# Heterotic String in Double Field Theory

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ArXiv:1411.3167, Ralph Blumenhagen, RS
ArXiv:1312.0719, Blumenhagen, Fuchs, Haßler, Lüst, RS
ArXiv:1304.2784, Blumenhagen, Deser, Plauschinn, Rennecke, Schmid

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# Double Field Theory Formulation of Heterotic String

## Effective action of Heterotic String

- The low-energy effective action for heterotic string in massless bosonic sector is described by

$$S = \int dx \sqrt{g}\, e^{-2\phi} \left( R + 4(\partial\phi)^2 - \frac{1}{12} H^{ijk} H_{ijk} - \frac{1}{4} G^{ij}{}_{\alpha} G_{ij}{}^{\alpha} \right)$$

  which is extended with $n$ gauge fields $A_i{}^{\alpha}, \alpha = 1, ..., n$.

- The field strength of the non-abelian gauge fields is defined as

$$G_{ij}{}^{\alpha} = \partial_i A_j{}^{\alpha} - \partial_j A_i{}^{\alpha} + g_0 [A_i, A_j]^{\alpha}$$

- The strength of the Kalb-Ramond field is modified by the Chern-Simons three-form,

$$H_{ijk} = 3\left( \partial_{[i} B_{jk]} - \kappa_{\alpha\beta} A_{[i}{}^{\alpha} \partial_j A_{k]}{}^{\beta} - \frac{1}{3} g_0\, \kappa_{\alpha\beta}\, A_{[i}{}^{\alpha} [A_j, A_{k]}]^{\beta} \right)$$

# Heterotic Double Field Theory

- **T-duality** is an important symmetry of string theory, as a consequence, the **non-geometric string background** also called much attention both on theory and phenomenology side. This leads to

**Flux Backgrounds Chain**

$$H_{abc} \to F^a{}_{bc} \to Q_c{}^{ab} \to R^{abc}$$

[Dabholkar, Hull, Shelton, Taylor and Wecht '02-06]

A natural question for heterotic string would be: what is the T-dual of a gauge flux $G_{ij} \to \ldots$?

- The global symmetry group of heterotic Double Field Theory is $O(D, D+n)$, as a generalization of T-duality group of heterotic string. [Hohm and Ki Kwak'13], see also [Siegel, Hull, Zwiebach, Aldazabal, Marques, Nunez, Lust, Andriot, Larfors, Patalong, Blumenhagen, Betz, Berman and Thompson, et al]
- Heterotic DFT lives on $2D+n$ dimensional space, coordinates $X^M = (\tilde{x}_i, x^i, y^\alpha)$, and $X^M$ transform as an $O(D, D+n)$ vector $X'^M = h^M{}_N X^N$, $h \in O(D, D+n)$. The gauge field $A^\alpha$ depends on the gauge coordinate $y^\alpha$.
- The heterotic DFT action is expressed in terms of generalized metric $H_{MN}$ and an $O(D, D+n)$ invariant dilation $d$, defined by $e^{-2d} = \sqrt{g}e^{-2\phi}$.

- The **abelian** bosonic subsector of heterotic action can be expressed under the so-called strong constraint(which annihilates the winding and gauge coordinates dependence) $\tilde{\partial}^i = \partial_\alpha = 0$, by

$$S = \int dx e^{-2d} \left( \frac{1}{8} H^{ij} \partial_i H^{KL} \partial_j H_{KL} - \frac{1}{2} H^{Mi} \partial_i H_{Kj} \partial_j H_{MK} - 2\partial_i d \partial_j H^{ij} + 4H^{ij} \partial_i d \partial_j d \right)$$

- As expected, the generalized metric is parametrized as in terms of the metric $g_{ij}$, the Kalb-Ramond field $B_{ij}$ and the gauge fields $A_i{}^\alpha$ as

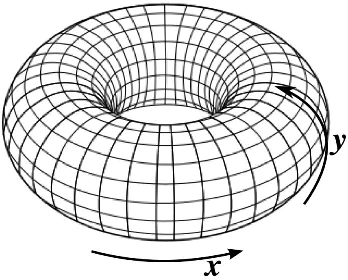
$$H_{MN} = \begin{pmatrix} g^{ij} & -g^{ik} C_{kj} & -g^{ik} A_{k\beta} \\ -g^{jk} C_{ki} & g_{ij} + C_{ki} g^{kl} C_{lj} + A_i{}^\gamma A_{j\gamma} & C_{ki} g^{kl} A_{l\beta} + A_{i\beta} \\ -g^{jk} A_{k\alpha} & C_{kj} g^{kl} A_{l\alpha} + A_{j\alpha} & \delta_{\alpha\beta} + A_{k\alpha} g^{kl} A_{l\beta} \end{pmatrix}$$

in which $C_{ij} = B_{ij} + \frac{1}{2} A_i{}^\alpha A_{j\alpha}$.

# T-duality in Heterotic DFT

## Non-geometric backgrounds of heterotic DFT

- Recall that under a global $O(D, D+n)$ transformation the coordinates and the generalized metric behave as $H^{'} = h^t H h\,, \quad X^{'} = hX\,, \quad \partial^{'} = (h^t)^{-1}\partial\,.$




- Consider a torus $T^2$ with flat metric $g_{ij} = \delta_{ij}$, vanishing Kalb-Ramond $B$-field and a constant abelian gauge flux $G_{ij}$. For the corresponding gauge field $A$, we choose $A_1 = f\,y, A_2 = 0$. This gives the field strength $G_{12} = -(\partial_1 A_2 - \partial_2 A_1) = f\,.$

- Apply T-duality in the $x$-direction, which in heterotic DFT can be implement by conjugation $\mathcal{H}' = \mathcal{T}_1^T \, \mathcal{H} \, \mathcal{T}_1$ with $O(2,3)$ transformation.
- Read off the new metric, B-field and the gauge field directly from the transformed generalized metric $\mathcal{H}'$, we have

$$g' = \begin{pmatrix} \frac{1}{1+(fy)^2+\frac{(fy)^4}{4}} & 0 \\ 0 & 1 \end{pmatrix}, \quad B' = 0, \quad A' = \begin{pmatrix} -\frac{(fy)}{1+\frac{(fy)^2}{2}} \\ 0 \end{pmatrix}.$$

- Similar as for the type II DFT, after two T-dualities there appears a non-trivial functional dependence in the denominators. By performing a proper **field redefinition**, the new non-geometric $J$-flux

$$J^1{}_2 = -\partial_2 \tilde{A}^1 = -f$$

- Applying another T-duality in the $y$ direction changes $y \to \tilde{y}$ in the generalized metric, so like in the $R$-flux background (locally non-geometric) we obtain a non-geometric gauge $\tilde{G}$-flux

$$\tilde{G}^{12} = -(\tilde{\partial}^1 \tilde{A}^2 - \tilde{\partial}^2 \tilde{A}^1) = f$$

# $O(D, D+n)$-induced field redefinition

- By comparing the components of the transformed generalized metric $\mathcal{H}'$ with the original $\mathcal{H}$, we are lead to make the field redefinition as

$$\tilde{g} = g + C^t g^{-1} C + A^2$$
$$\tilde{C} = \tilde{g}^{-1} C^t g^{-1}$$
$$\tilde{A} = -(\tilde{g}^{-1} + \tilde{C}) A$$

- We know the **heterotic Buscher transformations** is the standard rules for how the fields transform under T-duality in string theory. Thus we analogously performed Buscher rules step by step for comparing, we get exact the same results as perform $O(D, D+n)$ transformation on torus in each step.
- Namely, for heterotic DFT we are safe to use the $O(D, D+n)$ transformation to arrive the same results as Buscher rules but in a much simpler way.
- Furthermore, the first order $\alpha'$ correction of Buscher rules is naturally included in the form of the gauge field terms. [Serone and Trapletti'05], [Bedoya, Marques, and Nunez '14]

## A The Buscher rules derived from heterotic DFT

Using the implementation of T-duality in heterotic DFT, one can now quite generally (re-)derive the Buscher from the conjugation of the generalized metric with the corresponding T-duality matrix. Carrying out this procedure for a T-duality in the $x^\theta$ direction, we get precisely the $\alpha'$ corrected Buscher rules presented in [42]

$$\begin{aligned}
G'_{\theta\theta} &= \frac{G_{\theta\theta}}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)^2} \\
G'_{\theta i} &= -\frac{G_{\theta\theta}B_{\theta i} + \frac{\alpha'}{2}G_{\theta i}A_\theta^2 - \frac{\alpha'}{2}G_{\theta\theta}\,A_\theta A_i}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)^2} \\
G'_{ij} &= G_{ij} - \frac{G_{\theta i}G_{\theta j} - B_{\theta i}B_{\theta j}}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)} \\
&\quad - \frac{1}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)^2}\Bigg(G_{\theta\theta}\Big[\frac{\alpha'}{2}B_{\theta j}A_\theta A_i + \frac{\alpha'}{2}B_{\theta i}A_\theta A_j - \frac{\alpha'^2}{4}A_\theta A_i\,A_\theta A_j\Big] \\
&\qquad + \frac{\alpha'}{2}A_\theta^2\Big[(G_{\theta i} - B_{\theta i})(G_{\theta j} - B_{\theta j}) + \frac{\alpha'}{2}(G_{\theta i}A_\theta A_j + G_{\theta j}A_\theta A_i)\Big]\Bigg) \qquad \text{(A.1)} \\
B'_{\theta i} &= -\frac{G_{\theta i} + \frac{\alpha'}{2}A_\theta A_i}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)} \\
B'_{ij} &= B_{ij} - \frac{(G_{\theta i} + \frac{\alpha'}{2}A_\theta A_i)B_{\theta j} - (G_{\theta j} + \frac{\alpha'}{2}A_\theta A_j)B_{\theta i}}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)} \\
A'_\theta{}^\alpha &= -\frac{A_\theta{}^\alpha}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)} \\
A'_i{}^\alpha &= A_i{}^\alpha - A_\theta{}^\alpha\frac{G_{\theta i} - B_{\theta i} + \frac{\alpha'}{2}A_\theta A_i}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)}
\end{aligned}$$

where e.g. $A_\theta A_i = A_\theta^\alpha A_{i\alpha}$. Here the metric and the Kalb-Ramond field have dimension $[l]^0$ and the gauge field $[l]^{-1}$.

- Applying the field redefinitions, we have the generalized metric parametrized by the new metric $\tilde{g}_{ij}$, bi-vector $\tilde{C}^{ij}$ and (one-)vector $\tilde{A}^i$ as

$$\mathcal{H}_{MN} = \begin{pmatrix} \tilde{g}^{ij} + \tilde{C}^{ki}\,\tilde{g}_{kl}\,\tilde{C}^{lj} + \tilde{A}^i{}_\gamma \tilde{A}^{j\gamma} & -\tilde{g}_{jk}\,\tilde{C}^{ki} & \tilde{C}^{ki}\,\tilde{g}_{kl}\tilde{A}^l{}_\beta + \tilde{A}^i{}_\beta \\ -\tilde{g}_{ik}\,\tilde{C}^{kj} & \tilde{g}_{ij} & -\tilde{g}_{ik}\tilde{A}^k{}_\beta \\ \tilde{C}^{kj}\,\tilde{g}_{kl}\tilde{A}^l{}_\alpha + \tilde{A}^j{}_\alpha & -\tilde{g}_{jk}\tilde{A}^k{}_\alpha & \delta_{\alpha\beta} + \tilde{A}^k{}_\alpha\,\tilde{g}_{kl}\tilde{A}^l{}_\beta \end{pmatrix}$$

where $\tilde{C}^{ij} = \beta^{ij} + \frac{1}{2}\tilde{A}^i{}_\alpha \tilde{A}^{j\alpha}$, where $\beta^{ij}$ is the antisymmetric bi-vector.

- The definitions of the heterotic fluxes $\mathcal{F}_{ABC} = E_{CM}\mathcal{L}_{E_A} E_B{}^M$

**Geometric Gauge Fluxes**

$$G_{\alpha ij} = -2D_{[\underline{i}}A_{\underline{j}]\alpha} - D_\alpha B_{ij} + D_\alpha A_{[\underline{i}}{}^\gamma A_{\underline{j}]\gamma}$$

$$J^j{}_{\alpha i} = \tilde{\partial}^j A_{i\alpha}, \qquad K_{\alpha\beta i} = 2\,D_{[\underline{\alpha}}A_{i\underline{\beta}]}$$

**Non-geometric Gauge Fluxes**

$$\tilde{G}_\alpha{}^{ij} = -2\tilde{D}^{[\underline{i}}\tilde{A}^{\underline{j}]}{}_\alpha - D_\alpha \beta^{ij} + D_\alpha \tilde{A}^{[\underline{i}\gamma}\tilde{A}^{\underline{j}]}{}_\gamma$$

$$J^j{}_{\alpha i} = -\partial_i \tilde{A}^j{}_\alpha, \qquad \tilde{K}^{\alpha\beta i} = 2\,D^{[\underline{\alpha}}\tilde{A}^{i\underline{\beta}]}$$

- Thus we can complete the **gauge fluxes chain** under T-dualities

$$G_{\alpha ij} \to J^j{}_{\alpha i} \to \tilde{G}_\alpha{}^{ij}\ ; \qquad K_{\alpha\beta i} \to \tilde{K}^{\alpha\beta i}$$

# Lie algebroid for heterotic field redefinitions

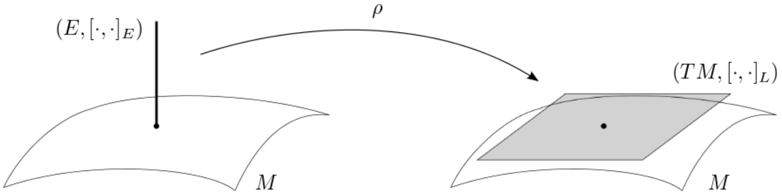



Figure 1: Illustration of a Lie algebroid. On the left, one can see a manifold $M$ together with a bundle $E$ and a bracket $[\cdot,\cdot]_E$. This structure is mapped via the anchor $\rho$ to the tangent bundle $TM$ with Lie bracket $[\cdot,\cdot]_L$, which is shown on the right.

[Blumenhagen, Deser, Plauschinn,Rennecke and Schmid'13]

- A Lie algebroid is specified by three pieces of information:
  a vector bundle $E$ over a manifold $M$,
  a bracket $[\cdot,\cdot]_E : E \times E \to E$,
  a homomorphism $\rho : E \to TM$ called the anchor.
- In **generalized geometry** for abelian sector of heterotic string, we considers a D-dimensional manifold $M$ with usual coordinates $x^i$, equipped with a generalized bundle $E = TM \oplus T^*M \oplus V$. [Hitchin'02; Gualtieri'04]

- On this bundle $E$ one defines a generalized metric $\mathscr{H}_{MN}$ in terms of the fundamental fields $g_{ij}$, $B_{ij}$ and ${A_i}^{\alpha}$ etc.
- An $O(D,D+n)$ transformation $\mathscr{M}$ acts on the generalized metric via conjugation, i.e. $\hat{H}(\hat{g},\hat{B},\hat{A}) = \mathscr{M}^t H(g,B,A)\,\mathscr{M}$, $\mathscr{M}^t\,\mathscr{M} = 1$, and therefore defines a field redefinition $(g,B,A) \longrightarrow (\hat{g},\hat{B},\hat{A})$.
- Can this be connected to DFT field redefinitions? Yes!

  By choosing $\mathscr{M} = \begin{pmatrix} a & b & m \\ c & d & n \\ p & q & z \end{pmatrix} = \begin{pmatrix} 0 & \tilde{g} & 0 \\ \tilde{g}^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

  with $\tilde{g} = g + C^t g^{-1} C + A^2$.
- We reproduced the heterotic DFT field redefinitions,

$$\hat{g} = \rho^t g \rho = \tilde{g}$$
$$\hat{C} = \rho^t \mathfrak{C} \rho = C^t g^{-1} \tilde{g}$$
$$\hat{A} = \rho^t \mathfrak{A} = -(1 + C^t g^{-1})A$$

$$\rho^* = (\rho^t)^{-1} = -(g + C)\,\tilde{g}^{-1}$$
$$\delta = -\tilde{g}$$
$$\mathfrak{A} = A$$

# The redefined heterotic action

- Recall that the NS-sector of the heterotic string action is

$$\mathscr{S} = \int dx \sqrt{g} e^{-2\phi} \left( R + 4(\partial\phi)^2 - \frac{1}{12} H^{ijk} H_{ijk} - \frac{1}{4} G^{ij\alpha} G_{ij\alpha} \right)$$

with $H = dB - \frac{1}{2}\delta_{\alpha\beta} A^\alpha \wedge dA^\beta$ and $G^\alpha = dA^\alpha$.

- Gravitational quantities transform as

$$\hat{R}^q{}_{mnp} = (\rho^{-1})^q{}_l \rho^i{}_m \rho^j{}_n \rho^k{}_p R^l{}_{ijk}\,, \qquad \hat{R}_{mn} = \rho^i{}_m \rho^j{}_n R_{ij}\,,$$
$$\hat{R} = R\,, \qquad \sqrt{|\hat{g}|} = \sqrt{|g|}|\rho^t|\,, \qquad \hat{\phi} = \phi\,, \qquad D_i = (\rho^t)_i^j \partial_j$$

[Blumenhagen, Deser, Plauschinn,Rennecke and Schmid'13]

- For the gauge field strength $G = dA$, we have $(\Lambda^2 \rho^*) d_E \hat{A} = d(\rho^* \hat{A}) = dA$

**Field Strength transforms as**

$$\hat{G} := d_E \hat{A} = (\Lambda^2 \rho^t) G$$

**Three-form Flux transforms as**

$$\hat{H} := d_E \hat{B} - \frac{1}{2} \hat{A} \wedge d_E \hat{A} = (\Lambda^3 \rho^t) H$$

- so that the action in the redefined fields can be expressed as

$$\mathscr{S} = \int dx \sqrt{\hat{g}}\, |\rho^*|\, e^{-2\phi} \left( \hat{R} + 4(D\phi)^2 - \frac{1}{12} \hat{H}^{ijk} \hat{H}_{ijk} - \frac{1}{4} \hat{G}^{ij\alpha} \hat{G}_{ij\alpha} \right)$$

# Conclusion

- By applying $O(D, D+n)$ transformation in heterotic DFT, we obtain the results from **Heterotic Buscher rules** for heterotic string theory. This also naturally includes the first order $\alpha'$ correction.
- As we arrive in non-geometric frame after the T-duality, we obtain the **Gauge vector** transformed from the gauge field. Then we also complete the field redefinition including the gauge field.
- From the heterotic DFT flux definition, we complete the **gauge fluxes chain** under T-dualities, which could be interesting input in string phenomenology or model building.
- As a parallel section, we construct an $O(D, D+n)$ kind **anchor** in **generalized geometry** and reproduce the field redefinition we made for heterotic DFT. By choosing different anchors, one can arrive **a sequence of equivalent actions** for heterotic supergravity.


Thank you

# EXTRA

- The $O(2,3)$ transformation on $T^2$ torus for heterotic DFT

$$\mathscr{T}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The upper $4 \times 4$ dimensional part of the metric is the same as the T-duality transformation for type II DFT.

# The fluxes of heterotic DFT

- We derive the general form of the components of the heterotic fluxes by $\mathscr{F}_{ABC} = E_{CM}\mathscr{L}_{E_A}E_B{}^M = \Omega_{ABC} + \Omega_{CAB} - \Omega_{BAC}$ .
- In order to treat geometric and non-geometric components at the same time, we use the general extended form of the generalized vielbein $E^A{}_M = \begin{pmatrix} e_a{}^i & -e_a{}^k C_{ki} & -e_a{}^k A_{k\beta} \\ -e^a{}_k \tilde{C}^{ki} & e^a{}_i + e^a{}_k \tilde{C}^{kj} C_{ji} & -e^a{}_k \tilde{A}^k{}_\beta \\ \tilde{A}^{i\alpha} & A_i{}^\alpha & \delta^\alpha{}_\beta \end{pmatrix}$

- In abelian heterotic generalized geometry, we considers a D-dimensional manifold $M$ with usual coordinates $x^i$, equipped with a generalized bundle $E = TM \oplus T^*M \oplus V$, whose sections are formal sums $\xi + \tilde{\xi} + \lambda$ of vectors, $\xi = \xi^i(x)\,\partial_i$, one-forms, $\tilde{\xi} = \tilde{\xi}_i(x)\,dx^i$ and gauge transformations, $\lambda = (\lambda_1(x), \ldots \lambda_n(x))$.
- For each non-geometric local $O(D, D+n)$ transformation this action is based on the differential geometry of a corresponding Lie algebroid.
- The anchor property and the corresponding formula for the **de Rahm differential** allow to compute
$\Big(\big(\Lambda^{n+1}\rho^*\big)(d_E\,\theta^*)\Big)(X_0,\ldots,X_n) = \big(d_E\,\theta^*\big)\big(\rho^{-1}(X_0),\ldots,\rho^{-1}(X_n)\big) =$
$d\big((\Lambda^n\rho^*)(\theta^*)\big)(X_0,\ldots,X_n)$
with the dual anchor $\rho^* = (\rho^t)^{-1}$ and for sections $X_i \in \Gamma(TM)$. The relation describes how exact terms translate in general.
- Moreover, any Lie algebroid can be equipped with a nilpotent exterior derivative as follows
$$d_E\,\theta^*(s_0,\ldots,s_n) = \sum_{i=0}^n (-1)^i \rho(s_i)\theta^*(s_0,\ldots,\hat{s}_i,\ldots,s_n) + \sum_{i<j}(-1)^{i+j}\theta^*([s_i,s_j]_E, s_0,\ldots,\hat{s}_i,\ldots,\hat{s}_j,\ldots,s_n)\,,$$
where $\theta^* \in \Gamma(\Lambda^n E^*)$ is the analog of an $n$-form on the Lie algebroid and $\hat{s}_i$ denotes the omission of that entry. The Jacobi identity of the bracket $[\cdot,\cdot]_E$ implies that satisfies $(d_E)^2 = 0$.