Topological Strings and Siegel Modular Forms

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Outline

1 Introduction

2 Modularity for mirror curves of genus one

3 Modularity for mirror curves of genus two

4 Conclusion

Topological String Theory

$\mathcal{N} = (2,2)$ nonlinear σ-models on Riemann surfaces Σ_g

- Path integral localises on fixed points $\delta\psi = 0$ of SUSY transformations, BUT: this needs covariantly constant spinors!
- Solution: Modify generator of Lorentz group (Twisting) $\rightarrow$ Some fermions become "scalars"
- On Calabi-Yau 3-folds two different twists lead to

  A-Model - depends on complexified Kähler class of target space

  B-Model - depends on complex structure of target space

- Sum over genera + integration over worldsheet complex structure

  $\rightarrow$ Topological string theory

$$\text{Free energy} \quad F = \sum_g \lambda^{2-2g} F_g$$

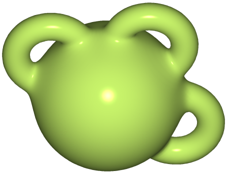
Local Mirror Symmetry

A-Model on toric non-compact CY 3-fold

$\updownarrow$ Mirror Symmetry

B-Model geometry essentially defined by Riemann surface Σ_g

- So far only models on mirror curves of genus one have been solved for $F_g, g \geq 0$
- Modular structure in these cases well known
- We calculated $F_g, g = 0, ..., 3$ for mirror curve of genus two
- Generalised modular to Siegel modular Structure + Found genus two analogue for E_2



B-Model at genus 0 (and 1)

- F_0^B essentially special Geometry on $\mathcal{M}_{c.s.}$
- Choice of complex structure
  $\leftrightarrow$ Integrals of meromorphic 1-form over A-cycles on Σ
- Periods over symplectic basis $\alpha_i, \beta_i \in H_1, \quad i = 1, ..., g$

$$\Pi = (1,\ t_i,\ t_i^D)$$

- $\partial_{t_i} F_0^B = t_i^D$ and flat coordinates t_i identify $\mathcal{M}_{c.s} = \mathcal{M}_{K\ddot{a}hl.}$

$$F_A^X(t) = F_B^Y(t)$$

- Periods are annihilated by "Picard-Fuchs" operators, e.g.

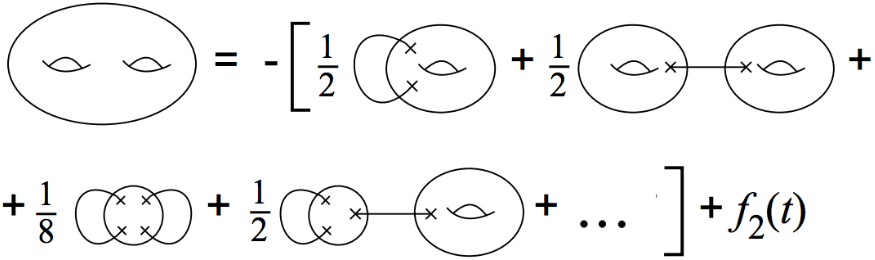
$$\left[\Theta^3 + 3z(3\Theta - 2)(3\Theta - 1)\Theta\right] \Pi_i = 0, \quad \Theta = z\partial_z$$

- Operators can be read off from A-Model Diagram
- F_1^B can be fixed from boundary behaviour

Direct Integration

B-Model Free energies at higher genus can be obtained from
holomorphic anomaly equation

$$\frac{\partial F_g^B}{\partial S^{ij}} = \frac{1}{2}\left(D_i \partial_j F_{g-1}^B + \sum_{0<g'<g} \partial_i F_{g'}^B \partial_j F_{g-g'}^B \right)$$



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$$D_i S^{kl} = -C_{inm} S^{km} S^{ln} + f_i^{kl}, \quad \Gamma_{ij}^k = -C_{ijl} S^{kl} + \tilde{f}_{ij}^k,$$

$$\partial_i F_1 = \frac{1}{2} C_{ijk} S^{jk} + A_i \quad \text{with} \quad C_{ijk} = \partial_i \partial_j \partial_k F_0$$

BCOV [hep-th/9309140], M.-x. Huang, A. Klemm [1009.1126]
B. Haghighat, A. Klemm, M. Rauch [0809.1674]

Riemann surfaces of genus one

- Weierstrass form of defining equation is

$$y^2 = 4x^3 - g_2(z, m_i)x - g_3(z, m_i)$$

- Note: $(g_2, g_3) \sim (r^2 g_2, r^3 g_3)$ for $r \in \mathbb{C}^*$
- On the other hand $\Sigma_1(\tau) = \mathbb{C}/(1\mathbb{Z} + \tau\mathbb{Z})$, $\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0$
- Action of $\mathrm{Sp}(2, \mathbb{Z})/\{\pm 1\}$ on τ leaves Σ_1 invariant

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau + b}{c\tau + d}$$

- Modular forms of weight k on $\mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ are holomorphic "functions" satisfying

$$f(\gamma\tau) = (c\tau + d)^k f(z)$$

Modular forms for $PSp(2,\mathbb{Z})$

- Eisenstein series of weight k=4,6,8,...

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{1}{(c\tau + d)^k}$$

- Can be regularised for k=2 to give form E_2 transforming as

$$E_2(\gamma\tau) = (c\tau + d)^2 E_2(\tau) - \frac{6}{\pi} ic(c\tau + d)$$

- Modular discriminant (with $q = \exp(2\pi i \tau)$)

$$\Delta_{\text{mod}}(\tau) = \frac{1}{1728}(E_4(\tau)^3 - E_6(\tau)^2) = q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

$$\eta = \Delta_{mod}^{1/24}, \qquad E_2 \sim \partial_\tau \log(\eta), \quad E_4 = r^4 g_2, \quad E_6 = r^6 g_3$$

Modularity and Direct Integration

- For genus one mirror curves everything can be expressed in modular forms:

$$\frac{\partial t}{\partial z} = \sqrt{\frac{E_6(\tau) g_2(z, m_i)}{E_4(\tau) g_3(z, m_i)}}$$

$$\frac{\partial^2}{\partial t^2} F_0(t, m_i) = -\frac{C}{2\pi i}\tau(t, m_i), \quad C_{ttt} = -\frac{C}{2\pi i}\frac{\partial \tau}{\partial t}$$

$$\partial_t F_1 - A_t = \frac{1}{2} C_{ttt} S^{tt} = -\partial_t \log(\eta)$$

$$S^{tt} = \frac{c_0}{12} E_2 \sim \partial_\tau \log(\eta)$$

M.-x. Huang, A.-K. Kashani-Poor, A. Klemm [1109.5728]

- How does this work for mirror curves of genus two?

Riemann surfaces of genus two

- Defined by hyperelliptic equation

$$y^2 = v_0 x^5 - v_1 x^4 + v_2 x^3 - v_3 x^2 + v_4 x - v_5$$

- Weierstrass functions g_2, g_3 replaced by Igusa invariants

$$[I_2(v_i) : I_4(v_i) : I_6(v_i) : I_{10}(v_i)] \in \mathbb{P}^{(1,2,3,5)}$$

- τ replaced by 2x2 matrix τ_{ij} in Siegel upper half plane

$$\mathcal{H}_2 = \{\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \in \mathrm{Mat}(2 \times 2, \mathbb{C}) : \tau^t = \tau, \mathrm{Im}(\tau) > 0\}$$

- $\mathrm{PSp}(2, \mathbb{Z})$ replaced by $\mathrm{PSp}(4, \mathbb{Z})$, ie. isometries of symplectic intersections

$$\begin{pmatrix} 0 & \mathbb{I}_{2x2} \\ -\mathbb{I}_{2x2} & 0 \end{pmatrix}$$

Siegel modular forms

- Holomorphic on $\mathcal{H}_2$ and satisfy $f(\gamma\tau) = \det(C + \tau D)^k f(\tau)$
- Siegel Eisenstein series are defined as

$$E_k^{(2)} = \sum_{(C,D)} \det(C\tau + D)^{-k}$$

- Series also diverges for k=2 and cannot be regularised!
- Cusp forms

$$\begin{aligned} \chi_{10} &\sim E_4 E_6 - E_{10} \\ \chi_{12} &\sim 441 E_4^3 + 250 E_6^2 - 691 E_{12} \end{aligned}$$

- $E_4 = r^4 I_4, \quad E_6 = r^6 I_6' = r^6 \frac{1}{2}(I_2 I_4 - 3 I_6), \quad \chi_{10} = r^{10} I_{10}$

G. van der Geer [math/0605346]

Siegel modularity and direct integration

- Note: Intersection matrix of integral set of periods on mirror curve is not symplectic! *S. Hosono [hep-th/0404043]*

$$\text{Instead:} \qquad \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}$$

- We found for B-Model on genus two mirror curve of $\mathbb{C}^3/\mathbb{Z}_5$

$$\det\left(\frac{\partial t_j}{\partial z_i}\right) = \sqrt{\frac{E_6(q_1,q_2,r)I_4(q_1,q_2,r)}{E_4(q_1,q_2,r)I_6'(q_1,q_2,r)z_1^{\frac{6}{5}}z_2^{\frac{8}{5}}}}$$

$$-\tau = CKC \quad \text{with} \quad K_{ij} = \partial_{t_i}\partial_{t_j}F_0$$

- And one can choose A_i so that

$$\partial_i F_1 = -\frac{1}{20}\partial_i \log(\chi_{10}) + A_i, \quad S^{t_i t_j} = \frac{1}{10}\left(C\frac{\partial \log(\chi_{10})}{\partial \tau}C\right)^{ij}$$

Generalised Ramanujan identity

- For genus one mirror curves

$$D_z S^{zz} = -C_{zzz} S^{zz} S^{zz} + f_z^{zz}$$

can be transformed into Ramanujan identity

$$\partial_\tau E_2 = \frac{1}{12}(E_2^2 - E_4)$$

- For genus two mirror curves

$$D_i S^{kl} = -C_{inm} S^{km} S^{ln} + f_i{}^{kl}$$

relates to

$$R_{\mathrm{sym}^2} S = \mathrm{t}(S \otimes S) + f_{\mathrm{R}S}(\tau)$$

with

$$f_i{}^{mn} = -C_{irs} C_k^r C_l^s C_o^m C_p^n (f_{\mathrm{RS}})^{kl,op}$$

Conclusion

- Modular expressions for B-Model quantities generalise to genus two
- Propagator can be chosen as logarithmic derivative of χ_{10}
  $\rightarrow$ Theory of almost meromorphic Siegel modular forms
- Constraint on propagator $\leftrightarrow$ generalised Ramanujan identity
- Modified intersection matrix becomes important
- Checked for $\mathbb{C}^3/\mathbb{Z}_5$ for $g = 0, 1, 2, 3$ and $\mathbb{C}^3/\mathbb{Z}_6$ for $g = 0, 1$

Thank you!