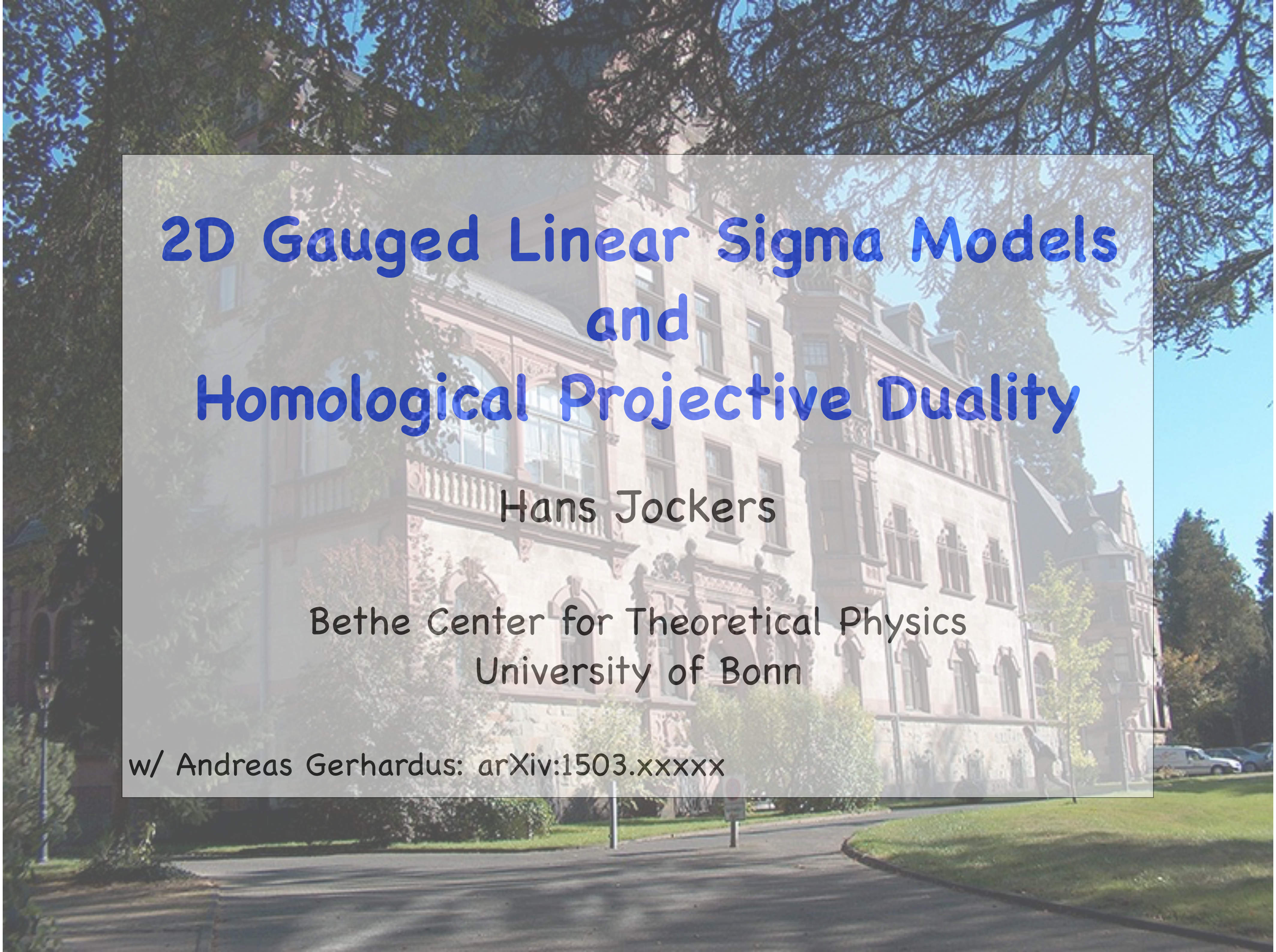

2D Gauged Linear Sigma Models and Homological Projective Duality

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Introduction

- Landscape of 4d type II string theories
  - **Miles Reid '87:**``The moduli space of 3-folds with K=0 may nevertheless be irreducible."
  - Unification of N=2 4d type II string theories

- New powerful tools
  - 2d gauge theory dualities for worldsheet theories
  - Localization techniques for quantum moduli spaces

- Unexpected topological brane equivalences
  - Discoveries of derived equivalences in algebraic geometry
  - Novel identification of CY 3-fold moduli spaces

2D Gauge Theories

- Spectrum of 2d N=(2,2) gauged linear sigma models
  - Gauge group G = $U(1)^k \times G_1 \times \ldots \times G_n$ & **vector multiplets** V_G
  - (Charged) matter **chiral multiplets** Φ
- Gauge linear sigma model Lagrangian

$$\mathcal{L}_{\mathrm{GLSM}}(V_G, \Phi) = \int d^4\theta \, \underbrace{\mathrm{tr}\, \overline{\Sigma}_G \Sigma_G}^{F_{\mu\nu}F^{\mu\nu}} + \int d^4\theta \, \overbrace{\overline{\Phi} e^{2\rho(V_G)} \Phi}^{\mathrm{tr}\left(\nabla_\mu \phi \cdot \nabla^\mu \bar{\phi}\right)}$$

$$+ \left(\int \underbrace{d^2\theta \, W_z(\Phi)}_{\text{superpotential}} + \int \underbrace{d^2\tilde{\theta} \, t \, \Sigma_{U(1)}}_{\substack{\text{complex} \\ \text{FI parameters } t}} + \text{c.c.} \right)$$

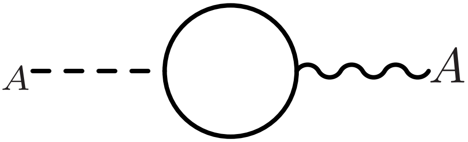
- Moduli dependence
  - **F-term moduli:** Parameters z in the superpotential W_z
  - **D-term moduli:** complexified FI parameters t

Renormalization Group Flow

- **RG flow:** UV 2d gauge theory to IR 2d N=(2,2) CFT

$$\Psi_{\mathrm{RG}} : \ \{\mathrm{GLSM}(t,z)\} \xrightarrow{\text{RG flow}} \{\mathrm{SCFT}(t,z)\}$$

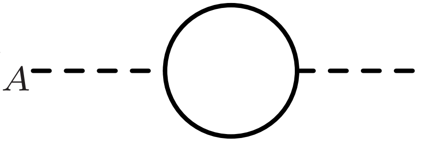
- U(1)$_A$ & U(1)$_V$ R-symmetry: U(1) current of IR N=2 SCA

$0 \sim j_A$  $A_G \sim \mathrm{Tr}_\rho A_G \quad \Rightarrow \quad \rho : V_\Phi \times G \to \mathrm{SL}(V_\Phi)$

axial anomaly

condition for axial anomaly cancelation

- Central charge c of IR N=2 SCA

$c/3 = \lim_{z\to 0} z\,\langle J(z)J(0)\rangle \sim j_A$  j_V

central charge

1-loop axial-vector diagram

N=2 4D Type IIA Strings

- N=2 4d type IIA worldsheet theory:

$$\underset{\text{4d space-time}}{\mathbb{R}^{3,1}} \times \underset{\text{“internal degrees of freedom”}}{\mathrm{SCFT}_{c=9}}$$

- N=2 4d type IIA moduli spaces:
  - **Moduli** of 2d N=(2,2) SCFT

$$\mathrm{SCFT}_{c=9}(t,z) \;\mapsto\; (t,z) \in \overset{\substack{\text{D-term}\\ \text{moduli}}}{\underset{\substack{\text{quantum Kähler}\\ \text{moduli space}}}{\boxed{\mathcal{M}_{qK}}}} \times \overset{\substack{\text{F-term}\\ \text{moduli}}}{\underset{\substack{\text{complex structure}\\ \text{moduli space}}}{\mathcal{M}_{cs}}}$$

  - **Marginal deformations** & analytic continuation

$$\mathrm{SCFT}_{c=9}(t,z) \;\mapsto\; \underset{\substack{\text{local special}\\ \text{Kähler manifold}}}{\mathcal{M}_{qK}}$$

Semi-Classical 2D GLSM

- RG flow to non-linear sigma model with CY_3 target space X
  - **Critical locus** of scalar potential $U_{t,z}(\Phi)$ as IR target space X

$$U_{t,z}(\phi) = |\text{F-term}|_z^2 + |\text{D-term}|_t^2$$

  - **Geometry:** Symplectic reduction (GIT quotient)

$$X \;=\; \underbrace{\{\operatorname{grad} W_z = 0\}}_{\text{complex structure moduli}} \subset \underbrace{V_\Phi /\!/_t \, G_{\mathbb{C}}}_{\text{Kähler moduli}}$$

- Semi-classical geometric reduction:
  - **Classical** non-linear sigma model phases (Witten,...):

$$\inf_{\phi \in X} \operatorname{rk} \operatorname{Hess} U_{t,z}(\phi) = \operatorname{codim} X$$

  - **Quantum** non-linear sigma model phases (Hori-Tong,...): Quantum IR phase of GLSM equiv. to a geom. CY_3 target space

B-Branes & CY Moduli Spaces

- N=2 4d type IIA strings: BPS particles/stable branes

$$(t,z) \in \mathcal{M}_{\mathrm{qK}} \times \mathcal{M}_{\mathrm{cs}} \mapsto \{\text{BPS branes}\}_{(t,z)\in\mathcal{M}_{\mathrm{qK}}\times\mathcal{M}_{\mathrm{cs}}} \subset \{\text{B branes}\}_{z\in\mathcal{M}_{\mathrm{cs}}}$$

D-branes with
notation of stability

"Holomorphic
branes"

- 2d twisted conformal field theory (TCFT):
  - Topological twist

$$\{\mathrm{SCFT}_{c=9}\}_{(t,z)\in\mathcal{M}_{\mathrm{qK}}\times\mathcal{M}_{\mathrm{cs}}} \xrightarrow{\text{twist}} \{\text{2D TCFT}\}_{z\in\mathcal{M}_{\mathrm{cs}}}$$

  - Topological B-branes — **no notion of stability**

$$\text{Func}: \ \mathrm{TCFT}_{z\in\mathcal{M}_{\mathrm{cs}}} \mapsto \{\text{B branes}\}_{z\in\mathcal{M}_{\mathrm{cs}}} \mapsto \mathcal{M}_{\mathrm{qK}}$$

category of
B-branes

Homological Projective Duality

- **Homological Projective Duality** (Kuznetsov)
  (over-simplified) working definition for our examples:
  - Semi-classical non-linear sigma models (geometric phases)

$$\mathrm{Set}(\mathrm{CY}_3) \ \subset \ \mathrm{Set}(\mathrm{TCFT})$$

  - Homological projective dual CYs: **Equivalent cat. of B-branes**

$$\mathrm{CY}_3\,X \ \sim_{\mathrm{HPD}} \ \mathrm{CY}_3\,Y \quad \Rightarrow \quad \mathrm{Cat}(\text{B-branes})_X \ \simeq \ \mathrm{Cat}(\text{B-branes})_Y$$

derived equivalence

- Implications:
  - B-brane moduli space of (stable) large volume D0-branes

$$\mathrm{D0}_X, \mathrm{D0}_Y \in \mathrm{Cat}(\text{B-branes}) \ \Rightarrow \ \mathcal{M}(\mathrm{D0}_X) = \mathrm{CY}_3\,X\,, \ \mathcal{M}(\mathrm{D0}_Y) = \mathrm{CY}_3\,Y$$

D0 brane moduli spaces

  - Quantum Kähler moduli spaces & large volume points

$$\mathrm{Func}(\mathrm{CY}_3\,X) \ = \ \mathrm{Func}(\mathrm{CY}_3\,Y)$$

large volume point for CY_3 X

large volume point for CY_3 Y

Example: Flop Transitions

- Homological proj. duality: **Birational CYs** (Kawamata)
  - Flop transition

$$X_1 \xleftrightarrow{\text{flop}} X_2 \;\Rightarrow\; X_1 \sim_{\text{HPD}} X_2 \Rightarrow \text{Cat(B-branes)}_{X_1} = \text{Cat(B-branes)}_{X_2}$$

  - Spectrum of D0 branes

$$\text{D0}_{X_1}, \text{D0}_{X_2} \in \text{Cat(B-branes)}$$

- Phases of **2d abelian GLSM** — toric geometry:
  - **Each** birational CY target phase X_k:
    Classical non-linear sigma model phase X_k
  - **Large volume points** in quantum Kähler moduli space:

$$p(X_k) \in \mathcal{M}_{\text{qK}}$$

large volume point
for each CY X_k

2D GLSM Dualities

- IR duality of 2d non-Abelian GLSMs A & B:

$$\Psi_{\mathrm{RG}}(A) \;=\; \Psi_{\mathrm{RG}}(B)$$

  - **2d Seiberg duals** (Hori; Kumar-Lapan-Morrison-Romo-HJ; ...):

    $SU(k) \longleftrightarrow SU(N_F-k)\,,\ USp(2k) \longleftrightarrow USp(2N_F-2k-2)\,,\ \ldots$

  - **Features:** Equivalent B-brane spectra
    Often exchange of classical & quantum GLSM phases

- **Proposal:**

  2d GLSM dualities, exchanging classical & quantum non-linear sigma model phases, yield CY target spaces that are (non-trivial) **homological projective duals**.

  - First example (Borisov-Căldăraru, Hori-Tong, Kuznetsov):
    Quantum Kähler phases of Rødland CY 3-fold

Skew Symmetric Sigma Models

c.f. talk by Andreas Gerhardus

- 1^{st} Skew Symmetric Sigma Model: **SSSM(X)**
  - Gauge group **USp(2)xU(1)** & charged matter multiplets
  - FI param. $\mathrm{Re}(t_X) \gg 0$: **Classical** non-linear sigma model phase X

$$X \;=\; \left\{ [\phi,\lambda] \in \mathbb{P}(\mathbb{C}^6, \Lambda^2\mathbb{C}^6) \middle| \operatorname{rk}\lambda \leq 2, \phi \in \ker\lambda \right\} \cap \mathbb{P}^5$$

  - FI param. $\mathrm{Re}(t_X) \ll 0$: **Quantum phase**

- 2^{nd} Skew Symmetric Sigma Model: **SSSM(Y)**
  - Gauge group **USp(4)xU(1)** & charged matter multiplets
  - FI param. $\mathrm{Re}(t_Y) \gg 0$: **Classical** non-linear sigma model phase Y

$$Y \;=\; \left\{ [\phi,\lambda] \in \mathbb{P}(\mathbb{C}^6, \Lambda^2\mathbb{C}^6) \middle| \operatorname{rk}\lambda \leq 4, \phi \in \ker\lambda \right\} \cap \mathbb{P}^{11}$$

  - FI param. $\mathrm{Re}(t_Y) \ll 0$: **Quantum phase**

Proposal

Claim:

(1) SSSM(Y) & SSSM(X) are **IR dual** families of GLSMs.

(2) The projective varieties **X and Y** are a (non-trivial) pair of **homological projective dual CY 3-folds**.

(3) Phases $\mathrm{Re}(t_X) \ll 0$ & $\mathrm{Re}(t_Y) \ll 0$ are **quantum non-linear sigma model phases** of SSSM(X) and SSSM(Y) with target space Y and X, respectively.

Remark: CY_3 X and CY_3 Y are **not birational**!

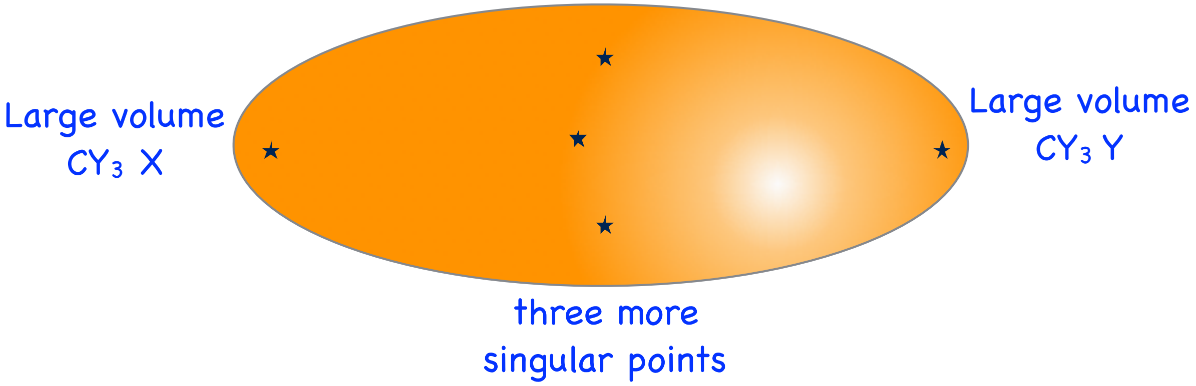
(b/c the cube root of the ratio deg(X)/deg(Y) is irrational)

GLSM Evidence

- `t Hooft anomaly matching & Witten index:

target space	proj. variety X	proj. variety Y
Central charge	$\frac{c}{3} = \dim X = 3$	$\frac{c}{3} = \dim Y = 3$
Axial anomaly	$\nabla j_A \sim c_1(X) = 0$	$\nabla j_A \sim c_1(Y) = 0$
Witten index	$(-1)^F = \chi(X) = -102$	$(-1)^F = \chi(Y) = -102$

- Identical singularities in quantum Kähler moduli space
  (c.f., Hofmann (Ph.D.thesis): Classification of PF operators;
  Miura: CY_3 X as miniscule Schubert variety)



Partition Function Evidence

- **Idea:** Match partition functions!
  - for 4D N=1 Seiberg dual theories (Römelsberger, Dolan-Osborn)
  - 2d N=(2,2) theories based on S^2 partition function (Benini-Cremonesi, Doroud-Gomis-LeFloch-Lee)
  - for 2d N=(2,2) GLSM duality of Rødland quantum Kähler phases (Lapan-Kumar-Morrison-HJ)

- SSSM(X) & SSSM(Y): see talk by Andreas Gerhardus

$$Z_{S^2}(\mathrm{SSSM}(X)) = Z_{S^2}(\mathrm{SSSM}(Y))$$

  - **Identification:** Non-trivial number-theoretic identities from higher dimensional Barnes-type integrals

Conclusions & Outlook

- **Proposal:** Dual pair of skew symmetric sigma models
  - **Physics proof** of a rare instance of homological projective duality for a (non-trivial) pair of CY 3-folds
  - Surprising identifications in CY 3-fold moduli spaces
- 2d Seiberg-like dualities & Homological Proj. Duality
  - **IR dual 2d gauge theories** yield derived **equivalences** for categories of **topological B-brane**
  - Implications for D0 brane moduli spaces (SYZ conjecture,...)
- Quantum Kähler moduli spaces & modularity
  - Properties of moduli spaces with several (non-birational) large volume limits (higher genus invariants,...)
  - 2d gauge theory dualities & mirror symmetry