{i=1}^{N_{\rm ev}} \lambda_i v^{(i)} v^{(i)\dagger}}_{\text{exact or other inexact treatment}} + \underbrace{\sum_{i=N_{\rm ev}+1}^{N} \lambda_i v^{(i)} v^{(i)\dagger}}_{\text{treat as a correction by noisey estimator}}$$

- exact treatment of the lowmodes: Comp. Phys Comm. 172 (2005) 145162
- inexact treatment of the lowmodes: JHEP07 (2007) 081
- the orthogonal complement can be corrected for by noise source techniques

The Dublin approach

Comp. Phys Comm. 172 (2005) 145162

- also two pieces for the quark propagator $Q^{-1} = \tilde{Q}_0 + \tilde{Q}_1$:
  $\tilde{Q}_0$ is low mode part and $\tilde{Q}_1 = Q^{-1}\mathcal{P}_1$

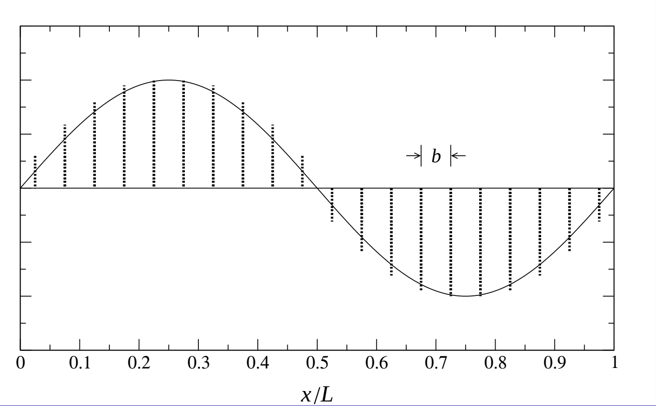
$$\mathcal{P}_1 = 1 - \mathcal{P}_0 = 1 - \sum_{j=1}^{N_{ev}} v^{(j)} v^{j\,\dagger}$$

- correct for $\tilde{Q}_1$ via $N_d \times L$ diluted noise vectors $\left\{\left(\eta_1^{(1)},...,\eta_L^{(1)}\right),\ldots,\left(\eta_1^{(N_d)},...,\eta_L^{(N_d)}\right)\right\}$ (due to diluation the noise vectors are mutually orthogonal before taking the noise average)
- the hybrid estimate for the all-to-all prop (for a single noise vector then is $\sum_{i=1}^{N_{ev+N_d}} u^{(i)}(\vec{x},x_0)w^{(i)}(\vec{y},y_0)^\dagger\gamma_5$ where
  $w^{(i)} = \left\{\frac{v^{(1)}}{\lambda_1},\ldots,\frac{v^{(N_{ev})}}{i\lambda_{N_{ev}}},\eta^{(1)},\ldots,\eta^{(N_d)}\right\}$ and
  $u^{(i)} = \left\{v^{(1)},\ldots,v^{(N_{ev})},\psi^{(1)},\ldots,\psi^{(N_d)}\right\}$
- construct observables from these vectors
- we don't know the optimal number of low modes (they are expensive to construct)
- again volume, quark mass and the observable under consideration do play a role

Approximate quark modes by constant modes

JHEP07 (2007) 081

- Lüscher splits the lattice of spatial extent L into b sub lattices (domain decomposition)
- on each sub-lattice there are 12 constant modes
- approximation of a global plane wave is already well described by this with small "deficit"
- works if fields are smooth on scale of block size b



Application to QCD

- free field far away from QCD
- helpful observation - local coherence:
  ”... a set of quark fields is referred to as locally coherent if the fields are locally well approximated by a relatively small numer of fields ...”
- this is in fact the case in lattice QCD - numerical test
  compute eigenmodes of $DD^{\dagger}$ and a domain decomposed subset of these low modes
- test on 64×32^3-lattice devided into 4^4-blocks and 12 out of 48 computed eigenmodes selected to construct the domain decomposed sub spaces
- result: the remaining 36 low modes are indeed ”coherent” with very small deficits
- open questions:
  economic/optimised way to construct deflation modes
  Lüscher uses relaxation by repeatedly applying the propagator to a random field - can one do better?
- can the low modes be applied effectively to the construction of correlation functions like in the Dublin approach?

i	$n_4n_3n_2n_1$	Γ_i	equivalent	sink insertion 3pt
0	0000	$\mathbb{1}$	$\mathbb{1}$	a0-a0
1	0001	γ_1	γ_1	a0-rho x 1
2	0010	γ_2	γ_2	a0-rho y 1
3	0011	$\gamma_1\gamma_2$	$\gamma_1\gamma_2$	a0-b1 z 1
4	0100	γ_3	γ_3	a0-rho z
5	0101	$\gamma_1\gamma_3$	$\gamma_1\gamma_3$	a0-b1 y 1
6	0110	$\gamma_2\gamma_3$	$\gamma_2\gamma_3$	a0-b1 x 1
7	0111	$\gamma_1\gamma_2\gamma_3$	$\gamma_5\gamma_4$	a0-pion 2
8	1000	γ_4	γ_4	a0-a0 2
9	1001	$\gamma_1\gamma_4$	$\gamma_1\gamma_4$	a0-rho x 2
10	1010	$\gamma_2\gamma_4$	$\gamma_2\gamma_4$	a0-rho y 2
11	1011	$\gamma_1\gamma_2\gamma_4$	$\gamma_3\gamma_5$	a0-a1 z 1
12	1100	$\gamma_3\gamma_4$	$\gamma_3\gamma_4$	a0-rho z 2
13	1101	$\gamma_1\gamma_3\gamma_4$	$\gamma_5\gamma_2$	a0-a1 y 1
14	1110	$\gamma_2\gamma_3\gamma_4$	$\gamma_1\gamma_5$	a0-a1 x 1
15	1111	γ_5	γ_5	a0-pion