Combining measurements using Best Linear Unbiased Estimators (BLUE)

Lecture 1/2 – BLUE combination basics

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About myself – context and disclaimer

- Context
  - Mainly worked as a HEP physicist until 2001
  - ALEPH and LEPEWWG, LEP2 4-fermion cross-section combinations
  - Mainly working in software and computing for HEP since 2002
  - Collaboration with TOP LHC WG on top mass combination in 2012
  - Not a member of ATLAS or CMS

- Disclaimer
  - No *direct* involvement in physics analyses for many years…
    - Will use some examples that are rather outdated
  - Will present personal opinions, not official views of LHC collaborations
  - Will only discuss precision measurements, not limits for searches
  - Will only use a frequentist approach, not a Bayesian approach

Outline #1 – BLUE basics

- Intro to combining measurements – why, how, who?
  - Simple academic examples and a glimpse of some real examples
- Best Linear Unbiased Estimators – basics
  - References, derivation of method, relation to other estimators
  - Effect of correlations – just a glimpse (discussed in detail tomorrow)
- BLUE combinations in practice – the multi-parameter case
  - Preparing the relevant inputs and extracting the results
  - A step-by-step example from LEP2 (statistically dominated)
- Intermediate wrap-up (between lectures #1 and #2)

Outline #2 – the effect of correlations

- Information in BLUE combinations
  - Today's example: LHC top mass 2012 (systematically dominated)
  - "Relative importance" of measurements in the presence of correlations
- Interpreting correlation effects and negative BLUE coefficients
  - "Low-correlation" and "high-correlation" regimes
- Estimating correlations in practice
  - "Conservative estimates" of correlations?
  - The risks in overestimating correlations and a few hints to avoid that
- Beyond BLUE?
- Wrap-up and conclusions (for lectures #1 and #2)

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Combining measurements – why, how?

- Why combine measurements?
  - Advancing knowledge (*scientia*) - ultimately, we are all in this game together!
  - Get better result (lower errors) combining knowledge (*information*)

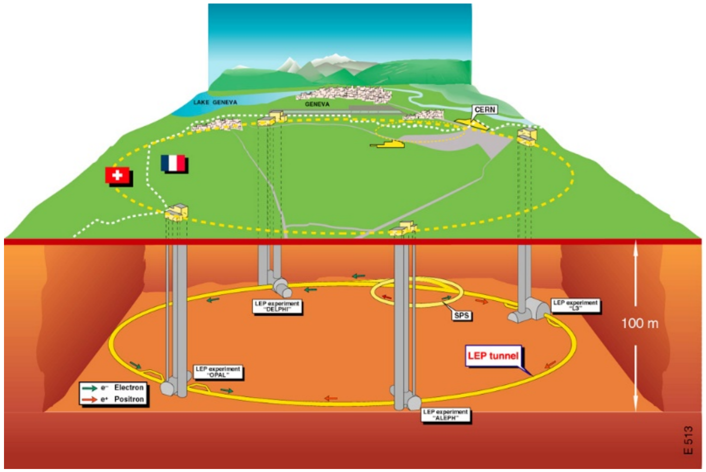
- What we what to know after combining results
  - Central value(s) of the unknown parameter(s)
  - Total combined error(s) – (and their correlations: the covariance matrix)
  - Contributions to the combined error/covariance from the different sources
  - How compatible input results are with one another
  - "How much" each measurement contributed (do we really need this?)

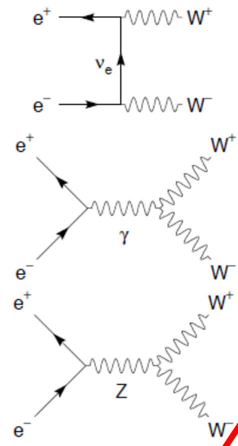
- What we need to provide as input
  - Central values of the input measurements
  - Model of observations – which measurement measures which parameter
  - Errors on each input measurement from every different error source
  - Covariances/Correlations between errors from every different error source
  - Some assumptions on the error distributions (Gaussian, Poisson...)
  - A choice of one or more combination methods (e.g. BLUE)

Combining measurements – who?

- First of all, different methods are combined within each experiment!
- Working groups of experiments at the same collider, such as:
  - The LEP Electro-Weak Working Group (LEPEWWG)
    - See the final reports on LEP1 and LEP2 physics from ALEPH, DELPHI, L3, OPAL
  - The Tevatron Electro-Weak Working Group (TEVEWWG)
  - The LHC Higgs boson Cross Section Working Group (LHCHXSWG)
  - The LHC Top quark Working Group (TOPLHCWG)
  - The DESY Terascale Working Group on SUSY/BSM Fits
  - *Healthy examples where competition rhymes with collaboration!*
- The ultimate reference of HEP knowledge: the Review of Particle Physics
  - Published every two years by the Particle Data Group
    - See the "Procedures" section on data treatment/selection, averages/fits, rounding
- All these groups interact with each other
  - Large overlap of people and expertise across different groups
  - Exact procedures and processes are different as the contexts are different

Example 1 – WW cross sections at LEP2



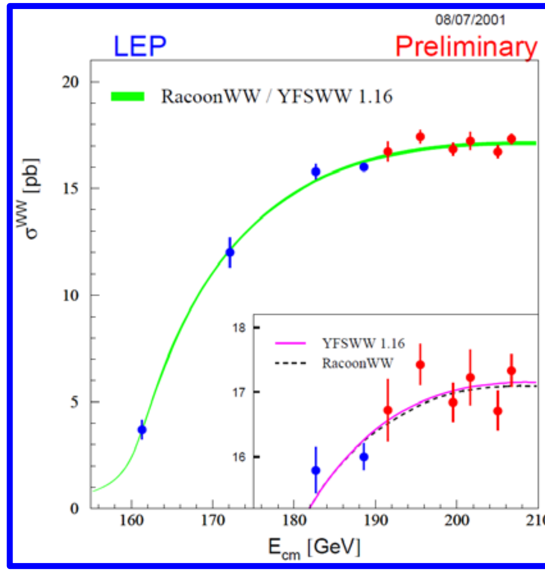


3 BLUE combinations

Inputs: 4+4+32 measurements

Results:1+1+8 parameters

$\sqrt{s}$ (GeV)	WW cross section (pb) ALEPH	DELPHI	L3	OPAL	LEP	χ^2/d.o.f.
161.3	$4.23 \pm 0.75^*$	$3.67\ ^{+\,0.99}_{-\,0.87}\ ^*$	$2.89\ ^{+\,0.82}_{-\,0.71}\ ^*$	$3.62\ ^{+\,0.94}_{-\,0.84}\ ^*$	3.69 ± 0.45	} 1.3 / 3
172.1	11.7 ± 1.3 *	11.6 ± 1.4 *	12.3 ± 1.4 *	12.3 ± 1.3 *	12.0 ± 0.7	} 0.22/ 3
182.7	$15.57 \pm 0.68^*$	$15.86 \pm 0.74^*$	$16.53 \pm 0.72^*$	$15.43 \pm 0.66^*$	15.79 ± 0.36	27.42/24
188.6	$15.71 \pm 0.38^*$	$15.83 \pm 0.43^*$	$16.24 \pm 0.43^*$	$16.30 \pm 0.38^*$	16.00 ± 0.21	
191.6	17.23 ± 0.91	16.90 ± 1.02	16.39 ± 0.93	16.60 ± 0.98	16.72 ± 0.48	
195.5	17.00 ± 0.57	17.86 ± 0.63	16.67 ± 0.60	18.59 ± 0.74	17.43 ± 0.32	
199.5	16.98 ± 0.56	17.35 ± 0.60	16.94 ± 0.62	16.32 ± 0.66	16.84 ± 0.31	
201.6	16.16 ± 0.76	17.67 ± 0.84	16.95 ± 0.88	18.48 ± 0.91	17.23 ± 0.42	
204.9	16.57 ± 0.55	17.44 ± 0.64	17.35 ± 0.64	15.97 ± 0.64	16.71 ± 0.31	
206.6	17.32 ± 0.45	16.50 ± 0.48	17.96 ± 0.51	17.77 ± 0.57	17.33 ± 0.25	



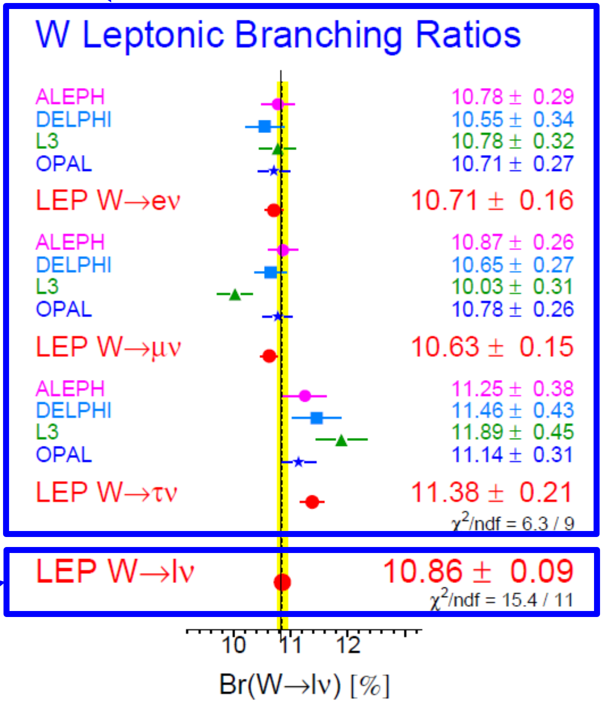
ADLO & LEPEWWG Preliminary Combination for Summer2001 Conferences

Example 2 – W branching ratios at LEP2

	Lepton non–universality		
Experiment	$\mathcal{B}(\mathrm{W} \to e\overline{\nu}_e)$ [%]	$\mathcal{B}(\mathrm{W} \to \mu\overline{\nu}_\mu)$ [%]	$\mathcal{B}(\mathrm{W} \to \tau\overline{\nu}_\tau)$ [%]
ALEPH	10.78 ± 0.29	10.87 ± 0.26	11.25 ± 0.38
DELPHI	10.55 ± 0.34	10.65 ± 0.27	11.46 ± 0.43
L3	10.78 ± 0.32	10.03 ± 0.31	11.89 ± 0.45
OPAL	10.71 ± 0.27	10.78 ± 0.26	11.14 ± 0.31
LEP	10.71 ± 0.16	10.63 ± 0.15	11.38 ± 0.21
χ^2/dof	6.3/9		

ADLO & LEPEWWG, LEP2 final report, Phys. Rep. 532 (2013) 119

BLUE combination
(Lepton non-universality)
Input: 12 measurements
Results:3 parameters



BLUE combination (Lepton universality)

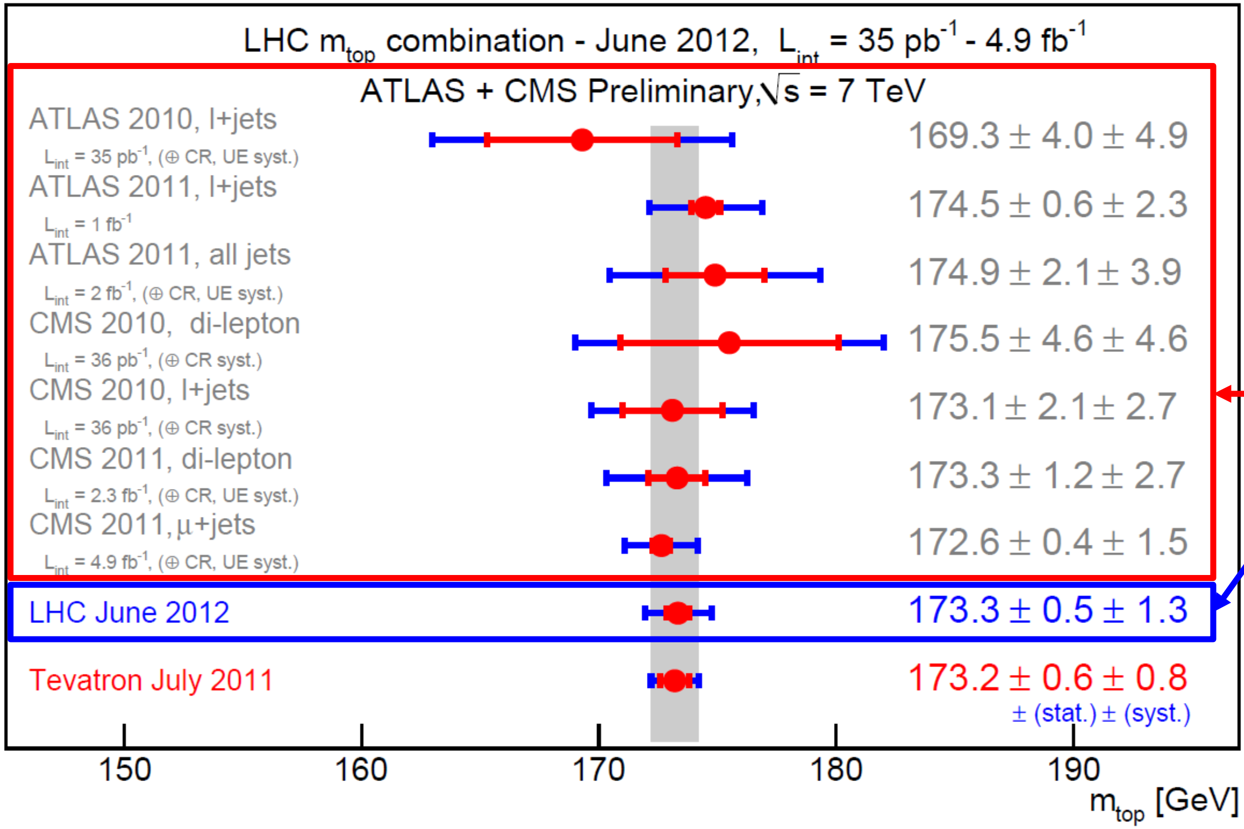
Reinterpret the same 12 measurements as measurements of a single parameter (different parametric model of observations)

Input: 12 measurements

Results:1 parameter

Example 3 – LHC m_{top} combination 2012

ATLAS-CONF-2012-095 / CMS PAS TOP-12-001



BLUE combination

Input: 7 measurements

Results:1 parameter

The simplest example of a combination

- How to combine two *independent* results 10±1 and 15±2?
  - from any basic statistics course – use the weighted average!
  - linear combination with weight proportional to 1/error^2
    - central value = $10 * \frac{\frac{1}{1}}{\frac{1}{1}+\frac{1}{4}} + 15 * \frac{\frac{1}{4}}{\frac{1}{1}+\frac{1}{4}} = 8 + 3 = 11.0$
  - combined error^2 is the inverse of the sum of 1/error^2
    - error = $\sqrt{\frac{1}{\frac{1}{1}+\frac{1}{4}}} = 0.9$
  - BLUE combinations are essentially a generalization of this method
    - also addressing measurements that are correlated and not independent

- Beware of some underlying assumptions
  - e.g. this method assumes that the errors are known a priori
  - how to combine 1±1 and 100±10? We'll discuss this later…

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- *Best Linear Unbiased Estimators – basics*
  - *References, derivation of method, relation to other estimators*
  - *Effect of correlations – just a glimpse (discussed in detail tomorrow)*

- BLUE combinations in practice – the multi-parameter case
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- Intermediate wrap-up (between lectures #1 and #2)

The BLUE method – Aitken (1934)

- Alexander Craig Aitken (1895-1967)
  - One of New Zealand's most famous mathematicians
  - Phenomenal memory and computational skills
  - Brilliant lecturer, athletics champion, fine musician

- The first published description of the BLUE method (AFAIK):
  - A. C. Aitken, *On Least Squares and Linear Combinations of Observations*, Proc. Roy. Soc. Edinburgh 55 (1935), 42
    - "Generalized least squares" estimation using the inverse of the covariance matrix of input measurements as the weighting matrix

BLUE – main references for this talk

- Papers
  - L. Lyons, D. Gibaut and P. Clifford, *How to combine correlated estimates of a single physical quantity*, NIM A270 (1988) 110 [link]
    - Basic formulas for a single parameter, including extensive discussion of correlations and negative BLUE coefficients for two measurements – a must read! ☺
  - A. Valassi, *Combining correlated measurements of several different physical quantities*, NIM A500 (2003) 391 [link]
    - Generalization of Lyons formulas and computation of individual error contributions for many parameters (e.g. LEP electroweak WG)
  - A. Valassi and R. Chierici, *Information and treatment of correlations in the combination of measurements using the BLUE method*, EPJ C74 (2014) 2717 [link]
    - Generalization of Lyons discussion of correlations and negative BLUE coefficients for more than two measurements of a single parameter (e.g. LHC top quark WG)
- Books
  - A. van den Bos, *Parameter Estimation for Scientists and Engineers*, Wiley-Interscience (2007) [ebook]
    - Very useful textbook covering information and estimation

BLUE – other useful references

- Papers
  - R. Nisius, *On the combination of correlated estimates of a physics observable*, EPJ C74 (2014) 3004 [link]
    - Another very useful discussion of correlations in BLUE combinations
- Books
  - O. Behnke and L. Moneta, *Parameter Estimation*, in *Data Analysis in HEP: a Practical Guide to Statistical Methods*, Wiley-VCH (2013) [ebook]
    - A modern reference also covering correlations in BLUE and PDG combinations!
  - F. James, *Statistical Methods in Experimental Physics (2nd Edition)*, World Scientific (2006)
    - Another very useful textbook covering information and estimation
  - G. Cowan, *Statistical Data Analysis*, Oxford University Press (1998)
    - Another very useful reference textbook covering estimation
  - S. Kay, *Fundamentals of Statistical Signal Processing (volume 1: Estimation Theory)*, Prentice Hall (1993)
    - A reference on signal detection and estimation theory for engineers

BLUE is a *point estimation* method

- Point estimation vs Interval estimation
  - *See Fred James's book or his 2015 DESY lectures at this School*

- Point estimation
  - Find the point in θ-space that gives the "best" estimate $\hat{\theta}$ of parameter θ, for a given vector of observations $\mathbf{y}$
    - Also determine the uncertainty on this estimate $\hat{\theta}$
    - If distributions are Gaussian, "value ± error" represents a 68.3% interval
  - During these lectures, θ will normally be a scalar and $\boldsymbol{\theta}$ **(bold)** a vector
  - Example: *precision measurements and their combinations (e.g. BLUE)*

- Interval estimation
  - Example: mass limits and their combinations in new particle searches
    - Beyond "estimate ± uncertainty": one-sided confidence intervals
    - Beyond Gaussians: confidence limits for arbitrarily complex distributions
  - *See Glen Cowan's 2015 DESY lectures at this School*

BLUE is a *frequentist* estimation method

- Frequentist estimation vs Bayesian estimation
  - *See Fred James's book or his 2015 DESY lectures at this School*

- Frequentist point estimation
  - The observations $\mathbf{y}$ are distributed as a p.d.f. $\mathrm{p}(\mathbf{y}|\boldsymbol{\theta})$ that depends on $\boldsymbol{\theta}$
  - Point estimation is an operational procedure to compute an estimator $\widehat{\boldsymbol{\theta}}$
  - Hence $\widehat{\boldsymbol{\theta}}$ is also distributed as a p.d.f that depends on the "true" $\boldsymbol{\theta}$

- BLUE can be used even if $\mathrm{p}(\mathbf{y}|\boldsymbol{\theta})$ is not Gaussian
  - But a Gaussian is generally assumed – see discussion later on

- ***I will use a frequentist approach throughout these lectures***

BLUE – assumptions and notation

- Given n observations $\mathbf{y} = y_i = \{y_i , ... y_n\}$
  - using Roman indices i, j, k... for observations
- Given N parameters $\boldsymbol{\theta} = \theta_\alpha = \{\theta_1 , ... \theta_N\}$ with $N \le n$
  - using Greek indices α, β, γ... for parameters
  - parametric model of observations: expectation values $E[y_i]=g_i(\boldsymbol{\theta})$
- Given the observation covariance matrix $\mathcal{M}$ (known a priori)
  - var(y_i) = $M_{ii} = \sigma_i^2 \rightarrow$ need errors on y_i
  - cov(y_i, y_j) = $M_{ij} = \sigma_i\sigma_j\rho_{ij} \rightarrow$ need correlations between y_i and y_j
  - *[NB can derive BLUE even if $\vec{y}$ are <u>not</u> multivariate Gaussian distributed]*
- Given a linear model expressed by the $n{\times}N$ “design matrix” $\mathcal{A}$
  - parametric model $E[y_i]=g_i(\boldsymbol{\theta})$ is linear: $E[y_i]=\Sigma_\alpha A_{i\alpha}\theta_\alpha$, i.e. $E[\mathbf{y}]=\mathcal{A}\boldsymbol{\theta}$
  - BLUE *combinations* are the simplest case of a linear model:
    - for BLUE *combinations*, $A_{i\alpha}=1$ if y_i is a measurement of θ_α , otherwise $A_{i\alpha}=0$
    - there are $n_\alpha = \{n_1 , ... n_N\}$ measurements of each parameter θ_α

BLUE – Best Linear Unbiased Estimator

- Build the <u>B</u>est <u>L</u>inear <u>U</u>nbiased <u>E</u>stimator $\hat{\theta}_\alpha$ for parameter θ_α ?
- $\hat{\theta}_\alpha$ must be a **Linear** combination of the observations y_i
  - $\hat{\theta}_\alpha = \Sigma_i \lambda_{\alpha i} y_i$, where $\lambda_{\alpha i}$ are the BLUE "coefficients" (better than "weights")
- $\hat{\theta}_\alpha$ must be **Unbiased** (given unbiased y_i with $E[y_i] = \Sigma_\alpha A_{i\alpha}\theta_\alpha$)
  - $E[\hat{\theta}_\alpha] = \theta_\alpha = \Sigma_i \lambda_{\alpha i} E[y_i] = \Sigma_{i\beta} \lambda_{\alpha i} A_{i\beta} \theta_\beta$, which requires $\Sigma_i \lambda_{\alpha i} A_{i\beta} = \delta_{\alpha\beta}\ \forall \alpha,\beta$
  - this is a normalization condition – in particular, for BLUE *combinations*:
    - sum of BLUE *combination* coefficients $\lambda_{\alpha i}$ (for the n_α meas. of θ_α) = 1
    - sum of BLUE *combination* coefficients $\lambda_{\alpha i}$ (for the n_β meas. of another θ_β) = 0
- $\hat{\theta}_\alpha$ must be the **Best** such estimator (that of minimum variance)
  - the covariance of the estimators is $\mathrm{cov}(\hat{\theta}_\alpha, \hat{\theta}_\beta) = \Sigma_{ij} \lambda_{\alpha i} \lambda_{\beta j} M_{ij}$
  - find the BLUE by minimizing $\mathrm{var}(\hat{\theta}_\alpha) = \Sigma_{ij} \lambda_{\alpha i} \lambda_{\alpha j} M_{ij}$

Build the BLUE!

- For each α, minimize $\Sigma_{ij}\lambda_{\alpha i}\lambda_{\alpha j}M_{ij}$ under N constraints $\Sigma_i\lambda_{\alpha i}A_{i\beta}=\delta_{\alpha\beta}$
  - Using N Lagrange multipliers $K_{\alpha\beta}$, differentiate with respect to $K_{\alpha\beta}$ and $\lambda_{\alpha i}$ the sum $[\Sigma_{ij}\lambda_{\alpha i}\lambda_{\alpha j}M_{ij} + 2\,\Sigma_\gamma K_{\alpha\gamma}(\delta_{\alpha\gamma} - \Sigma_j\lambda_{\alpha j}A_{j\gamma})]$
  - This gives N+n linear equations for the N+n unknowns $K_{\alpha\beta}$ and $\lambda_{\alpha i}$
- The BLUE coefficients are $\boxed{\lambda_{\alpha i} = \Sigma_\beta (\mathcal{A}^T\mathcal{M}^{-1}\mathcal{A})^{-1}{}_{\alpha\beta}(\mathcal{A}^T\mathcal{M}^{-1})_{\beta i}}$
  - this automatically satisfies the normalization $\Sigma_i\,\lambda_{\alpha i}A_{i\beta} = \delta_{\alpha\beta}\ \forall\alpha,\beta$
- The BLUE estimators are therefore $\boxed{\hat{\boldsymbol{\theta}} = (\mathcal{A}^T\mathcal{M}^{-1}\mathcal{A})^{-1}(\mathcal{A}^T\mathcal{M}^{-1})\,\mathbf{y}}$
- Their covariance is simply $\boxed{\mathrm{cov}(\hat{\theta}_\alpha, \hat{\theta}_\beta) = (\mathcal{A}^T\mathcal{M}^{-1}\mathcal{A})^{-1}{}_{\alpha\beta}}$

Breakdown of error contributions

- Measurement errors come from many different error sources
  - Typical for HEP: statistical errors and many sources of systematics
  - Details later on for some of the LEP2 and LHC examples we saw
  - The covariance matrix for the input measurements is just their sum

$$M_{ij} = \mathrm{cov}\,(y_i,y_j) = \Sigma_s\, \mathrm{cov}^{[s]}(y_i,y_j) = \Sigma_s\, M^{[s]}{}_{ij}$$

- Nice feature of BLUE: easy breakdown of combined error
  - Full covariance of BLUE estimators is $\mathrm{cov}(\hat{\theta}_\alpha, \hat{\theta}_\beta) = \Sigma_{ij}\, \lambda_{\alpha i}\, \lambda_{\beta j}\, M_{ij}$
  - Replace $\mathcal{M}$ by $\Sigma_s \mathcal{M}^{[s]}$ to get contributions from individual error sources, $\mathrm{cov}(\hat{\theta}_\alpha, \hat{\theta}_\beta) = \Sigma_s \mathrm{cov}^{[s]}(\hat{\theta}_\alpha, \hat{\theta}_\beta)$ where $\boxed{\mathrm{cov}^{[s]}(\hat{\theta}_\alpha, \hat{\theta}_\beta) = \Sigma_{ij}\, \lambda_{\alpha i}\, \lambda_{\beta j}\, M^{[s]}{}_{ij}}$

- Breakdown can also be obtained using numerical methods
  - Numerically compute partial derivatives $\partial\hat{\theta}_\alpha / \partial y_i$ – they are just the $\lambda_{\alpha i}$!

BLUE vs. Linear Least Squares

- Least Squares: minimize squared residual sum for $E[y_i]=g_i(\boldsymbol{\theta})$
  - Ordinary Least Squares (OLS): minimize $\Sigma_i(y_i-g_i(\hat{\boldsymbol{\theta}}))^2$
  - Generalized Least Squares (GLS): minimize $\Sigma_{ij}(y_i-g_i(\hat{\boldsymbol{\theta}}))R_{ij}(y_j-g_j(\hat{\boldsymbol{\theta}}))$
    - Not necessarily using the inverse covariance $\mathcal{M}^{-1}$ as weighting matrix $\mathcal{R}$
  - Linear Least Squares: linear model, minimize $(\mathbf{y}-\mathcal{A}\hat{\boldsymbol{\theta}})^T\mathcal{R}(\mathbf{y}-\mathcal{A}\hat{\boldsymbol{\theta}})$
    - It can be shown that the solution to the linear least squares problem is an estimator that is linear in the observations, $\hat{\boldsymbol{\theta}} = (\mathcal{A}^T\mathcal{R}\mathcal{A})^{-1}(\mathcal{A}^T\mathcal{R})\,\mathbf{y}$
    - If the inverse covariance $\mathcal{M}^{-1}$ is used as weighting matrix $\mathcal{R}$, the Linear Least Squares Estimator is the BLUE, $\hat{\boldsymbol{\theta}} = (\mathcal{A}^T\mathcal{M}^{-1}\mathcal{A})^{-1}(\mathcal{A}^T\mathcal{M}^{-1})\,\mathbf{y}$!

- Relation to χ^2 – *in addition, <u>if the y_i are Gaussian distributed</u>*:
  - $(\mathbf{y}-\mathcal{A}\boldsymbol{\theta})^T\mathcal{M}^{-1}(\mathbf{y}-\mathcal{A}\boldsymbol{\theta})$ for the true $\boldsymbol{\theta}$ is distributed as χ^2 with n d.o.f.
  - $(\mathbf{y}-\mathcal{A}\hat{\boldsymbol{\theta}})^T\mathcal{M}^{-1}(\mathbf{y}-\mathcal{A}\hat{\boldsymbol{\theta}})$ for the estimated $\hat{\boldsymbol{\theta}}$ is distributed as χ^2 with n-N d.o.f.

BLUE vs. Maximum Likelihood (MLE)

- Maximum Likelihood Estimator (MLE)
  - Need a priori knowledge of joint p.d.f. $\mathrm{p}(\mathbf{y}|\boldsymbol{\theta})$ for observation distributions
    - Covariance matrix $\mathcal{M}$ for observations $\mathbf{y}$ may depend on parameters $\boldsymbol{\theta}$
    - Example: good choice for Poisson distributions
  - Can be applied to non-linear parametric models of observations
  - Maximize (numerically if needed) log likelihood $\log \mathrm{L}(\mathbf{y}|\hat{\boldsymbol{\theta}})=\log \mathrm{p}(\mathbf{y}|\hat{\boldsymbol{\theta}})$
    - Notation: use $\mathrm{L}(\mathbf{y}|\boldsymbol{\theta})$ to indicate we are interested in the dependence on $\boldsymbol{\theta}$

- Best Linear Unbiased Estimator (BLUE)
  - Need a priori knowledge of covariance matrix $\mathcal{M}$ for observations $\mathbf{y}$
    - Example: not a good choice for Poisson distributions
    - Do not need a priori knowledge of joint p.d.f. $\mathrm{p}(\mathbf{y}|\boldsymbol{\theta})$
  - Only applies to linear parametric models of observations
  - Algebraic solution exists

- *For linear models and* *<u>Gaussian y_i</u>*, MLE and BLUE coincide!
  - Maximising $\log \mathrm{f}(\mathbf{y};\hat{\boldsymbol{\theta}})$ = $-(\mathbf{y}-\mathcal{A}\hat{\boldsymbol{\theta}})^{\mathrm{T}}\mathcal{M}^{-1}(\mathbf{y}-\mathcal{A}\hat{\boldsymbol{\theta}})$ ⇔ Linear Least Squares!

How to combine 1±1 and 100±10?

- Counting experiment – a max likelihood approach is better!
  - this is a Poisson process with measurements $n_1 \pm \sqrt{n_1}$ and $n_2 \pm \sqrt{n_2}$
    - p.d.f. $p(n|\mu) = \mu^n e^{-\mu}/n!$, hence likelihood $L(n_1,n_2|\mu) = (\mu^{n_1} e^{-\mu}/n_1!)(\mu^{n_2} e^{-\mu}/n_2!)$
  - solving $dL(n_1,n_2|\mu)/d\mu = 0$ gives max likelihood estimator $\hat{\mu} = (n_1+n_1)/2$
  - variance on $\hat{\mu}$ is simply $\sigma^2_{\hat{\mu}} = (\sigma_1{}^2 + \sigma_2{}^2)/4 = (n_1+n_1)/4$
    - intuitively: population variance estimate divided by 2 (central limit theorem)
  - in our example, combining 1±1 and 100±10 gives 50.5±5.0

- The point here is that the errors depend on the central values
  - Different errors are estimated for different values 1±1 and 100±10
  - But the underlying population variance is the same for both

[Thanks to Louis Lyons from whom I stole the idea for this slide!]

BLUE – Fits vs. Combinations

- Recall our initial assumptions and notation
  - given n observations $\mathbf{y} = y_i = \{y_i , \ldots y_n\}$
  - given N parameters $\boldsymbol{\theta} = \theta_\alpha = \{\theta_1 , \ldots \theta_N\}$ with $N \leq n$
  - parametric model of observations: expectation values $E[y_i]=g_i(\boldsymbol{\theta})$
  - all this may apply both for (complex) fits and (simpler) combinations

- Fits of curves – introduce yet another dimension
  - given n observation locations $\mathbf{x} = x_i = \{x_i , \ldots x_n\}$
  - the observations $\mathbf{y}$ are measurements of a function $f(\mathbf{x};\boldsymbol{\theta})$ at those points
    - parametric model of observations is $E[y_i] = f(x_i;\boldsymbol{\theta}) = g_i(\boldsymbol{\theta})$
    - linear model example (may use BLUE!): fit of polynomials $f(x;\boldsymbol{\theta})= \Sigma_i \theta_i x^i$

- Combinations are much easier – the simplest linear model
  - there are $n_\alpha = \{n_1 , \ldots n_N\}$ measurements of each parameter θ_α
    - it is like fitting the heights of N steps in a ladder-like curve

- Why this slide? Because everyone uses a different notation!
  - hopefully this will ease your navigation of different books and papers

Effects of correlations – just a glimpse

- Correlations between measurements of different parameters
  - If =0, the combination is equivalent to N combinations of the n_α measurements of each parameter θ_α, one at a time $(\alpha=1,..N)$
    - The full χ^2 with n-N d.o.f. is just the sum of N individual χ^2 with n_α-1 d.o.f
  - If $\neq$0, whether positive or negative, *they always reduce the variance* on each parameter estimate (with respect to the case where they are 0)
    - Minima in (n-N)-D space vs in (n_α-1)-D subspaces where $\lambda_{\alpha i}=0$ if $A_{i\alpha}=0$
    - These correlations always add positive *information* on each parameter

- Correlations between measurements of the same parameter
  - They may reduce or increase the variance on that parameter
  - This will be ***covered in detail tomorrow*** *– preview in the next two slides*
    - Will focus only on the simpler case of N measurements of one parameter
    - Analytic study for the simplest case of 2 measurements of one parameter

n measurements of 1 parameter (preview)

- Given n observations $\mathbf{y} = y_i = \{y_i , \ldots y_n\}$ of a single parameter Y
  - simpler notation: use Y instead of θ as all y_i are measurements of Y
  - parametric model of observations: expectation values $E[y_i]=g_i(Y)=Y=U_i Y \ \forall i$
    - in matrix notation $E[\mathbf{y}]=\mathbf{U}Y$ where the $n{\times}1$ "design matrix" is the unit vector $U_i=1 \ \forall i$

- The formulas we saw for N parameters simplify as follows for 1 parameter:
  - BLUE central value $\boxed{\hat{Y} = \tilde{\boldsymbol{\lambda}}\mathbf{y} = \sum_{i=1}^{n} \lambda_i \, y_i}$

    Note: Using $\tilde{\mathbf{v}}=\mathbf{v}^{\mathrm{T}}$ here to indicate the transpose of a vector
  - BLUE coefficients (given the $n{\times}n$ measurement covariance) $\boxed{\lambda_i = \frac{(\mathcal{M}^{-1}\mathbf{U})_i}{(\tilde{\mathbf{U}}\mathcal{M}^{-1}\mathbf{U})}}$
  - normalization condition for the BLUE coefficients $\boxed{\tilde{\mathbf{U}}\boldsymbol{\lambda} = \sum_{i=1}^{n} \lambda_i = 1}$
  - BLUE variance $\boxed{\mathrm{var}(\hat{Y}) = \sigma_{\hat{Y}}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \, \mathcal{M}_{ij} \, \lambda_j \ = \frac{1}{(\tilde{\mathbf{U}}\mathcal{M}^{-1}\mathbf{U})}}$

2 measurements of 1 parameter (preview)

- Given two measurements $y_A \pm \sigma_A$ and $y_B \pm \sigma_B$ of one parameter Y
  - given their correlation ρ, their covariance matrix $\mathcal{M}$ is $\begin{pmatrix} \sigma_A^2 & \rho\sigma_A\sigma_B \\ \rho\sigma_A\sigma_B & \sigma_B^2 \end{pmatrix}$
  - invert $\mathcal{M}$ analytically to find the BLUE coefficients and the BLUE variance:

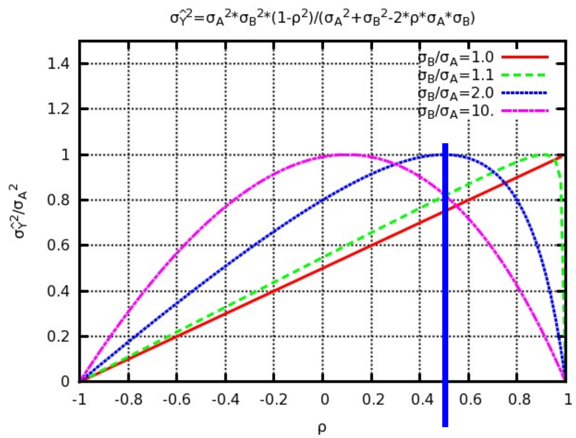
$$\boxed{\hat{Y} = \lambda_A y_A + \lambda_B y_B}$$

$$\boxed{\lambda_A = \frac{\sigma_B^2 - \rho\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}}$$

$$\boxed{\lambda_B = \frac{\sigma_A^2 - \rho\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}}$$

$$\boxed{\sigma_{\hat{Y}}^2 = \frac{\sigma_A^2\sigma_B^2(1-\rho^2)}{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}}$$

- The effect of correlations was extensively discussed in the Lyons paper. Fixing σ_A and σ_B and varying ρ, the combined error $\sigma_{\hat{Y}}$ has a maximum (equal to σ_A) for $\rho=\sigma_A/\sigma_B$ – which is where λ_B flips sign from >0 to <0

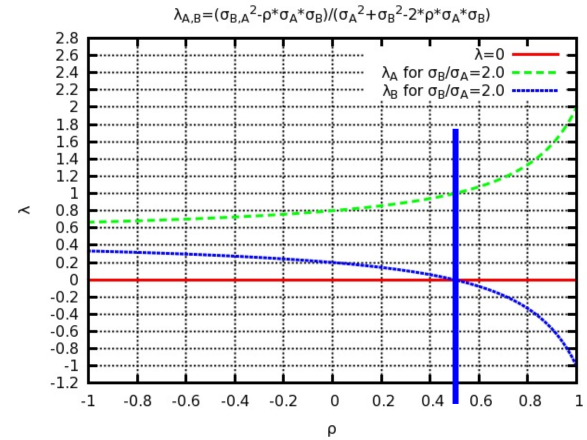


Example for σ_B/σ_A=2:

- all λ>0 for ρ<0.5

- $\sigma_{\hat{Y}}$ is maximum for ρ=0.5

- λ_B<0 for ρ>0.5



Outline #1 – BLUE basics

- Intro to combining measurements – why, how, who?
  - Simple academic examples and a glimpse of some real examples
- Best Linear Unbiased Estimators – basics
  - References, derivation of method, relation to other estimators
  - Effect of correlations – just a glimpse (discussed in detail tomorrow)
- *BLUE combinations in practice – the multi-parameter case*
  - *Preparing the relevant inputs and extracting the results*
  - *A step-by-step example from LEP2 (statistically dominated)*
- Intermediate wrap-up (between lectures #1 and #2)

W branching ratios at LEP2 – detailed

ADLO & LEPEWWG, LEP2 final report, *Phys. Rep.* 532 (2013) 119

Decay channel	$\mathcal{B}$	$\Delta\mathcal{B}^{stat}$	(unc) $\Delta\mathcal{B}^{syst}$	(cor) $\Delta\mathcal{B}^{syst}$	$\Delta\mathcal{B}^{syst}$	$\Delta\mathcal{B}$	3×3 correlation for $\Delta\mathcal{B}$		
ALEPH [93]									
$\mathcal{B}(W \to e\overline{\nu}_e)$	10.78	±0.27	±0.09	±0.04	±0.10	±0.29	1.000	-0.009	-0.332
$\mathcal{B}(W \to \mu\overline{\nu}_\mu)$	10.87	±0.25	±0.07	±0.04	±0.08	±0.26	-0.009	1.000	-0.268
$\mathcal{B}(W \to \tau\overline{\nu}_\tau)$	11.25	±0.32	±0.19	±0.05	±0.20	±0.38	-0.332	-0.268	1.000
DELPHI [94]									
$\mathcal{B}(W \to e\overline{\nu}_e)$	10.55	±0.31	±0.13	±0.05	±0.14	±0.34	1.000	0.030	-0.340
$\mathcal{B}(W \to \mu\overline{\nu}_\mu)$	10.65	±0.26	±0.06	±0.05	±0.08	±0.27	0.030	1.000	-0.170
$\mathcal{B}(W \to \tau\overline{\nu}_\tau)$	11.46	±0.39	±0.17	±0.09	±0.19	±0.43	-0.340	-0.170	1.000
L3 [95]									
$\mathcal{B}(W \to e\overline{\nu}_e)$	10.78	±0.29	±0.10	±0.08	±0.13	±0.32	1.000	-0.016	-0.279
$\mathcal{B}(W \to \mu\overline{\nu}_\mu)$	10.03	±0.29	±0.10	±0.07	±0.12	±0.31	-0.016	1.000	-0.295
$\mathcal{B}(W \to \tau\overline{\nu}_\tau)$	11.89	±0.40	±0.17	±0.11	±0.20	±0.45	-0.279	-0.295	1.000
OPAL [96]									
$\mathcal{B}(W \to e\overline{\nu}_e)$	10.71	±0.25	±0.09	±0.06	±0.11	±0.27	1.000	0.135	-0.303
$\mathcal{B}(W \to \mu\overline{\nu}_\mu)$	10.78	±0.24	±0.07	±0.07	±0.10	±0.26	0.135	1.000	-0.230
$\mathcal{B}(W \to \tau\overline{\nu}_\tau)$	11.14	±0.31	±0.16	±0.06	±0.17	±0.35	-0.303	-0.230	1.000

LEP Average (without lepton universality assumption)									
$\mathcal{B}(W \to e\overline{\nu}_e)$	10.71	±0.14	±0.05	±0.05	±0.07	±0.16	1.000	0.136	-0.201
$\mathcal{B}(W \to \mu\overline{\nu}_\mu)$	10.63	±0.13	±0.04	±0.05	±0.07	±0.15	0.136	1.000	-0.122
$\mathcal{B}(W \to \tau\overline{\nu}_\tau)$	11.38	±0.17	±0.09	±0.07	±0.11	±0.21	-0.201	-0.122	1.000
χ^2/dof	6.3/9								

LEP Average (with lepton universality assumption)						
$\mathcal{B}(W \to \ell\overline{\nu}_\ell)$	10.86	±0.06	±0.03	±0.06	±0.07	±0.09
$\mathcal{B}(W \to \text{had.})$	67.41	±0.18	±0.10	±0.17	±0.20	±0.27
χ^2/dof	15.4/11					

Between experiments: COR=100%, UNC=0%
Between channels within one experiment:
e.g. -27% W→τν / W→μν selection
cross-contamination in ALEPH

BLUE combination

Detailed input:
12 measured values, plus
12(×3) errors(×sources),
and all correlations

Detailed results:
3 parameter values, plus
3(×3) errors(×sources),
all correlations, and χ^2

Detailed results:
1 parameter value, plus
1(×3) errors(×sources),
and χ^2

WW cross sections at LEP2 – detailed

ADLO & LEPEWWG Preliminary Combination for Summer2001 Conferences

$\sqrt{s}$	σ_{WW}	$\Delta\sigma_{WW}^{stat}$	(LCEC) $\Delta\sigma_{WW}^{syst}$	(LUEU) $\Delta\sigma_{WW}^{syst}$	(LUEC) $\Delta\sigma_{WW}^{syst}$	$\Delta\sigma_{WW}^{syst}$	$\Delta\sigma_{WW}$
ALEPH [6, 10, 14, 19, 74]							
182.7 GeV	15.57	±0.62	±0.09	±0.09	±0.26	±0.29	±0.68
188.6 GeV	15.71	±0.34	±0.05	±0.09	±0.15	±0.18	±0.38
191.6 GeV	17.23	±0.89	±0.05	±0.09	±0.15	±0.18	±0.91
195.5 GeV	17.00	±0.54	±0.05	±0.09	±0.15	±0.18	±0.57
199.5 GeV	16.98	±0.53	±0.05	±0.09	±0.15	±0.18	±0.56
201.6 GeV	16.16	±0.74	±0.05	±0.09	±0.15	±0.18	±0.76
204.9 GeV	16.57	±0.52	±0.05	±0.09	±0.15	±0.18	±0.55
206.6 GeV	17.32	±0.41	±0.05	±0.09	±0.15	±0.18	±0.45
DELPHI [7, 11, 15, 20, 74]							
182.7 GeV	15.86	±0.69	±0.09	±0.07	±0.24	±0.27	±0.74
188.6 GeV	15.83	±0.38	±0.07	±0.05	±0.18	±0.20	±0.43
191.6 GeV	16.90	±1.00	±0.07	±0.06	±0.20	±0.22	±1.02
195.5 GeV	17.86	±0.59	±0.07	±0.06	±0.20	±0.22	±0.63
199.5 GeV	17.35	±0.56	±0.07	±0.06	±0.20	±0.22	±0.60
201.6 GeV	17.67	±0.81	±0.08	±0.07	±0.21	±0.23	±0.84
204.9 GeV	17.44	±0.60	±0.06	±0.05	±0.21	±0.22	±0.64
206.6 GeV	16.50	±0.43	±0.06	±0.05	±0.20	±0.21	±0.48
L3 [8, 12, 18, 21, 74]							
182.7 GeV	16.53	±0.67	±0.08	±0.14	±0.21	±0.26	±0.72
188.6 GeV	16.24	±0.37	±0.04	±0.08	±0.20	±0.22	±0.43
191.6 GeV	16.39	±0.90	±0.08	±0.08	±0.21	±0.24	±0.93
195.5 GeV	16.67	±0.55	±0.08	±0.08	±0.21	±0.24	±0.60
199.5 GeV	16.94	±0.57	±0.08	±0.08	±0.21	±0.24	±0.62
201.6 GeV	16.95	±0.85	±0.08	±0.08	±0.21	±0.24	±0.88
204.9 GeV	17.35	±0.59	±0.08	±0.08	±0.21	±0.24	±0.64
206.6 GeV	17.96	±0.45	±0.08	±0.08	±0.21	±0.24	±0.51
OPAL [9, 13, 16, 17, 22, 74]							
182.7 GeV	15.43	±0.61	±0.14	±0.00	±0.22	±0.26	±0.66
188.6 GeV	16.30	±0.34	±0.07	±0.00	±0.17	±0.18	±0.38
191.6 GeV	16.60	±0.88	±0.12	±0.00	±0.40	±0.42	±0.98
195.5 GeV	18.59	±0.60	±0.12	±0.00	±0.41	±0.43	±0.74
199.5 GeV	16.32	±0.54	±0.10	±0.00	±0.37	±0.38	±0.66
201.6 GeV	18.48	±0.81	±0.12	±0.00	±0.40	±0.42	±0.91
204.9 GeV	15.97	±0.52	±0.10	±0.00	±0.36	±0.37	±0.64
206.6 GeV	17.77	±0.42	±0.09	±0.00	±0.37	±0.38	±0.57

LEP/Energy Correlated/Uncorrelated (LCEC, LUEU, LUEC)
Between LEP experiments: LC=100%, LU=0%
Between Energies within one experiment: EC=100%, EU=0%

BLUE combination

Detailed input: 32 measured values, plus 32(×4) errors(×sources), and all correlations

Detailed results:8 parameter values, plus 8(×4) errors(×sources), all correlations, and χ^2

$\sqrt{s}$/ GeV	182.7	188.6	191.6	195.5	199.5	201.6	204.9	206.6
182.7	1.000	0.197	0.113	0.169	0.169	0.128	0.166	0.201
188.6	0.197	1.000	0.134	0.200	0.200	0.150	0.196	0.239
191.6	0.113	0.134	1.000	0.119	0.119	0.090	0.118	0.143
195.5	0.169	0.200	0.119	1.000	0.177	0.133	0.174	0.211
199.5	0.169	0.200	0.119	0.177	1.000	0.133	0.175	0.212
201.6	0.128	0.150	0.090	0.133	0.133	1.000	0.131	0.159
204.9	0.166	0.196	0.118	0.174	0.175	0.131	1.000	0.209
206.6	0.201	0.239	0.143	0.211	0.212	0.159	0.209	1.000

$\sqrt{s}$	σ_{WW}	$\Delta\sigma_{WW}^{stat}$	(LCEC) $\Delta\sigma_{WW}^{syst}$	(LUEU) $\Delta\sigma_{WW}^{syst}$	(LUEC) $\Delta\sigma_{WW}^{syst}$	$\Delta\sigma_{WW}^{syst}$	$\Delta\sigma_{WW}$	χ^2/d.o.f.
LEP Averages								
182.7 GeV	15.79	±0.32	±0.10	±0.04	±0.11	±0.15	±0.36	
188.6 GeV	16.00	±0.18	±0.05	±0.03	±0.08	±0.10	±0.21	
191.6 GeV	16.72	±0.46	±0.07	±0.03	±0.11	±0.13	±0.48	
195.5 GeV	17.43	±0.29	±0.07	±0.04	±0.10	±0.13	±0.32	27.42/24
199.5 GeV	16.84	±0.28	±0.07	±0.04	±0.10	±0.13	±0.31	
201.6 GeV	17.23	±0.40	±0.07	±0.04	±0.10	±0.13	±0.42	
204.9 GeV	16.71	±0.28	±0.07	±0.04	±0.10	±0.13	±0.31	
206.6 GeV	17.33	±0.22	±0.06	±0.04	±0.10	±0.12	±0.25	

BLUE – software implementations

- Many different software implementations exist for BLUE combinations
  - FORTRAN code used for LEP (one for 2f, another for 4f, and many more…)
  - C++ code used for LHC (one by R. Nisius, another by myself described below)
  - Basic formulas are really easy (you could prepare your own in Python to play!)
    - Most boring and lengthy part is reading the inputs and displaying the results
    - Different implementations have different features, also for modifying the inputs

- In these lectures I will show screenshots from BlueFin
  - C++ translation of FORTRAN code I wrote for 4f at LEP2
  - With many features added for the studies I did with Roberto Chierici
    - Several schemes to reduce correlations – and an automatic full dump to latex/pdf
  - Designed more as a private tool than for wider adoption – but can be improved
    - Some ugly implementation choices (Boost matrices)
    - An even uglier configuration setup (need access to CERN AFS or CVMFS)
    - Only really tested for a limited number of specific examples
  - Do feel free to try it out if you want! https://svnweb.cern.ch/trac/bluefin

WW cross sections at LEP2 – again (1)

Appendix A1. Input data.

```
#==========================================================================
#==== BlueFin input data file =============================================
#==========================================================================

# The file is expected to have the following format.
# Blank lines and lines with only empty spaces are ignored.
# Lines starting by '#' are reserved for comments and are ignored.
# Data lines are composed of fields separated by one or more empty spaces.
# Fields cannot contain empty spaces, with the exception of the title line.

# The next line must have 2 fields: 'TITLE' and the title of the
# BlueFin combination, which must be enclosed within double quotes
# and may contain only alphanumeric characters or spaces or hyphens.
TITLE "LEP2 WW cross sections"

# The next line must have 2 fields: 'NOBS' and the number of observables.
NOBS 8

# The next line must have 2 fields: 'NMEA' and the number of measurements.
NMEA 32

# The next line must have 2 fields: 'NERR' and the number of error sources.
NERR 5

# The next NERR+3 lines must have NMEA+1 fields in this format:
# - in the 1st line: 'MEANAME' followed by NMEA distinct measurement names
#   (measurement names may contain only alphanumeric characters or spaces);
# - in the 2nd line: 'OBSNAME' followed by the NMEA names (with NOBS distinct
#   values) of the observables measured by the corresponding measurements
#   (observable names may contain only alphanumeric characters or spaces
#   and should preferably be at most 3 characters long);
# - in the 3rd line: 'MEAVAL' followed by the NMEA measured central values;
# - in each of the last NERR lines: the error source name followed by the
#   NMEA partial errors for each measurement due to the given error source
#   (error source names may contain only alphanumeric characters or spaces).
# === From echo `cat sww.in | egrep '\+(A|D|L|O)' | awk '{print substr($1,2,1)$2}'` | sed 's| |  |g'
MEANAME  A183  A189  A192  A196  A200  A202  A205  A207  D183  D189  D192  D196  D200  D202  D205  D207  L183  L189  L192  L196  L200  L202  L205  L207  O183  O189
    O192  O196  O200  O202  O205  O207
# === From echo `cat sww.in | egrep '\+(A|D|L|O)' | awk '{print "C"$2}'` | sed 's| |  |g'
OBSNAME   183   189   192   196   200   202   205   207   183   189   192   196   200   202   205   207   183   189   192   196   200   202   205   207   183   189
      192   196   200   202   205   207
# === From echo `cat sww.in | egrep '\+(A|D|L|O)' | awk '{print $3}'`
MEAVAL  15570 15710 17230 17000 16980 16160 16570 17320 15860 15830 16900 17860 17350 17670 17440 16500 16530 16240 16390 16670 16940 16950 17350 17957 15430 16300
    16600 18590 16320 18480 15970 17770
# === From echo `cat sww.in | egrep '\+(A|D|L|O)' | awk '{print $4}'` | sed 's| |   |g'
Stat      620   340   890   540   530   740   520   410   690   380   1000  590   560   810   600   430   670   370   900   550   570   850   590   450   610   340
      880   600   540   810   520   420
# === From echo `cat sww.in | egrep '\+(A|D|L|O)' | awk '{print $7}'` | sed 's| |      |g'
LCEU        0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0     0
        0     0     0     0     0     0
# === From echo `cat sww.in | egrep '\+(A|D|L|O)' | awk '{print $8}'` | sed 's| |     |g'
LCEC       90    50    50    50    50    50    50    50    85    70    70    70    70    80    60    60    80    44    80    80    80    80    80    80   140    65
      120   120    99   115    96    92
# === From echo `cat sww.in | egrep '\+(A|D|L|O)' | awk '{print $9}'` | sed 's| |     |g'
```

INPUT DATA FILE

#parameters is 8

#measurements is 32

#error sources is 5 (really 4, one is empty)

measurement name for 32 measurements

parameter name for 32 measurements

central values for 32 measurements

error #1 (of 5) for 32 measurements

LEP Combination for the Summer 2001 Conferences (http://lepewwg.web.cern.ch/LEPEWWG/lepww/4f/Summer01). The output results may differ because of rounding errors.

BlueFin 01.00.04 17 LEP2 WW cross sections (sww.bfin)

WW cross sections at LEP2 – again (2)

```
LUEU        93    89    89    89    89    89    89    89    65    50    60    60    60    65    54    50   138    76    84    84    84    84    84    84     0     0
         0     0     0     0     0     0
# === From echo ‘cat sww.in | egrep ’\+(A|D|L|O)’ | awk ’{print $10}’‘ | sed ’s| |    |g’
LUEC       256   148   148   148   148   148   148   148   235   181   200   200   200   206   205   195   205   202   210   210   210   210   210   210   219   168
       402   413   367   404   357   369

# The next NMEA*(NMEA-1)/2+1 rows must have NERR+2 fields in this format:
# - in the 1st line: ’CMEA1’ ’CMEA2’ (correlations between 2 measurements)
#   followed by the NERR error source names in the same order used above;
# - in each of the NMEA*(NMEA-1)/2 last lines: the names of two distinct
#   measurements followed by the NERR correlations between the partial
#   errors on the two measurements due to corresponding error source.
#   Measurements must appear in the same order listed above.
# === From: adlo="A D L O"; ecms="183 189 192 196 200 202 205 207"; for a1 in $adlo; do for e1 in $ecms; do for a2 in $adlo; do for e2 in $ecms; do m1=$a1$e1; m2=
    $a2$e2; lueu=0; lcec=1; if [ "$e1" == "$e2" ]; then lceu=1; else lceu=0; fi; if [ "$a1" == "$a2" ]; then luec=1; else luec=0; fi; if [[ $m2 > $m1 ]]; then echo "
    $m1    $m2     0     $lceu     $lcec     $lueu     $luec"; fi; done; done; done; done
CMEA1 CMEA2 Stat LCEU LCEC LUEU LUEC
A183   A189    0    0    1    0    1
A183   A192    0    0    1    0    1
A183   A196    0    0    1    0    1
A183   A200    0    0    1    0    1
A183   A202    0    0    1    0    1
A183   A205    0    0    1    0    1
A183   A207    0    0    1    0    1
A183   D183    0    1    1    0    0
A183   D189    0    0    1    0    0
A183   D192    0    0    1    0    0
A183   D196    0    0    1    0    0
A183   D200    0    0    1    0    0
A183   D202    0    0    1    0    0
A183   D205    0    0    1    0    0
A183   D207    0    0    1    0    0
A183   L183    0    1    1    0    0
A183   L189    0    0    1    0    0
A183   L192    0    0    1    0    0
A183   L196    0    0    1    0    0
A183   L200    0    0    1    0    0
A183   L202    0    0    1    0    0
A183   L205    0    0    1    0    0
A183   L207    0    0    1    0    0
A183   O183    0    1    1    0    0
A183   O189    0    0    1    0    0
A183   O192    0    0    1    0    0
A183   O196    0    0    1    0    0
A183   O200    0    0    1    0    0
A183   O202    0    0    1    0    0
A183   O205    0    0    1    0    0
A183   O207    0    0    1    0    0
A189   A192    0    0    1    0    1
A189   A196    0    0    1    0    1
A189   A200    0    0    1    0    1
A189   A202    0    0    1    0    1
A189   A205    0    0    1    0    1
A189   A207    0    0    1    0    1
A189   D183    0    0    1    0    0
A189   D189    0    1    1    0    0
A189   D192    0    0    1    0    0
```

INPUT DATA FILE (continued)

5 x correlations for 32*31/2 measurement pairs!

(dumped to input file from a one-line bash script…)

LEP Combination for the Summer 2001 Conferences (http://lepewwg.web.cern.ch/LEPEWWG/lepww/4f/Summer01). The output results may differ because of rounding errors.

BlueFin 01.00.04   18   LEP2 WW cross sections (sww.bfin)

WW cross sections at LEP2 – again (3)

	A183	A189	A192	A196	A200	A202	A205	A207	D183	D189	D192	D196	D200	D202	D205	D207	L183	L189	L192	L196	L200	L202	L205	L207	O183	O189	O192	O196	O200	O202	O205	O207
A183	0.07	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A189	0.04	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A192	0.04	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A196	0.04	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A200	0.04	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A202	0.04	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A205	0.04	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A207	0.04	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D183	0	0	0	0	0	0	0	0	0.06	0.04	0.05	0.05	0.05	0.05	0.05	0.05	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D189	0	0	0	0	0	0	0	0	0.04	0.03	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D192	0	0	0	0	0	0	0	0	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D196	0	0	0	0	0	0	0	0	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D200	0	0	0	0	0	0	0	0	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D202	0	0	0	0	0	0	0	0	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D205	0	0	0	0	0	0	0	0	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
D207	0	0	0	0	0	0	0	0	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
L183	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0
L189	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0
L192	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0
L196	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0
L200	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0
L202	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0
L205	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0
L207	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0	0	0	0	0	0	0	0
O183	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.05	0.04	0.09	0.09	0.08	0.09	0.08	0.08
O189	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.04	0.03	0.07	0.07	0.06	0.07	0.06	0.06
O192	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.09	0.07	0.16	0.17	0.15	0.16	0.14	0.15
O196	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.09	0.07	0.17	0.17	0.15	0.17	0.15	0.15
O200	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.08	0.06	0.15	0.15	0.13	0.15	0.13	0.14
O202	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.09	0.07	0.16	0.17	0.15	0.16	0.14	0.15
O205	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.08	0.06	0.14	0.15	0.13	0.14	0.13	0.13
O207	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.08	0.06	0.15	0.15	0.14	0.15	0.13	0.14

Table 8: Partial input covariance between measurements. Error source #4: LUEC. Values /1M are displayed.

this is LEP-Uncorrelated Energy-Correlated (LUEC):

block matrix with 4 separate 8x8 matrices for ALEPH, DELPHI, L3, OPAL

Display INPUT DATA
Covariance matrix for one error source

LEP Combination for the Summer 2001 Conferences (http://lepewwg.web.cern.ch/LEPEWWG/lepww/4f/Summer01). The output results may differ because of rounding errors.

BlueFin 01.00.04 8 LEP2 WW cross sections (sww.bfin)

WW cross sections at LEP2 – again (4)

	A183	A189	A192	A196	A200	A202	A205	A207	D183	D189	D192	D196	D200	D202	D205	D207	L183	L189	L192	L196	L200	L202	L205	L207	O183	O189	O192	O196	O200	O202	O205	O207
A183	0.47	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
A189	0.04	0.15	0.02	0.02	0.02	0.02	0.02	0.02	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	0.01	~0	0.01	0.01	~0	0.01	~0	~0
A192	0.04	0.02	0.82	0.02	0.02	0.02	0.02	0.02	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	0.01	~0	0.01	0.01	~0	0.01	~0	~0
A196	0.04	0.02	0.02	0.32	0.02	0.02	0.02	0.02	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	0.01	~0	0.01	0.01	~0	0.01	~0	~0
A200	0.04	0.02	0.02	0.02	0.31	0.02	0.02	0.02	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	0.01	~0	0.01	0.01	~0	0.01	~0	~0
A202	0.04	0.02	0.02	0.02	0.02	0.58	0.02	0.02	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	0.01	~0	0.01	0.01	~0	0.01	~0	~0
A205	0.04	0.02	0.02	0.02	0.02	0.02	0.30	0.02	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	0.01	~0	0.01	0.01	~0	0.01	~0	~0
A207	0.04	0.02	0.02	0.02	0.02	0.02	0.02	0.20	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	0.01	~0	0.01	0.01	~0	0.01	~0	~0
D183	0.01	~0	~0	~0	~0	~0	~0	~0	0.54	0.05	0.05	0.05	0.05	0.06	0.05	0.05	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
D189	0.01	~0	~0	~0	~0	~0	~0	~0	0.05	0.18	0.04	0.04	0.04	0.04	0.04	0.04	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01
D192	0.01	~0	~0	~0	~0	~0	~0	~0	0.05	0.04	1.05	0.04	0.04	0.05	0.05	0.04	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01
D196	0.01	~0	~0	~0	~0	~0	~0	~0	0.05	0.04	0.04	0.40	0.04	0.05	0.05	0.04	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01
D200	0.01	~0	~0	~0	~0	~0	~0	~0	0.05	0.04	0.04	0.04	0.36	0.05	0.05	0.04	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01
D202	0.01	~0	~0	~0	~0	~0	~0	~0	0.06	0.04	0.05	0.05	0.05	0.71	0.05	0.04	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
D205	0.01	~0	~0	~0	~0	~0	~0	~0	0.05	0.04	0.05	0.05	0.05	0.05	0.41	0.04	~0	~0	~0	~0	~0	~0	~0	~0	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01
D207	0.01	~0	~0	~0	~0	~0	~0	~0	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.23	~0	~0	~0	~0	~0	~0	~0	~0	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01
L183	0.01	~0	~0	~0	~0	~0	~0	~0	0.01	0.01	0.01	0.01	0.01	0.01	~0	~0	0.52	0.04	0.05	0.05	0.05	0.05	0.05	0.05	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
L189	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	~0	0.04	0.19	0.05	0.05	0.05	0.05	0.05	0.05	0.01	~0	0.01	0.01	~0	0.01	~0	~0
L192	0.01	~0	~0	~0	~0	~0	~0	~0	0.01	0.01	0.01	0.01	0.01	0.01	~0	~0	0.05	0.05	0.87	0.05	0.05	0.05	0.05	0.05	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
L196	0.01	~0	~0	~0	~0	~0	~0	~0	0.01	0.01	0.01	0.01	0.01	0.01	~0	~0	0.05	0.05	0.05	0.36	0.05	0.05	0.05	0.05	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
L200	0.01	~0	~0	~0	~0	~0	~0	~0	0.01	0.01	0.01	0.01	0.01	0.01	~0	~0	0.05	0.05	0.05	0.05	0.38	0.05	0.05	0.05	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
L202	0.01	~0	~0	~0	~0	~0	~0	~0	0.01	0.01	0.01	0.01	0.01	0.01	~0	~0	0.05	0.05	0.05	0.05	0.05	0.78	0.05	0.05	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
L205	0.01	~0	~0	~0	~0	~0	~0	~0	0.01	0.01	0.01	0.01	0.01	0.01	~0	~0	0.05	0.05	0.05	0.05	0.05	0.05	0.41	0.05	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
L207	0.01	~0	~0	~0	~0	~0	~0	~0	0.01	0.01	0.01	0.01	0.01	0.01	~0	~0	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.26	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
O183	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.44	0.05	0.10	0.11	0.09	0.10	0.09	0.09
O189	0.01	~0	~0	~0	~0	~0	~0	~0	0.01	~0	~0	~0	~0	0.01	~0	~0	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01	0.05	0.15	0.08	0.08	0.07	0.08	0.07	0.07
O192	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.10	0.08	0.95	0.18	0.16	0.18	0.16	0.16
O196	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.11	0.08	0.18	0.54	0.16	0.18	0.16	0.16
O200	0.01	~0	~0	~0	~0	~0	~0	~0	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01	0.09	0.07	0.16	0.16	0.44	0.16	0.14	0.14
O202	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.10	0.08	0.18	0.18	0.16	0.83	0.16	0.16
O205	0.01	~0	~0	~0	~0	~0	~0	~0	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01	0.09	0.07	0.16	0.16	0.14	0.16	0.41	0.14
O207	0.01	~0	~0	~0	~0	~0	~0	~0	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	~0	0.01	0.01	0.01	0.01	0.01	0.01	0.09	0.07	0.16	0.16	0.14	0.16	0.14	0.32

Table 3: Full input covariance between measurements (summed over error sources). Values /1M are displayed.

Display INPUT DATA
Full covariance matrix (sum of all error sources)

LEP Combination for the Summer 2001 Conferences (http://lepewwg.web.cern.ch/LEPEWWG/lepww/4f/Summer01). The output results may differ because of rounding errors.

WW cross sections at LEP2 – again (5)

CVW (central value weight) is just the BLUE coefficient here

Measurements		CVW183/%	CVW189/%	CVW192/%	CVW196/%	CVW200/%	CVW202/%	CVW205/%	CVW207/%	Stat	LCEU	LCEC	LUEU	LUEC
A183	15.57 ± 0.68	25.58	-0.58	-0.36	-0.42	-0.40	-0.50	-0.47	-0.40	0.62	0	0.09	0.09	0.26
D183	15.86 ± 0.74	22.26	0.04	0.29	0.16	0.09	0.09	0.19	0.06	0.69	0	0.09	0.07	0.23
L183	16.53 ± 0.72	23.49	0.30	0.50	0.46	0.56	0.67	0.60	0.57	0.67	0	0.08	0.14	0.20
O183	15.43 ± 0.66	28.68	0.25	-0.42	-0.20	-0.25	-0.27	-0.31	-0.24	0.61	0	0.14	0	0.22
BLUE 183	15.79 ± 0.36	100.00	0	0	0	0	0	0	0	0.32	0	0.10	0.04	0.11

Measurements		CVW183/%	CVW189/%	CVW192/%	CVW196/%	CVW200/%	CVW202/%	CVW205/%	CVW207/%	Stat	LCEU	LCEC	LUEU	LUEC
A189	15.71 ± 0.38	-0.86	26.72	1.11	0.98	0.95	0.80	0.80	0.94	0.34	0	0.05	0.09	0.15
D189	15.83 ± 0.43	-1.39	21.33	-0.65	-0.93	-1.07	-1.18	-0.79	-1.06	0.38	0	0.07	0.05	0.18
L189	16.24 ± 0.43	0.16	22.12	-0.69	-0.80	-0.43	-0.18	-0.30	-0.36	0.37	0	0.04	0.08	0.20
O189	16.30 ± 0.38	2.09	29.83	0.23	0.76	0.55	0.56	0.30	0.48	0.34	0	0.07	0	0.17
BLUE 189	16.00 ± 0.21	0	100.00	0	0	0	0	0	0	0.18	0	0.05	0.03	0.08

Measurements		CVW183/%	CVW189/%	CVW192/%	CVW196/%	CVW200/%	CVW202/%	CVW205/%	CVW207/%	Stat	LCEU	LCEC	LUEU	LUEC
A192	17.23 ± 0.91	0.47	0.45	27.07	0.77	0.75	0.75	0.73	0.74	0.89	0	0.05	0.09	0.15
D192	16.90 ± 1.02	0.18	0.12	21.25	0.26	0.22	0.22	0.27	0.22	1.00	0	0.07	0.06	0.20
L192	16.39 ± 0.93	0.14	-0.01	25.85	0.09	0.17	0.21	0.21	0.19	0.90	0	0.08	0.08	0.21
O192	16.60 ± 0.97	-0.79	-0.56	25.83	-1.11	-1.14	-1.18	-1.21	-1.14	0.88	0	0.12	0	0.40
BLUE 192	16.72 ± 0.48	0	0	100.00	0	0	0	0	0	0.46	0	0.07	0.04	0.10

Measurements		CVW183/%	CVW189/%	CVW192/%	CVW196/%	CVW200/%	CVW202/%	CVW205/%	CVW207/%	Stat	LCEU	LCEC	LUEU	LUEC
A196	17.00 ± 0.57	1.17	1.15	2.03	29.30	1.87	1.86	1.80	1.83	0.54	0	0.05	0.09	0.15
D196	17.86 ± 0.63	0.43	0.28	0.80	23.94	0.52	0.52	0.65	0.51	0.59	0	0.07	0.06	0.20
L196	16.67 ± 0.60	0.29	-0.10	0.22	26.61	0.31	0.42	0.40	0.37	0.55	0	0.08	0.08	0.21
O196	18.59 ± 0.74	-1.89	-1.33	-3.06	20.15	-2.70	-2.79	-2.86	-2.71	0.60	0	0.12	0	0.41
BLUE 196	17.43 ± 0.32	0	0	0	100.00	0	0	0	0	0.29	0	0.07	0.04	0.10

Measurements		CVW183/%	CVW189/%	CVW192/%	CVW196/%	CVW200/%	CVW202/%	CVW205/%	CVW207/%	Stat	LCEU	LCEC	LUEU	LUEC
A200	16.98 ± 0.56	0.98	1.03	1.92	1.79	28.27	1.74	1.70	1.73	0.53	0	0.05	0.09	0.15
D200	17.35 ± 0.60	0.26	0.17	0.72	0.51	24.56	0.41	0.57	0.41	0.56	0	0.07	0.06	0.20
L200	16.94 ± 0.62	0.07	-0.23	0.04	-0.06	23.20	0.23	0.23	0.19	0.57	0	0.08	0.08	0.21
O200	16.32 ± 0.66	-1.32	-0.98	-2.67	-2.23	23.96	-2.38	-2.49	-2.34	0.54	0	0.10	0	0.37
BLUE 200	16.84 ± 0.31	0	0	0	0	100.00	0	0	0	0.28	0	0.07	0.04	0.10

Measurements		CVW183/%	CVW189/%	CVW192/%	CVW196/%	CVW200/%	CVW202/%	CVW205/%	CVW207/%	Stat	LCEU	LCEC	LUEU	LUEC
A202	16.16 ± 0.76	0.67	0.63	1.12	1.05	1.04	30.03	1.00	1.02	0.74	0	0.05	0.09	0.15
D202	17.67 ± 0.84	0.06	0.04	0.29	0.19	0.15	24.54	0.23	0.15	0.81	0	0.08	0.07	0.21
L202	16.95 ± 0.88	0.15	-0.03	0.11	0.07	0.15	22.28	0.20	0.18	0.85	0	0.08	0.08	0.21
O202	18.48 ± 0.91	-0.88	-0.64	-1.53	-1.31	-1.34	23.16	-1.43	-1.35	0.81	0	0.12	0	0.40
BLUE 202	17.23 ± 0.42	0	0	0	0	0	100.00	0	0	0.40	0	0.07	0.04	0.10

LEP Combination for the Summer 2001 Conferences (http://lepewwg.web.cern.ch/LEPEWWG/lepww/4f/Summer01). The output results may differ because of rounding errors.

BlueFin 01.00.04

Display RESULTS (more on next slide)

LEP2 WW cross sections (sww.bfin)

WW cross sections at LEP2 – again (6)

Measurements		CVW183/%	CVW189/%	CVW192/%	CVW196/%	CVW200/%	CVW202/%	CVW205/%	CVW207/%	Stat	LCEU	LCEC	LUEU	LUEC
A205	16.57 ± 0.55	0.93	1.01	1.94	1.80	1.77	1.76	29.66	1.75	0.52	0	0.05	0.09	0.15
D205	17.44 ± 0.64	0.35	0.19	0.72	0.52	0.44	0.44	22.00	0.43	0.60	0	0.06	0.05	0.20
L205	17.35 ± 0.64	∼ 0	-0.26	∼ 0	-0.10	0.08	0.17	22.07	0.15	0.59	0	0.08	0.08	0.21
O205	15.97 ± 0.64	-1.28	-0.95	-2.67	-2.22	-2.30	-2.37	26.27	-2.33	0.52	0	0.10	0	0.36
BLUE 205	16.71 ± 0.31	0	0	0	0	0	0	100.00	0	0.28	0	0.06	0.03	0.10

Measurements		CVW183/%	CVW189/%	CVW192/%	CVW196/%	CVW200/%	CVW202/%	CVW205/%	CVW207/%	Stat	LCEU	LCEC	LUEU	LUEC
A207	17.32 ± 0.45	1.35	1.54	2.95	2.73	2.70	2.66	2.61	29.14	0.41	0	0.05	0.09	0.15
D207	16.50 ± 0.48	0.86	0.54	1.54	1.19	1.03	1.04	1.28	25.87	0.43	0	0.06	0.05	0.20
L207	17.96 ± 0.51	-0.11	-0.49	-0.11	-0.27	0.05	0.20	0.21	22.40	0.45	0	0.08	0.08	0.21
O207	17.77 ± 0.57	-2.10	-1.59	-4.38	-3.65	-3.78	-3.90	-4.10	22.58	0.42	0	0.09	0	0.37
BLUE 207	17.33 ± 0.25	0	0	0	0	0	0	0	100.00	0.22	0	0.06	0.03	0.10

BLUE for one of the 8 parameters

Table 1: BLUE's of the combination (χ^2/ndof = 27.42/24). Values /1% are displayed. For each input measurement i, the central value weight CVW or λ_i^α with which that measurement contributes to the BLUE for observable α is listed.

Central values and total errors
(4 input, 1 output)

Individual error sources (4 input, 1 output)

Note how most errors decrease except those correlated between all 32 measurements (LCEC)

	183	189	192	196	200	202	205	207
183	1.00	0.20	0.11	0.17	0.17	0.13	0.17	0.20
189	0.20	1.00	0.13	0.20	0.20	0.15	0.20	0.24
192	0.11	0.13	1.00	0.12	0.12	0.09	0.12	0.14
196	0.17	0.20	0.12	1.00	0.18	0.13	0.17	0.21
200	0.17	0.20	0.12	0.18	1.00	0.13	0.17	0.21
202	0.13	0.15	0.09	0.13	0.13	1.00	0.13	0.16
205	0.17	0.20	0.12	0.17	0.17	0.13	1.00	0.21
207	0.20	0.24	0.14	0.21	0.21	0.16	0.21	1.00

Table 2: Correlations between the BLUE's.

8x8 correlations between the 8 parameter BLUEs

χ^2 for the global combination
(32-8 = 24 degrees of freedom)

BLUE coefficients for the 4 measurements of THIS parameter
(normalization: sum=1)

BLUE coefficients for the 4 measurements of another parameter
(normalization: sum=0)

- Today we almost ignored BLUE coefficients
  – And note they are not even published, e.g. in the LEP2 reports
- Tomorrow we will discuss them a lot!
  – What they can tell us (about measurements and their correlations)

Display RESULTS

LEP Combination for the Summer 2001 Conferences (http://lepewwg.web.cern.ch/LEPEWWG/lepww/4f/Summer01). The output results may differ because of rounding errors.

BLUEFIN

BLUEFIN 01.00.04 2 LEP2 WW cross sections (sww.bfin)

Outline #1 – BLUE basics

- Intro to combining measurements – why, how, who?
  - Simple academic examples and a glimpse of some real examples
- Best Linear Unbiased Estimators – basics
  - References, derivation of method, relation to other estimators
  - Effect of correlations – just a glimpse (discussed in detail tomorrow)
- BLUE combinations in practice – the multi-parameter case
  - Preparing the relevant inputs and extracting the results
  - A step-by-step example from LEP2 (statistically dominated)
- *Intermediate wrap-up (between lectures #1 and #2)*

Conclusions #1 – BLUE basics

- Combining measurements
  - Minimize errors on combined results – examples LEPEWWG, PDG etc
- Best Linear Unbiased Estimators
  - Exact matrix formulas for central value, error and error breakdown
    - Linear combination of all measurements, also those of other parameters
    - Coefficients can be positive or negative, subject to normalization condition
  - Must know covariance matrix of measurements, no need to know p.d.f.
    - Prefer MLE if you know the p.d.f. distribution and it is not Gaussian
  - BLUE has many nice additional properties if distributions are Gaussian
    - Sum of squared residuals is c2 distributed, BLUE and MLE coincide
- BLUE combinations in practice
  - Multi-parameter LEP2 examples and details from BlueFin software
  - Need input correlations between all measurements of all parameters
    - More tomorrow on the effect of correlations and estimating them in practice

Entr’acte

And now for something completely different

Still about statistics,

but forget about BLUE combinations for a while

Us and them – in HEP we are lucky! (spoilt?)

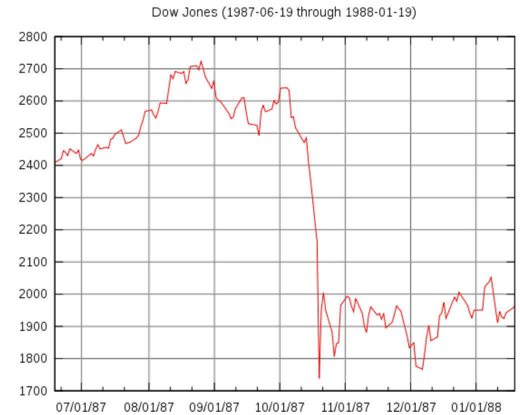
- Statistics is used in many different fields in different ways
  - Different challenges and issues imply different practices and buzzwords
    - e.g. Chris Blake's Statistics Lectures for astronomers
    - e.g. Natasha Devroye's Detection and Estimation course for engineers
    - e.g. Nature "Points of Significance" open access column for biologists
  - "Descriptive" vs "inferential" statistics, "internal" vs "external" validity...
  - Advice for us too: make sure that people understand what is being done
    - e.g. Belia, Fidler, Williams, Cumming, Psychological Methods, 2005, "Researchers Misunderstand Confidence Intervals and Standard Error Bars"

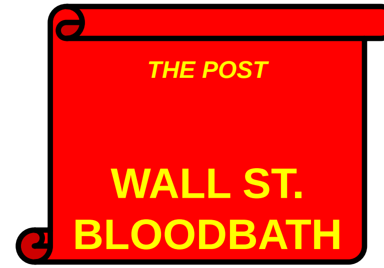
- **High-energy physicists are lucky!**
  - Data sampling? Any LHC period is *representative* of the Laws of Nature!
  - Moving targets? The Laws of Nature (in practice) *do not vary in time*!
  - Black swans? Event fluctuations are *Poissonian* in quantum mechanics!
  - Distributions? Simulate models (mostly with *few parameters*) using MC!

  $\rightarrow$ Result: spectacular agreement of experiments and statistical predictions

Others are not so lucky!

19 October 1987

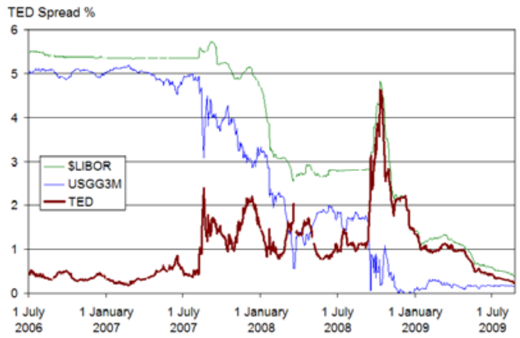




- Beware of your assumptions! And beware of “magic” tools!
  – “The Black-Scholes equation was the mathematical justification for the trading that plunged the world's banks into catastrophe [….] On 19 October 1987, Black Monday, *the world's stock markets lost more than 20% of their value within a few hours. An event this extreme is virtually impossible under the model's assumptions* […] Large fluctuations in the stock market are far more common than Brownian motion predicts. The reason is *unrealistic assumptions – ignoring potential black swans*. But usually the model performed very well, so as time passed and confidence grew, many bankers and traders *forgot the model had limitations*. They used the equation as a kind of *talisman*, a bit of *mathematical magic* to protect them against criticism if anything went wrong.” [I. Stewart, The Guardian, 2012]

"Outliers" and "black swans"

2007-2008



- What is hidden in one "sigma"? (Not everything is Gaussian!)
  - "By any historical standard, the financial crisis of the past 18 months has been extraordinary. Some suggested it is the worst since the early 1970s; others, the worst since the Great Depression; others still, the worst in human history. […] Back in August 2007, *the CFO of Goldman Sachs commented to the FT «We are seeing things that were 25-standard deviation moves, several days in a row». […] A 25-sigma event would be expected to occur once every 6x10^{124} lives of the universe.* That is quite a lot of human histories. […] Fortunately, there is a simpler explanation – *the model was wrong*. [...] I have outlined some elements of an agenda to address some of the failures exposed by the crisis. These measures [...] also involve *much greater transparency* to the wider world about risk metrics and accompanying management actions." [A. Haldane, Bank of England, 2009, "Why banks failed the stress test"]

Lessons learnt? **(also for BLUE?)**

- Statistics is used in many different fields in different ways
  - Life seems easier in HEP, but it's not a reason to lower your guard
- 1. Never stop questioning (deconstructing) your assumptions!
  - Especially when they are kind of long-established common sense
  - Especially when these assumptions end up kind of hidden elsewhere
- 2. Never stop questioning your tools!
  - Do not treat them as "talismans" or "mathematical magic"!
  - Especially when they kind of help you hide uncomfortable assumptions
- My personal preference/suggestion: aim for transparency
  - Uncomfortable results may be there to ring alarm bells and help you...
  - *Should negative BLUE coefficients make us feel uncomfortable?*
  - *I think instead that they can teach us a lot… more tomorrow!*

openclipart.org

Entr’acte - finished

Back to the main show...

tomorrow!

Conclusions #1 – BLUE basics

- Combining measurements
  - Minimize errors on combined results – examples [illegible] PDG etc
- Best Linear Unbiased Estimators
  - Exact matrix formulas for centra[illegible] breakdown
    - Linear combination of all [illegible] of other parameters
    - Coefficients can be [illegible] to normalization condition
  - Must know covar[illegible]ements, no need to know p.d.f.
    - Prefer M[illegible]istribution and it is not Gaussian
  - BLUE [illegible]al properties if distributions are Gaussian
    - [illegible]als is c2 distributed, BLUE and MLE coincide
- BLU[illegible]ations in practice
  - Mul[illegible]arameter LEP2 examples and details from BlueFin software
  - Need input correlations between all measurements of all parameters
    - More tomorrow on the effect of correlations and estimating them in practice

QUESTIONS?