Introduction to Effective Field Theories

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School on Precision Measurements
in top-quark and bottom-quark physics

Meinerzhagen, 24. September 2015

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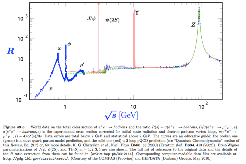

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- Large Logarithms and Fermi's theory (of weak interactions)
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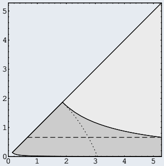

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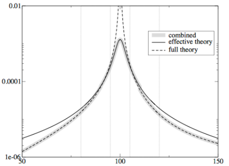
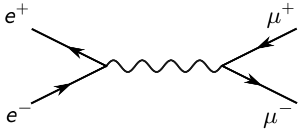


- Summary

Part I: Introductory Lesson

There are a few lessons we need to learn before diving into the techniques of Effective Field Theories. To illustrate consider

$e^+e^- \to \mu^+\mu^-$
in the limit $m_e \to 0$.



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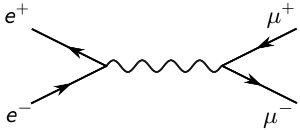
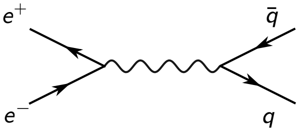


$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3(2E)^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right) + \mathcal{O}(\alpha^3)$$
$$= \frac{4\pi\alpha^2}{3s}\left[1+\mathcal{O}(\alpha^3, \frac{m_\mu^2}{s})\right]$$

where $s = (p+p')^2 = (2E)^2$, with E the energy of the muon in the center-of-mass frame.

The second line is a good approximation for energies far above production threshold $(E \gg m_\mu)$.



What changes when considering $e^+e^- \to q\bar{q}$ instead?

1. Replace coupling $e \longrightarrow Q|e|$, with $Q = -\frac{1}{3}, +\frac{2}{3}$.
2. Count each quark 3 times, one for each colour.
3. Include strong interaction effects (gluon exchange between the quarks, hadronisation, ...)

- In the limit $E \gg m_q \gg \Lambda_{\rm QCD}$ the formula changes to

$$\sigma(e^+e^- \to q\bar{q}) = 3Q^2 \frac{4\pi\alpha^2}{3s}\left[1 + \frac{\alpha_s(\mu)}{4\pi}(\dots)\right]$$

- At which scale μ do we evaluate $\alpha_s(\mu)$? [Actually a quite complicated problem...]
  As long as $\mu \gg \Lambda_{\rm QCD}$ we have $\alpha_s(\mu)/(4\pi) \ll 1$ (asymptotic freedom).

With the approximations made we have

R-ratio far away from production thresholds

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \longrightarrow 3\sum_{i=1}^{n} Q_i^2 = R_n$$

i	flavour	Q_i
1	up	$+\frac{2}{3}$
2	down	$-\frac{1}{3}$
3	strange	$-\frac{1}{3}$
4	charm	$+\frac{2}{3}$
5	bottom	$-\frac{1}{3}$

n	R_n
3	2.00
4	3.33
5	3.67

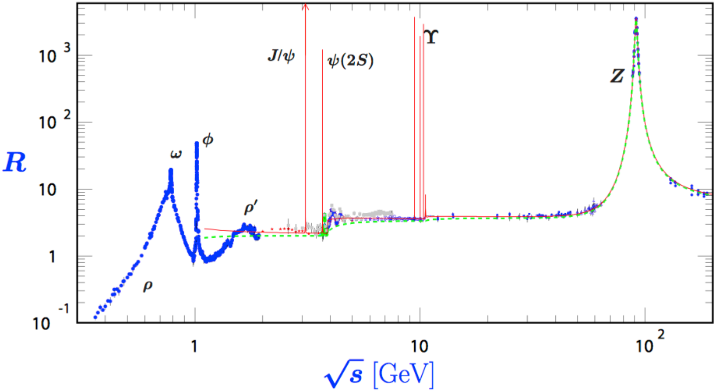


Figure 49.5: World data on the total cross section of $e^+e^- \to hadrons$ and the ratio $R(s) = \sigma(e^+e^- \to hadrons, s)/\sigma(e^+e^- \to \mu^+\mu^-, s)$. $\sigma(e^+e^- \to hadrons, s)$ is the experimental cross section corrected for initial state radiation and electron-positron vertex loops, $\sigma(e^+e^- \to \mu^+\mu^-, s) = 4\pi\alpha^2(s)/3s$. Data errors are total below 2 GeV and statistical above 2 GeV. The curves are an educative guide: the broken one (green) is a naive quark-parton model prediction, and the solid one (red) is 3-loop pQCD prediction (see "Quantum Chromodynamics" section of this *Review*, Eq. (9.7) or, for more details, K. G. Chetyrkin *et al.*, Nucl. Phys. **B586**, 56 (2000) (Erratum *ibid.* **B634**, 413 (2002)). Breit-Wigner parameterizations of J/ψ, $\psi(2S)$, and $\Upsilon(nS), n = 1, 2, 3, 4$ are also shown. The full list of references to the original data and the details of the R ratio extraction from them can be found in `[arXiv:hep-ph/0312114]`. Corresponding computer-readable data files are available at `http://pdg.lbl.gov/current/xsect/`. (Courtesy of the COMPAS (Protvino) and HEPDATA (Durham) Groups, May 2010.)

Part I: Introductory Lesson

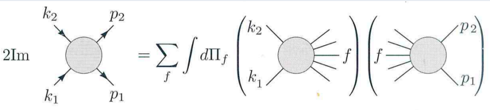
Straight-forward (quantum-mechanic) way to calculate the cross section:

1. Compute all Feynman diagrams to the desired precision to get the amplitude.
2. Square the amplitude
3. Integrate over all final-state phase space.

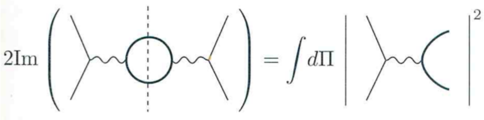
There is a short-cut to this recipe . . .

From amplitudes to cross section/decay rate:

The Optical Theorem

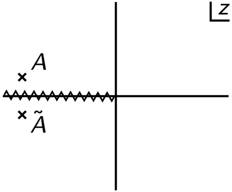
- The optical theorem is a straight-forward consequence of the unitarity of the S-matrix.
- $$2\text{Im}\mathcal{A}(a \to b) = \sum_f \int d\Pi_f\, \mathcal{A}(a \to f)\mathcal{A}^*(b \to f)$$
- 

In the case at hand $e^+e^- \to q\bar{q}$ we have a one-loop Feynman diagram.



We need the "Discontinuity" of the forward-scattering amplitude [$a = b$].

- The following buzzwords are useful to know when talking to theorists: *Discontinuity, Branch-cut, Cutkosky rules*.
- They refer to the fact that $e^{2\pi i} = 1$, or $\ln \tilde{z} = \ln z + n \cdot 2\pi i$.
- We need to agree on a single prescription (branch) of the logarithm.
- Example:

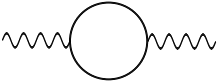


$$\begin{aligned} A &= -r + i\delta = r\, e^{i(\pi-\epsilon)} \\ \tilde{A} &= -r - i\delta = r\, e^{-i(\pi-\epsilon)} \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} \ln A &= \ln r + i\pi - i\epsilon \\ \ln \tilde{A} &= \ln r - i\pi + i\epsilon \end{aligned}$$

- $$\Rightarrow \quad \ln A - \ln \tilde{A} \overset{\epsilon \to 0}{\longrightarrow} \underbrace{2\pi i}_{\text{Disc}} = 2i\,\text{Im}(\ln A)$$

Which particles can run in the loop before taking the discontinuity?

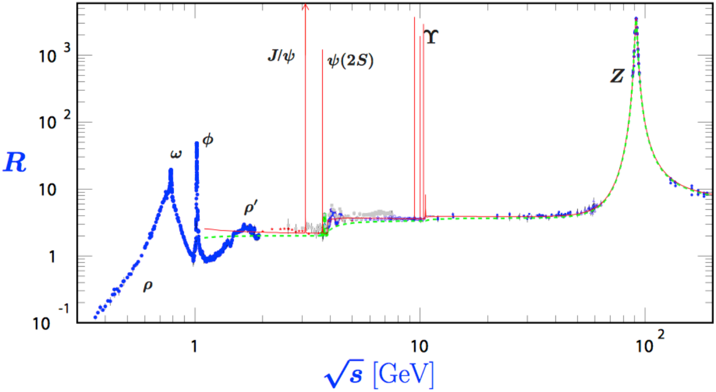


$$\sim \; -\frac{2\alpha Q^2}{\pi}\int_0^1 dx\, x(1-x)\, \ln \frac{m^2}{m^2 - x(1-x)s}$$

has branch-cut, where $m^2 - x(1-x)s < 0$,
starting at $m^2 - \frac{1}{4}s < 0$ or $s > (2m)^2$.

Answer:

Any particle can run, but only the ones kinematically allowed to exist on-shell contribute to the physical cross section.
(i.e. gives a nonzero Disc.)



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So let us pick s well below the $b\bar{b}$ threshold, say $\sqrt{s} = 7$ GeV, and compute $\sigma(e^+e^- \to q\bar{q})$ in **two different theories**:

1. $n_f = 4$, namely u, d, s, c: $\Rightarrow$ $R = 3.33$.
2. $n_f = 4$ plus massive b-quark: $\Rightarrow$ $R = 3.33$, too.

Both theories describe the same IR physics, but look different in the UV.

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Both theories describe the same IR physics, but look different in the UV.

This is in the spirit of EFTs: Particles that (are so massive that they) don't appear as final states are "integrated out".

Full Theory	process energy	Effective Theory
New Physics	$E \ll \Lambda_{\mathrm{NP}}$	Standard Model
Standard Model	$E \ll M_W$	Weak eff. Hamiltonian
⋮		

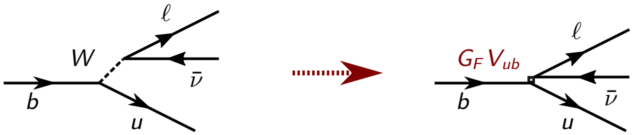
[Enrico Fermi's Manhattan Project badge]

Before we explore EFTs further, a short interlude . . .

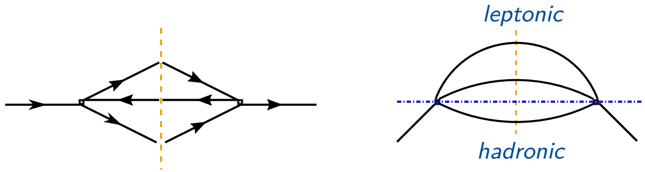
Inclusive B decays

We look at the semileptonic $B \to X_u \ell \bar{\nu}$ as an example.
The same procedure also works for $B \to X_s \gamma$ and $B \to X_s \ell^+ \ell^-$ FCNCs, ...

Amplitude: Integrate out the W boson.

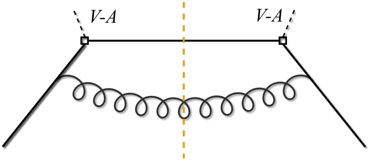


Optical Theorem:



Precision requires us to include strong-interaction effects!
Focus on hadronic part of the diagram.

Example:



Here: $X_u \ni (u + g)$

[These are the type of diagrams theorists think of when discussing inclusive B decays.]

Recap

Part I: Lessons from the *R*-ratio

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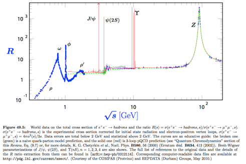
- Optical Theorem ...
- ... and inclusive B decays



Part II: Philosophy of EFTs

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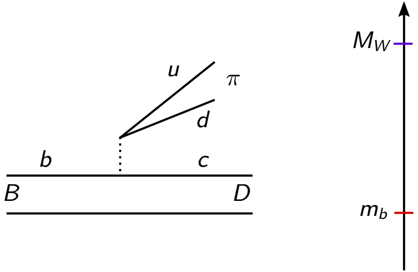
Quantum corrections

Loops and logs

- Consider a process that happens at energy scales much smaller than $M_W \sim \mathcal{O}(100\,\text{GeV})$, e.g. weak $B \to D\pi$ decays



- **Problem for precision:** Strong interactions with multiple (vastly different) scales can lead to uncontrolled perturbative series:

$$P(M_W, m_b) = 1 + \alpha_s \left(\# \ln \frac{M_W}{m_b} + * \right) + \alpha_s^2 \left(\# \ln^2 \frac{M_W}{m_b} + * \right) + \ldots$$

Part II: Philosophy of EFT usage

$$P(M_W, m_b) = 1 + \alpha_s \left(\# \ln \frac{M_W}{m_b} + * \right) + \alpha_s^2 \left(\# \ln^2 \frac{M_W}{m_b} + * \right) + \dots$$

is uncontrolled, because $(\alpha_s \ln \frac{M_W}{m_b})^n$ is not small due to large logs. The perturbative series needs to be reorganised, and all such factors resummed.

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Solution

1. **Match!** We need to achieve a separation of scales, sometimes also called "Factorization".
2. **Run!** As far as you can. [⟵ Attn: clever double entendre.]
3. (Keep going.) [⟵ OK, that's just lame.]

Part II: Philosophy of EFT usage

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Step 1: Match P to this product of two series:

$$\left[1 + \alpha_s \left(\# \ln \frac{M_W}{\mu} + * \right) + \dots \right] \cdot \left[1 + \alpha_s \left(\# \ln \frac{\mu}{m_b} + * \right) + \dots \right]$$

$$P(M_W, m_b) = C(M_W, \mu) D(m_b, \mu)$$

at the cost of introducing a "factorization scale" μ.

Step 2a:
While the physical observable $P(M_W, m_b)$ is formally μ-independent, the factors C and D by themselves are not. They obey

$$\text{RGEs:} \left\{ \begin{array}{rcl} \mu\frac{d}{d\mu}C(M_W,\mu) & = & \gamma(\mu)\,C(M_W,\mu) \\ \mu\frac{d}{d\mu}D(M_W,\mu) & = & -\gamma(\mu)\,D(M_W,\mu) \end{array} \right\} \Rightarrow \mu\frac{d}{d\mu}(CD) = 0$$

[“C and D run with μ.”]

Step 2b:
Solve the Renormalization-Group Equations and evolve:

$$\begin{array}{rcl} C(M_W,\mu) & = & C(M_W,\mu_{\rm high})\,U(\mu_{\rm high},\mu) \\ D(m_b,\mu) & = & D(m_b,\mu_{\rm low})\,U(\mu,\mu_{\rm low}) \end{array}$$

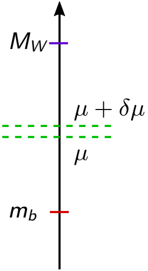
Part II: Philosophy of EFT usage

- One picks $\mu_{\rm high} \sim M_W$, so that $C(M_W, \mu_{\rm high})$ does not have large logs.
- Similarly $\mu_{\rm low} \sim m_b$.
- The scale μ can be anything, e.g. $\mu = \mu_{\rm low}$.
- Therefore $\boxed{P(M_W, m_b) = \underbrace{C(M_W, \mu_{\rm high}) U(\mu_{\rm high}, \mu_{\rm low})}_{C_{\rm RGimproved}(M_W, \mu_{\rm low})} D(m_b, \mu_{\rm low})}$
- $U(\mu_{\rm high}, \mu_{\rm low})$ is generally an exponential, which resums $\left(\alpha_s \ln \frac{\mu_{\rm high}}{\mu_{\rm low}}\right)^n$.
- This is called **Renormalization-Group improved Perturbation Theory**
- The accuracy is labelled LO, NLO, NNLO, etc. [Exercise]

Part II: Philosophy of EFT

- The ingredients in the factorized physical observable $P = C \cdot D$ are connected to

$$\langle \text{Full theory} \rangle = C(M_W, \mu)\, \langle \text{EFT}, \mu \rangle$$



$$\sum_i C_i(\mu + \delta\mu)\, \langle O_i(\mu + \delta\mu) \rangle$$

$$\updownarrow$$

$$\sum_i C_i(\mu)\, \langle O_i(\mu) \rangle$$

Reshuffle [modes]

- The EFT reproduces the IR physics of the Full Theorie to any desired precision.
- The couplings [Ken Wilson coefficients $C_i(\mu)$] capture the UV completion of the Full Theorie.

Part II: Philosophy of EFT

- *This is the modern view of renormalization. All renormalisable theories are EFTs.*
- Case in point: QCD and the strong coupling.

$$\frac{d}{d \ln \mu} \alpha_s(\mu) = \beta(\mu) = -2\alpha_s(\mu) \sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_s(\mu)}{4\pi} \right)^{n+1}$$

with $\beta_0 = 11 - \frac{2}{3} n_f, \ \ldots$ [David Gross, Frank Wilczek, David Politzer]

- Explicit solution can be derived as an exercise.

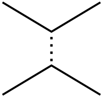
$$\alpha_s(\mu) = \frac{2\pi}{\beta_0 \ln \frac{\mu}{\Lambda_{\mathrm{QCD}}}} + \mathcal{O}\left(\frac{1}{\ln^2 \frac{\mu}{\Lambda_{\mathrm{QCD}}}} \right)$$

- $\Rightarrow$ Landau pole for $\mu \to \Lambda_{\mathrm{QCD}}$, asymptotic freedom for $\mu \to \infty$

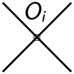
Operator Product Expansion

Quick and Dirty® OPE

Non-local interactions can be expanded in local operators:



$$= \sum_i C_i(M_W)$$



$$\frac{-1}{k^2 - M_W^2} = \frac{1}{M_W^2}\left(1 + \frac{k^2}{M_W^2} + \frac{k^4}{M_W^4} + \dots\right)$$

$$\mathcal{L}_{\rm eff} \ni \frac{g^2}{M_W^2}(\bar\psi\psi)(\bar\psi\psi) + \frac{g^2}{M_W^4}(\bar\psi\psi)(i\partial)^2(\bar\psi\psi) + \dots$$

- In general any operator abiding by the symmetries can appear.
- It's an infinite sum, but ordered by power counting. ["OPE in $1/M_W$"]

Part II: Philosophy of EFT – OPE

Other important example: Non-locality in coordinate space can manifest itself as an integral over a finite interval in momenum space:

$$\int_0^M dk\, \frac{2m^2 k}{(k^2+m^2)^2} = \frac{M^2}{M^2+m^2} = 1 - \frac{m^2}{M^2} + \frac{m^4}{M^4} + \dots$$

- So Integrals can be expanded in an OPE in 1/interval.
- We will see explicit examples in B decays shortly.
- Disclaimer: There are lots of caveats, which we don't need for our lesson on Quick and Dirty® OPE.

Part II: Philosophy of EFT

Lesson learned:

When calculating processes involving a hierarchy of scales, e.g. weak B decays,

1. Integrate out the weak gauge bosons and top quark from the SM at a large scale $\mu_{\rm high} \sim \mathcal{O}(100\,\text{GeV})$. **MATCH!**
2. Evolve the coefficient functions down to $\mu_{\rm low} \sim \mathcal{O}(5\,\text{GeV})$. **RUN!**
3. If more scales exist in the problem (m_c, **experimental cuts**, ...), do it again. **REPEAT!**

Keep going, until energy scales are so low that $\alpha_s(\mu)$ is not perturbative anymore.

- Besides precision through **improved Perturbation Theory**, we can also systematically include **Power Corrections**.

Recap

Part II: Philosophy of EFTs

- Large Logarithms and Fermi's theory (of weak interactions)
- "Integrating out" degrees of freedom (massive particles, modes)
- Resummation
- Renormalization and the running of $\alpha_s(\mu)$
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Part III: Inclusive B decays

- Cutting the background away
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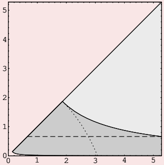
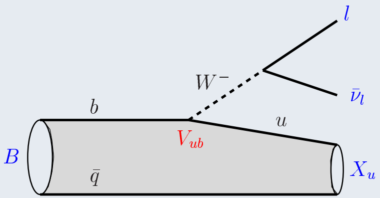


Part III: Inclusive B decays

Semileptonic decay



Kinematics

First we need to familiarize ourselves with the kinematics.

- $M_B v = P_X + \underbrace{P_\ell + P_{\bar\nu}}_{q}$, where q is the momentum of the lepton pair,
- v is the 4-velocity of the B meson, $M_B = 5.279$ GeV, and
- $P_X = \begin{pmatrix} E_X \\ \vec{P}_X \end{pmatrix}$ is the hadronic final state's momentum.

Part III: Inclusive B decays

- Any one event is characterized by 3 independent kinematic variables (scalar variables).
- Example: q^2, E_ℓ, $M_X^2 = P_X^2 = E_X^2 - |\vec{P}_X|^2$
- All choices are equally valid, but my favorite set is the "light-cone components" build from $E_X, |\vec{P}_X|, E_\ell$,

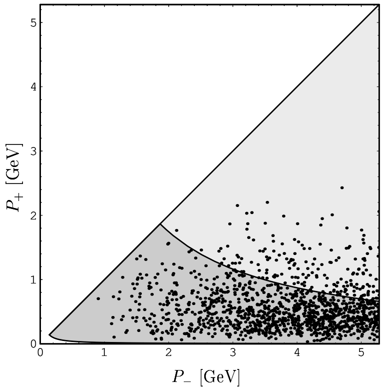
Choice of kinematic variables

$$\begin{aligned} P_+ &= E_X - |\vec{P}_X| \\ P_- &= E_X + |\vec{P}_X| \\ P_\ell &= M_B - 2E_\ell \end{aligned}$$

- I'll defend my choice shortly, and also explain why the signs seem messed-up.
- Note that $M_X^2 = E_X^2 - |\vec{P}_X|^2 = \underbrace{(E_X - |\vec{P}_X|)}_{P_+}\underbrace{(E_X + |\vec{P}_X|)}_{P_-} = P_+ P_-$.

- By definition we have $P_+ \leq P_-$, with equality only for $\vec{P}_X = 0$.
- One reason for liking this choice is that the hadronic phase space is particularly simple: *a triangle.*

$$\frac{m_\pi^2}{P_-} \leq P_+ \leq P_- \leq M_B$$



- The dots shall give an impression on the distribution of events in the phase space.
- Besides the border $P_+P_- = m_\pi^2$ as the lightest X_u state we also show the lightest X_c state in $B \to X_c \ell \bar{\nu}$: $P_+P_- = m_D^2$.
- This is where the background makes measurements very difficult. $\frac{\text{background}}{\text{signal}} \approx \frac{|V_{cb}|^2}{|V_{ub}|^2} \approx 100$

Part III: Inclusive B decays – Phase Space

- Exercise: Show that in $B \to X_s\gamma$ decays the available phase space is

$$\frac{m_K^2}{P_-} \le P_+ \le P_- = M_B \quad \text{with} \quad P_+ = M_B - 2E_\gamma \, , P_- = M_B \, .$$

- Exercise: Show that the charged-lepton energy in $B \to X_u \ell \bar{\nu}$ satisfies

$$\frac{m_\pi^2}{P_-} \le P_+ \le P_\ell \le P_- \le M_B \quad \text{with} \quad P_\ell = M_B - 2E_\ell \, .$$

- Exercise: Show that $q^2 = (M_B - P_+)(M_B - P_-)$.

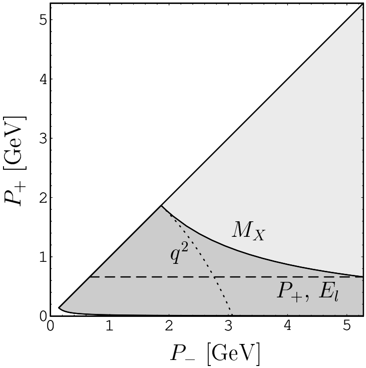
Taming the background

So how can we cut away the $b \to c$ background?

1. Cut on $q^2 \gtrsim (M_B - M_D)^2$
2. Cut on $M_X^2 \lesssim M_D^2$
3. Cut on $P_+ \lesssim M_D^2/M_B$ or perhaps Cut on $P_\ell \lesssim M_D^2/M_B$
4. Some clever combination thereof?

Experimental Challenge:

1. Cut on $q^2 \gtrsim (M_B - M_D)^2$
2. Cut on $M_X^2 \lesssim M_D^2$
3. Cut on $P_+ \lesssim M_D^2/M_B$
   or perhaps $P_\ell \lesssim M_D^2/M_B$
4. Some clever combination?



Theoretical Challenge:

- What can we calculate with what precision?
- How can Effective Field Theories help us?

(1) and (2): Cut on high q^2 and/or low M_X^2.

- Let's look at the cut on q^2. We need to calculate the differential decay rate and integrate over the allowed region in $P_\pm$.

$$\left.\begin{array}{ll} (i) & \displaystyle\int_{m_\pi}^{2M_D - \frac{M_D^2}{M_B}} dP_- \ldots \\ (ii) & \displaystyle\int_{m_\pi}^{M_D} dP_+ \ldots \end{array}\right\} \Rightarrow \quad \text{OPE in} \quad \frac{1}{m_c}$$

[Remember the lesson on Quick and Dirty® OPE]

- $\Gamma(q^2 \geq (M_B - M_D)^2) = \Gamma_{\text{total}} \left[\# + \frac{\bar{\Lambda}}{m_c}\# + \frac{\lambda_1}{m_c^2}\# + \frac{\lambda_2}{m_c^2}\# + \mathcal{O}(\frac{\Lambda_{\text{QCD}}^3}{m_c^3})\right]$
- Can we improve on the precision of this calculation by relaxing the q^2 cut and combining it with an M_X^2 cut? "BLL" (C. Bauer, Z. Ligeti, M. Luke)
- OPE on the P_- integration can be improved, but OPE on P_+ integration remains in $1/m_c$.
  Only extension into the charmed region yields relief.

Part III: Inclusive B decays

(0): Let's take a step back and calculate the total rate Γ_{total} first.

- Then the integrals in $P_\pm$ are over intervals of size $\mathcal{O}(m_b)$.
- $\Gamma_{\text{total}} = \frac{G_F^2 |V_{ub}|^2 m_b^5}{192\pi^3}\left[\# + \frac{\lambda_1}{m_b^2}\# + \dots\right]$.
- The leading-power hashtag is calculated by sending $m_b \to \infty$, while keeping the residual momentum k in

$$p_b = m_b v + k$$

dynamical and of order Λ_{QCD}.

There is a field for that!

Heavy-quark effective theory **(HQET)**: The m_b dependence gets banned into Wilson coefficients (UV), and the operators deal with the residual momentum k (IR). Power corrections are in $1/m_b$. [Howard Georgi, 1990s]

We therefore also get $1/m_b$ corrections on the previous slide.

For fans of Lagrangians:

Propagator:

$$\frac{1}{m_b\not{v} + \not{k} - m_b} = \frac{m_b\not{v} + \not{k} + m_b}{(m_b v + k)^2 - m_b^2} = \frac{1}{v \cdot k}\,\frac{1+\not{v}}{2} + \mathcal{O}(k/m_b)$$

Fields:

$$\psi(x) = e^{im_b v\cdot x}\left[\frac{1+\not{v}}{2}h_v(x) + \frac{1-\not{v}}{2}H_v(x)\right]$$

Lagrangian:

$$\mathcal{L}_{\mathrm{HQET}} = \bar{h}_v iv \cdot Dh_v + \ldots$$

(3): Cut on $P_+ \leq \Delta$.

- Let's try and fail with an OPE: [neglect m_π for now]

$$\left.\begin{array}{lll} (i) & \int_0^{M_B} dP_- \ldots & \text{O.K.} \\ (ii) & \int_0^{\Delta} dP_+ \ldots & \text{not!} \end{array}\right\} \Rightarrow \quad \text{OPE in} \quad \frac{\Lambda_{\rm QCD}}{\Delta}\,?$$

- But the ideal cut $\Delta = \frac{M_D^2}{M_B} = 660$ MeV is too small:
  All powers of $\frac{\Lambda_{\rm QCD}}{\Delta}$ are order 1.
- They need to be summed up into a function $\tilde{S}(\Delta)$, which is almost what we call the "Shape Function".

Part III: Inclusive B decays

- We aim at calculating the decay rate where $P_- \sim m_b$ and $P_+ \sim \Lambda_{QCD}$,
- which means that the X_u state is a JET!

Jet: spray of particles

- Both E_X and $|\vec{P}_X|$ are somewhere near $m_b/2$. **large energy**
- The invariant mass $M_X^2 = P_+ P_- \sim m_b \Lambda_{\mathrm{QCD}}$ is somewhat large, but much smaller than m_b^2. **smaller invariant mass**
- The particles in this jet can interact with soft gluons, and also with gluons collinear to this jet.

There are fields for that!

Soft-Collinear Effective Theory **(SCET)** describes
such particles and interactions.

[Bauer, Pirjol, Stewart; Luke, 2000s]

Part III: Inclusive B decays – SCET

Let us align the z axis with the total jet momenum $\vec{P}_X$. Define **reference 4-vectors**

$$n^\mu = (1,0,0,1)\ ,\ \bar{n}^\mu = (1,0,0,-1)$$

Light-cone coordinates

Any 4-vector p^μ can be decomposed as

$$p^\mu = \underbrace{n\cdot p}_{p_+}\frac{\bar{n}^\mu}{2} + \underbrace{\bar{n}\cdot p}_{p_-}\frac{n^\mu}{2} + p_\perp^\mu = (p_+, p_-, p_\perp) \quad , \quad p^2 = p_+p_- + p_\perp^2 .$$

- If p^μ is *mainly* in the n^μ direction, then $p_- \gg p_+$ and $p_- \gg p_\perp$.
- Define small parameter λ, s.t.

$$(p_+, p_-, p_\perp) \sim m_b(\lambda^2, 1, \lambda).$$

"hard-collinear momentum"

- Here: $\lambda^2 = \Lambda_{\rm QCD}/m_b$.

The light-cone coordinates explain, why the signs seem "reversed":
$p_+ = n \cdot p = p^0 - p^3$ and $p_- = \bar{n} \cdot p = p^0 + p^3$.

[Sometimes the reference vectors are even called $n_+ = n$, $n_- = \bar{n}$]

For fans of Lagrangians:

Propagator:

$$\frac{1}{\not{p}} = \frac{\not{p}}{p^2} = \frac{p_- \frac{\not{n}}{2} + \dots}{p_+ p_- + p_\perp^2} = \frac{1}{p_+ + \frac{p_\perp^2}{p_-}} \frac{\not{n}}{2} + \dots$$

Fields:

$$\psi = \frac{\not{n}\not{\bar{n}}}{4}\xi + \frac{\not{\bar{n}}\not{n}}{4}\eta$$

Lagrangian:

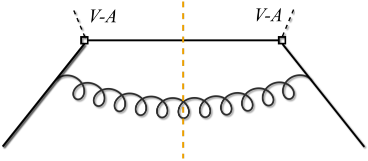
$$\mathcal{L}_{\text{SCET}} = \bar{\xi}\frac{\not{\bar{n}}}{2}\left[in \cdot D + i\not{D}_\perp \frac{1}{i\bar{n} \cdot D} i\not{D}_\perp\right]\xi + \dots$$

Similarly we can assign a power-counting rule for "soft momenta", e.g. the residual momentum $k \sim \Lambda_{\rm QCD}$ in $p_b = m_b v + k$.

Power counting

[hard	$m_b(1,1,1)$]
hard-collinear	$m_b(\lambda^2, 1, \lambda)$
soft	$m_b(\lambda^2, \lambda^2, \lambda^2)$

QCD sample diagram:



Soft momenta can couple to hard-collinear ones:

$$(\lambda^2, \lambda^2, \lambda^2) + (\lambda^2, 1, \lambda) \sim (\underbrace{\lambda^2 + \lambda^2}_{\mathcal{O}(\lambda^2)}, 1, \lambda)$$

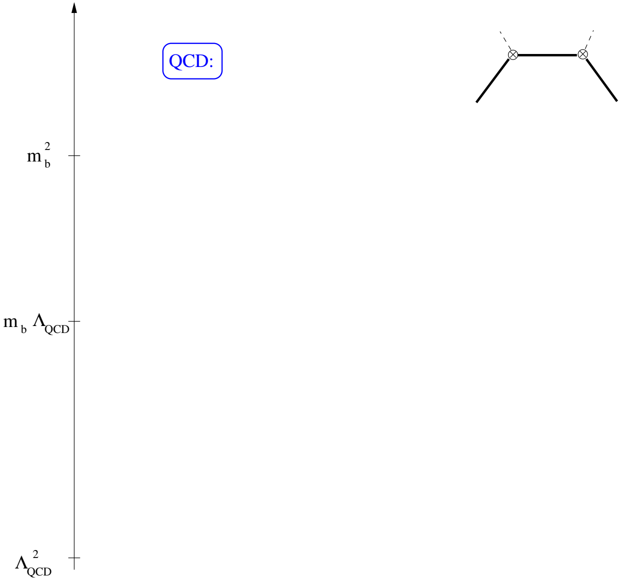
Dependence on the large scales m_b, E_X are absorbed in Wilson coefficients, while the scales $M_X \sim \sqrt{m_b \Lambda_{\rm QCD}}$ and $P_+ \sim \Lambda_{\rm QCD}$ remain dynamical.

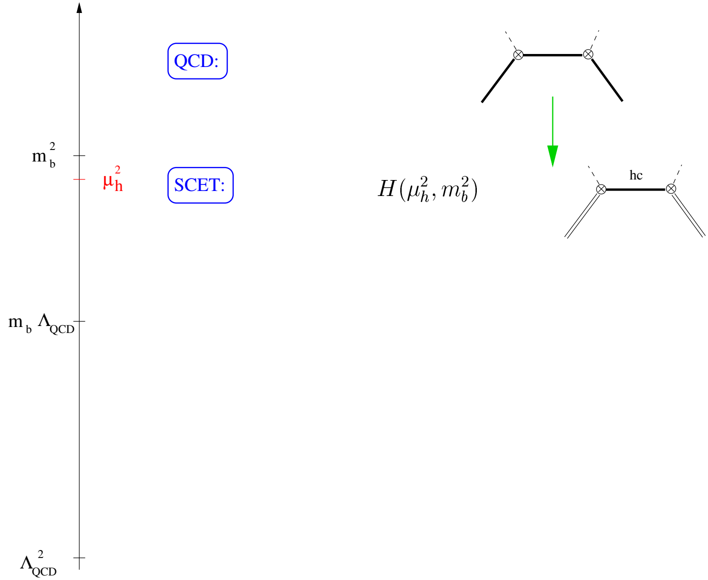
- We can also integrate out the "intermediate scale" $\sqrt{m_b \Lambda_{\mathrm{QCD}}}$ – which is still perturbative – and all collinear dynamics with it. We are left with only soft physics, i.e. pure **HQET**.
- This new Wilson "coefficient" is actually a distribution. Due to the discontinuity we have $\delta(p_X^2)$ at tree level.
- To make wordsmithing worse, this Wilson coefficient/distribution is called the **"jet function"**. It is universal (process independent) and known to high accuracy.
- Finally there is one (non-local) HQET operator left. Its matrix element between two B meson states is the **"shape function"**.

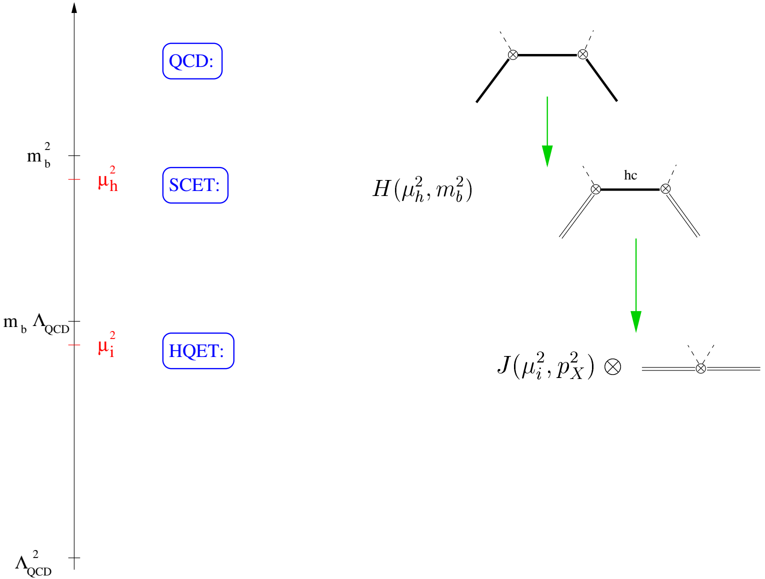
Match and Run

The procedure described above is sometimes called "Multi-Step Matching", and disentangles (factorizes) physics effects from different energy scales. The running resums large logs of the form $\ln(m_b^2/M_X^2)$.

Part III: Inclusive B decays – Multi-Step Matching

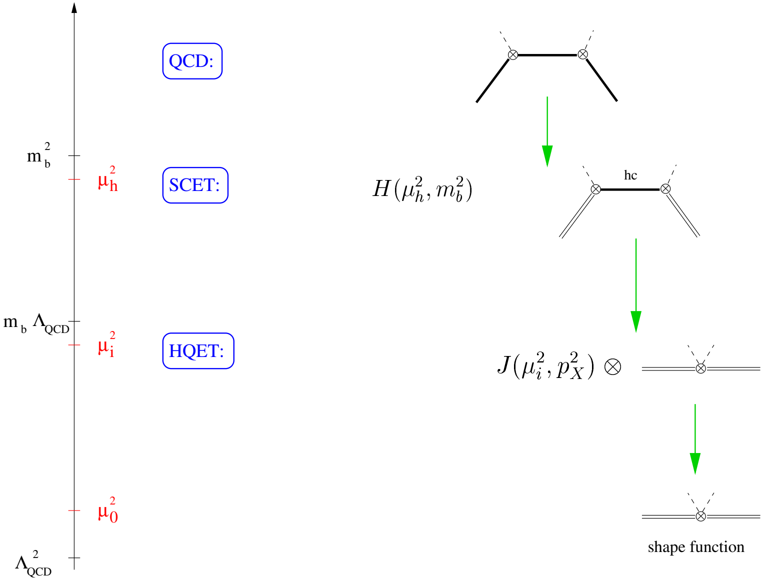



QCD:
SCET:
m_b^2
μ_h^2
$m_b \Lambda_{QCD}$
Λ_{QCD}^2
$H(\mu_h^2, m_b^2)$
hc

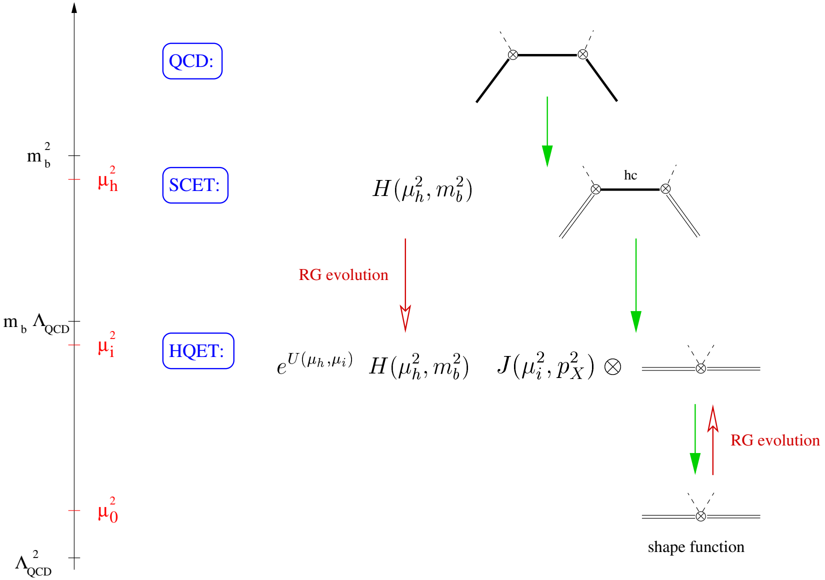


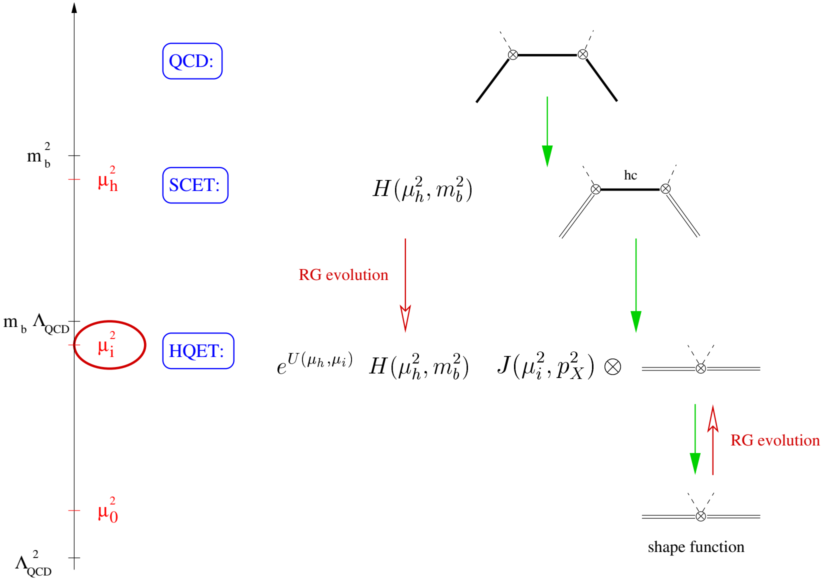
m_b^2
μ_h^2
$m_b \Lambda_{QCD}$
μ_i^2
Λ_{QCD}^2
QCD:
SCET:
HQET:
$H(\mu_h^2, m_b^2)$
hc
$J(\mu_i^2, p_X^2) \otimes$

Part III: Inclusive B decays – Multi-Step Matching



Part III: Inclusive B decays – Multi-Step Matching




QCD:
SCET:
HQET:
m_b^2
μ_h^2
$m_b \Lambda_{QCD}$
μ_i^2
μ_0^2
Λ_{QCD}^2
$H(\mu_h^2, m_b^2)$
hc
RG evolution
$e^{U(\mu_h,\mu_i)}$ $H(\mu_h^2, m_b^2)$ $J(\mu_i^2, p_X^2) \otimes$
RG evolution
shape function

The differential decay rate in the "shape-function region", i.e. where $P_+ \sim \Lambda_{\rm QCD}$ and $P_- \sim m_b$, is then expressed as the product of the leptonic and hadronic tensors,

$$\frac{1}{\Gamma_{\rm total}} \frac{d^3\Gamma}{dP_+ dP_- dP_\ell} = L_{\mu\nu} W^{\mu\nu},$$

where the hadronic tensor **factorizes:**

"BLNP" (S.Bosch, B.L., M.Neubert, G.Paz)

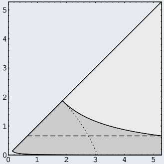
$$W^{\mu\nu} = \sum_{i,j} \mathrm{tr}\left[\Gamma_i^\mu \frac{\not{p}_-}{2} \Gamma_j^\nu \frac{1+\not{v}}{2}\right] e^{U(\mu_h,\mu_i)} H_{ij}(P_-,\mu_h) \int_0^{P_+} d\omega J(p_\omega^2,\mu_i) S(\omega,\mu_i)$$
$$+\mathcal{O}(\frac{1}{m_b})$$

The same structure is also found for power correction: subleading shape functions, etc.

Recap

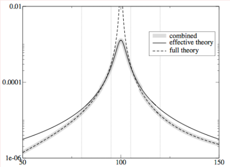
Part III: Inclusive B decays

- Cutting the background away
- HQET and SCET
- Factorization and Resummation



Part IV: Top physics

- Corrections to Breit-Wigner line-shape
- Unstable particle effective theory
- Top pair production (near threshold)
- NRQCD



Part IV: Top physics

M.Beneke, A.P.Chapovsky, A.Signer, G.Zanderighi; 2004

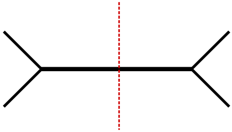
"Several higher-order calculations involving **unstable particles** have been performed in recent years, in particular for the line shape of the Z boson, W-pair production, and $t\bar{t}$ production. In these calculations the finite width of the particles have been treated in a variety of often pragmatic approaches. While this may be adequate for the present, it is certainly desirable to formulate a theoretical framework that would allow for systematic improvements of the accuracy of such calculations. Moreover, future precision experiments require that such a framework be developed."

Part IV: Top physics – Line Shape

- The instability of a particle with mass M leads to a shift of the pole location of the propagator into the complex plane.

$$\frac{1}{p^2 - M_*^2} \quad , \qquad M_*^2 = M^2 - iM\Gamma$$

- e.g. $e^+e^- \to Z_0 + X$ inclusive cross section



$$\sim \mathrm{Im} \frac{1}{p^2 - M_*^2} = -\frac{M\Gamma}{(p^2 - M^2)^2 + M^2\Gamma^2}$$

- Breit-Wigner:

$$\frac{M\Gamma}{(p^2 - M^2)^2 + M^2\Gamma^2} \quad \xrightarrow{\Gamma \to 0} \quad \pi\delta(p^2 - M^2)$$

Problem

Unstable particle fields have no asymptotic particle states, should not be cut!
Usually we get away with it, if Γ is negligible.

Part IV: Top physics – Line Shape

- But for weak bosons, top quark, $\Gamma \sim 2$ GeV $> \Lambda_{\mathrm{QCD}}$. The width gives sizable effects in strong interactions.
- Particularly important if p^2 is near M^2, i.e. $p^2 - M^2 \sim M\Gamma$ in the peak region, where the power-counting variable

$$\delta = \frac{p^2 - M^2}{M^2} \sim \frac{\Gamma}{M} \sim \mathcal{O}(0.01)$$

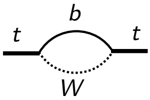
is small.

How can the mass parameter M_*^2 become complex?

- Self energy bubbles resummed into the propagator:

$$\frac{1}{p^2 - M^2} \longrightarrow \frac{1}{p^2 - M^2 - \Pi(p^2)}$$

- Remember from earlier: if loop particles can be on-shell, then loop gets imaginary part.



$+ \ldots$

- With $\Pi(p^2) \sim \Gamma M \sim g^2 M^2$, so two small parameters of the same order: $g^2 \sim \delta$.
- Propagator $\frac{1}{p^2 - M_*^2} \sim \frac{1}{M^2 \delta}$, so $(g^2/\delta)^n \sim \mathcal{O}(1)$ need to be resummed.

"Dyson resummation"

There is a field for that!

Unstable-particle effective field theory
begins just like HQET with $p^\mu = Mv^\mu + k^\mu$, but this time $k^\mu \sim \Gamma$.

[M.Beneke, A.P.Chapovsky, A.Signer, G.Zanderighi, 2010s]

Let us explore the logic behind the construction of this theory with scalar particles instead of fermions.

For fans of Lagrangians:

Propagator with $p = Mv + k$ and $M_*^2 - M^2 = M\Delta$

$$\frac{1}{p^2 - M_*^2} = \frac{1}{2M} \frac{1}{\left(v \cdot k - \frac{\Delta}{2} + \frac{\Delta^2}{8M} + \frac{k_\perp^2}{2M} + \dots\right)}$$

Fields:

$$\phi(x) = e^{iMv\cdot x}\phi_v(x)$$

Lagrangian:

$$\begin{aligned}\mathcal{L}_{\text{HSETfup}} &= 2M\phi_v^\dagger \left(iv \cdot D - \frac{\Delta^{(1)}}{2}\right)\phi_v \\ &\quad +2M\phi_v^\dagger \left(\frac{(iD_\perp)^2}{2M} + \frac{[\Delta^{(1)}]^2}{8M} - \frac{\Delta^{(2)}}{2}\right)\phi_v \\ &\quad + \dots\end{aligned}$$

We could explore a little more in the exercises.

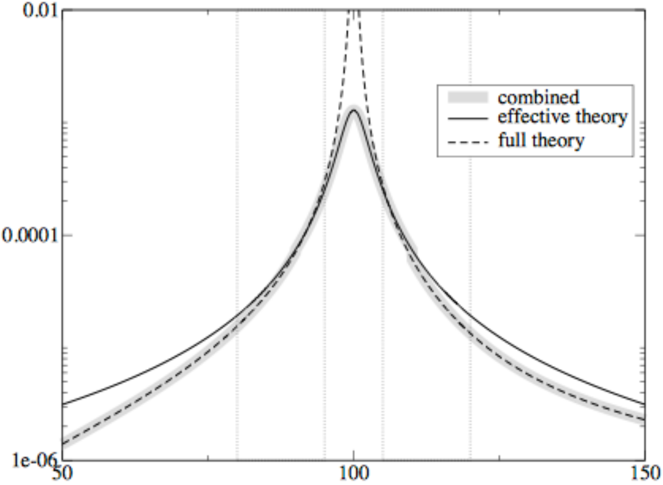
Part IV: Top physics – Line Shape

- The quantity Δ in the Lagrangian is actually a **Wilson coefficient**.
- It is determined by a **matching calculation** and is IR safe.
- The matching is performed by equating the self-energy

$$\Pi(p^2)\Big|_{\text{full theory}} \stackrel{!}{=} \Pi(p^2)\Big|_{\text{effective theory}}$$

- While the full theory gives a good description of the line shape $\sigma(p^2)$ away from the resonance $\delta = \frac{p^2 - M^2}{M^2} \sim \mathcal{O}(1)$, the effective theory is apropriate for $\delta \ll 1$.
- Full theory and EFT results are merged where both theories are valid.
- The precise definition of the mass ($\overline{\text{MS}}$, pole mass, ...) is important since the residual mass term adds to Δ.
- If we use the pole scheme, then $\Delta = -i\Gamma$ is purely imaginary.

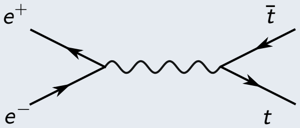
Line shape = Inclusive cross-section



This framework can be extended to any required precision.

Top pair production near threshold

This is a more complicated process, but the leading contribution comes from



- Besides α_s there is another small parameter $v \ll 1$, the velocity of the top.

- In the CoM frame the top-quark momentum p^μ has time component

$$p^0 = \frac{\sqrt{s}}{2} = M_t + \frac{1}{2} M_t v^2 + \ldots \quad \Rightarrow \quad \underbrace{\sqrt{s} - 2M_t}_{E} = M_t v^2$$

- Relevant energy scales:
  $M_t \sim 175$ GeV, $|\vec{p}_t| \sim M_t v \sim 20$ GeV, $E = M_t v^2 \sim \Gamma_t \sim 2$ GeV.
- In the threshold region we need to resum $(\alpha_s/v) \sim \mathcal{O}(1)$ to all orders.

There are fields for that!

Non-relativistic QCD **(NRQCD)** expands the fields in "modes": Momentum components scale like certain powers in $v \ll 1$. [Bodwin, Braaten, Lepage, 1995]

QCD fields are decomposed in the following modes, where momentum p^μ scales like

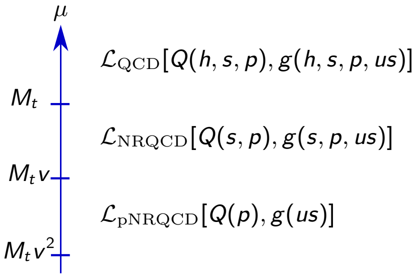
$$\begin{aligned}
\text{hard}(h) &: \quad p^0 \sim M_t \quad , \quad \vec{p} \sim M_t \\
\text{soft}(s) &: \quad p^0 \sim M_t v \quad , \quad \vec{p} \sim M_t v \\
\text{potential}(p) &: \quad p^0 \sim M_t v^2 \quad , \quad \vec{p} \sim M_t v \\
\text{ultrasoft}(us) &: \quad p^0 \sim M_t v^2 \quad , \quad \vec{p} \sim M_t v^2
\end{aligned}$$

When on-shell, only massless fields can be ultrasoft, and only heavy quarks can be potential.

- The virtualities for massless fields are of order
  - hard (h): $p^2 \sim M_t^2$
  - soft (s), potential (p): $p^2 \sim M_t^2 v^2$
  - ultrasoft (us): $p^2 \sim M_t^2 v^4$.
- When integrating out the scales M_t^2 we get NRQCD. When further integrating out $M_t^2 v^2$ "potential NRQCD".

Caveat: On-shell heavy quarks still have virtuality $\sim M_t^2 v^2$. pNRQCD is spatially non-local!

Two-step matching procedure:



Part IV: Top physics – NRQCD

Potential sector: Resummation of Feynman diagrams in threshold region

- Massless (e.g. gluon) propagators:

$$\frac{1}{k^2} = -\frac{1}{\vec{k}^2} + \ldots \quad , \quad \mathcal{O}(v^{-2})$$

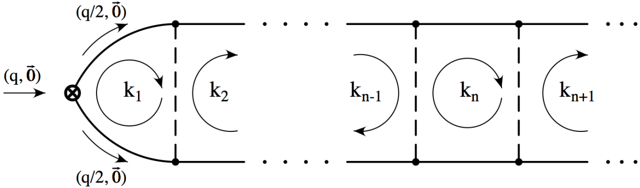
- Heavy-quark propagators: [$q = (\sqrt{s}, 0, 0, 0)$ = photon momentum]

$$\frac{1}{(q/2+k)^2 - M_t^2} = \frac{1}{2M_t}\frac{1}{E/2 + k^0 - \vec{k}^2/(2M_t)} + \ldots \quad , \quad \mathcal{O}(v^{-2})$$

- New rung on the ladder: add two heavy-quark propagators and one massless propagator $\mathcal{O}(v^{-6})$. Integration measure $d^4k \sim \mathcal{O}(v^5)$.

New rung on the ladder proportial to $(\alpha_s/v) \sim 1$, needs resummation.

Part IV: Top physics – NRQCD



The infinite sum builds up a Coulomb potential in which the top quarks act. PNRQCD resembles Quantum-Mechanics perturbation theory.

Problem: Width of the top $E \longrightarrow E + i\Gamma_t$

The correlator function (before taking the Disc.) has an uncancelled IR divergence of the form $\sim \alpha_s E/\epsilon$, which survives the cut as $\sim M_t \alpha_s \alpha_{\rm EW}/\epsilon$.

[M.Beneke, 1501.0737]

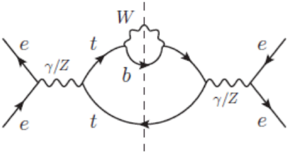


Figure 11: Hard, off-shell top diagram contribution to $e^+e^- \to bW^+\bar{t}$, which leads to a linear infrared divergence.

- **Solution: We have to look at the full physical process!**

$$\sigma_{e^+e^-\to W^+W^-b\bar{b}} = \underbrace{\sigma_{e^+e^-\to[t\bar{t}]_{\mathrm{res}}}}_{\mathrm{NRQCD}} + \sigma_{e^+e^-\to[W^+W^-b\bar{b}]_{\mathrm{nonres}}}$$

- IR divergences cancel between the two parts. **State of the art: NNLO**

Lessons from this lecture

0. There is almost always a hierarchy of scales. [Except for Conformal Field Theories.]
1. Particles that are too heavy to appear on-shell can be integrated out. Pretty much all Field Theories are Effective Field Theories.
   - We have seen this in $R(s) = \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$ explicitely.
   - Another prominent example is the Weak Effective Hamiltonian.
   - What really matters is not the mass, but the "off-shellness" $p^2 - m^2$.
   - Quantum effects from the heavy mass/off-shellness are captured in Wilson coefficients/couplings.

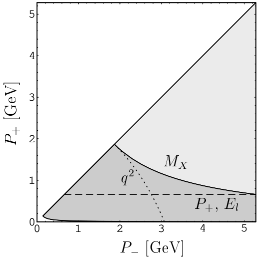
Lessons from this lecture

2. EFTs help us with accuracy.
   - We can separate physics effects from different energy scales: EFT operators reproduce the IR, Wilson coefficients capture UV effects.
   - The Renormalization Group allows us to resum the perturbative expansion, e.g. $\alpha_s \ln M/m$.
   - The Operator Product Expansion gives a systematic framework to improve results in a Power Series of power-counting parameters, e.g. $\lambda = m/M$.

Lessons from this lecture

3. Example in bottom-quark physics: $\Delta \ll \sqrt{\Delta m_b} \ll m_b \ll M_W \ll \Lambda_{\rm NP}$.

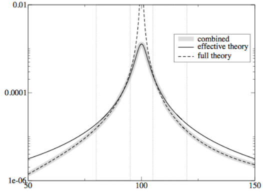
- These are the energy scales relevant to many inclusive B decays, including weak decays $B \to X\ell\bar{\nu}$, FCNC decays $B \to X\gamma$, $B \to X\ell^+\ell^-$.
- Introduced: HQET, SCET
- Using EFTs can reduce the μ-dependence.



Lessons from this lecture

4. Example in top-quark physics: $\Gamma_t \ll \sqrt{\Gamma_t M_t} \ll M_t,$
$M_t v^2 \ll M_t v \ll M_t.$

- The width of an unstable particle is a new (low) scale, which can be captured by EFTs.
- We touched on the line-shape and on top-pair production.
- Introduced: HSETfup, (p)NRQCD



Lessons from this lecture

5. EFTs are a lot of work, but fun!

Thank you for listening.