SL(2,C) Chern-Simons Theory, Quantum Curve,

and

4d Quantum Geometry

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Outline of Main Results:

- **Classical Correspondence:**

  Flat connections on M_3 = Simplicial geometries on M_4

- **Quantum Correspondence:**

  CS wave function on M_3 = Quantum geometries on M_4

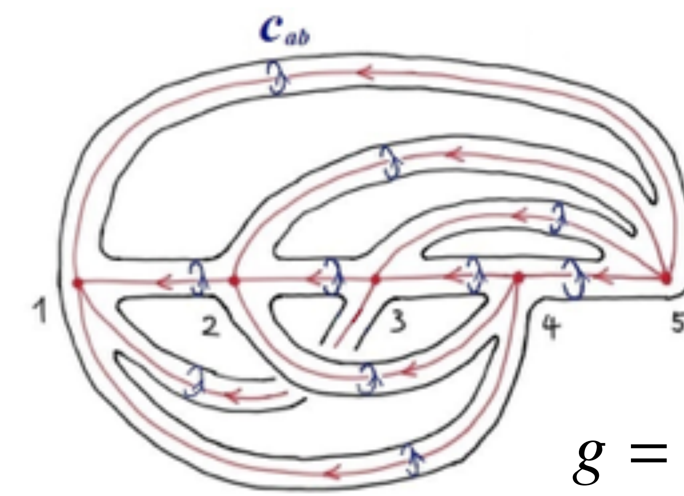
  relates to quantum simplicial gravity on M_4

- **3d SCFT Correspondence:**

  3d SCFT $\longrightarrow$ Quantum simplicial gravity on M_4

Moduli space of flat connections on 2-surface

Closed 2-surface Σ_g **e.g.**



$g = 6$

Phase space: $\mathcal{M} = \mathcal{M}_{flat}(\Sigma_g, \mathrm{SL}(2,\mathbb{C}))$ (A s.t. F_A=0)

- **Hyper-Kahler: complex structures** i (from 2-surface) j (from complex group) $k=ij$

- **Holomorphic Symplectic structure:** $\omega \sim \int_{\Sigma_g} \mathrm{tr}\,[\delta A \wedge \delta A]$ **(Atiyah-Bott-Goldman)**

Complex Fenchel-Nielsen coordinates $(x_c, y_c) \in (\mathbb{C}^\times)^2$

Logarithmic coordinates are symplectic $\omega = \sum_c \mathrm{d}\ln y_c \wedge \mathrm{d}\ln x_c$

Holomorphic Lagrangian submanifold

Dimofte 2011
Gukov, Saberi 2012
Dimofte, Gaiotto, van der Veen 2013

Let's consider a 3-manifold M_3 s.t. $\partial M_3 = \Sigma_g$

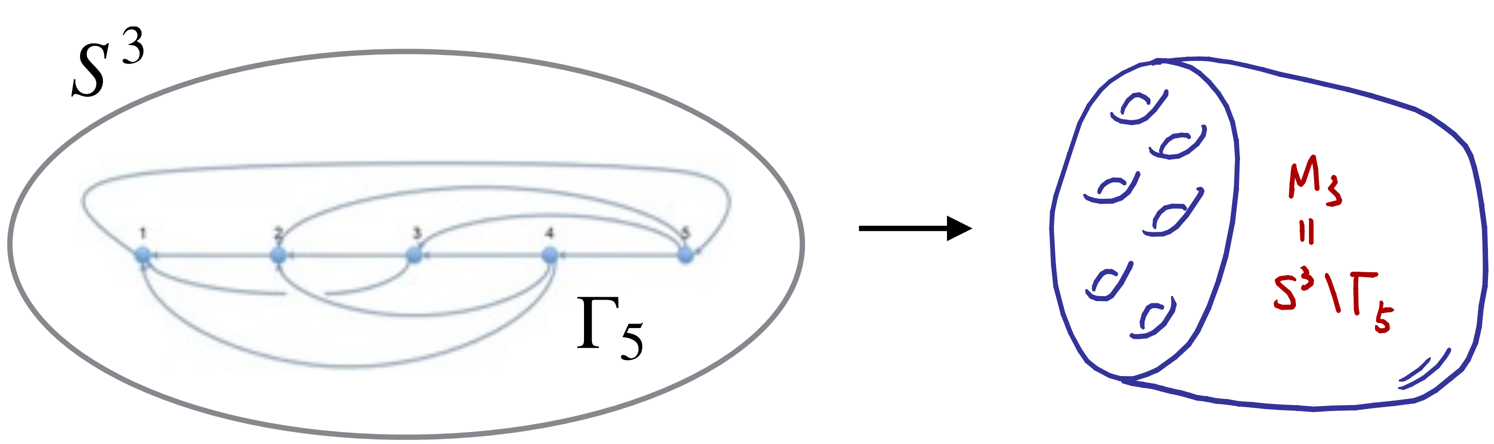
$$\mathcal{M}_{flat}(M_3, \mathrm{SL}(2,\mathbb{C})) \simeq \mathcal{L}_A \hookrightarrow \mathcal{M}_{flat}(\Sigma_g, \mathrm{SL}(2,\mathbb{C}))$$

Holomorphic polynomial eqns $\mathbf{A}_m(x_c, y_c) = 0, \quad m = 1, \cdots, 3g-3$

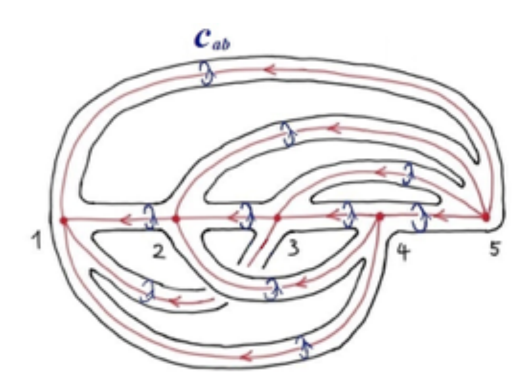
e.g. Knot complement (g=1): A-polynomial

We focus on *graph complement 3-manifold* in 3-sphere: removing a tubular open neighborhood of a graph embedded in 3-sphere.

$$M_3 = S^3 \setminus N(\Gamma) \equiv S^3 \setminus \Gamma$$



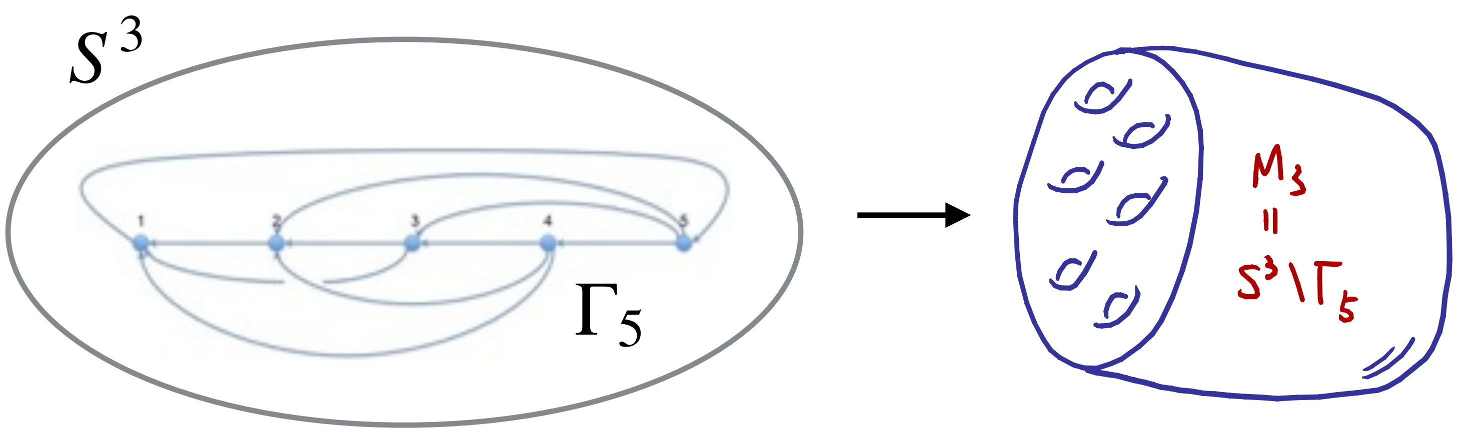
$$\partial M_3 =$$



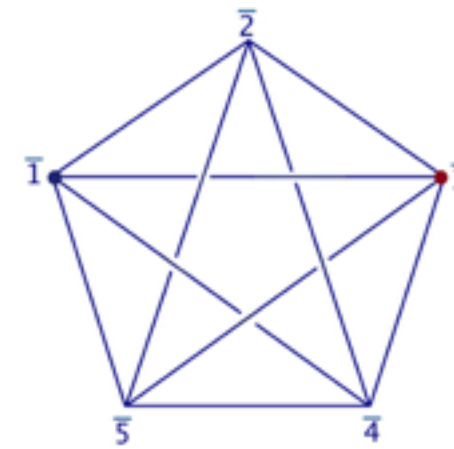
$\Sigma_{g=6}$

$$\mathcal{M}_{flat}(S^3 \setminus \Gamma_5, \mathrm{SL}(2,\mathbb{C})) \simeq \mathcal{L}_A \hookrightarrow \mathcal{M}_{flat}(\Sigma_{g=6}, \mathrm{SL}(2,\mathbb{C}))$$

Flat Connections in 3d *v.s.* Simplicial Geometry in 4d



Γ_5 graph complement



4-simplex

A class of $SL(2,\mathbb{C})$ flat connetion on $S^3 \setminus \Gamma_5$	=	Lorentzian 4-simplex geometries with constant curvature Λ

Haggard, MH, Kamiński, Riello 2014

A subset of branches in Lagrangian submanifold

Flat Connections in d-1 *v.s.* Discrete Geometry in d

$\pi_1(M_{d-1})$

Fundamental group of a $(d-1)$-manifold M_{d-1} (defected S^{d-1})

$\overset{S}{\simeq}$

$\pi_1(\mathrm{sk}(M_d))$

Fundamental group of the 1-skeleton of a d-dim polyhedron M_d

ω_{flat}

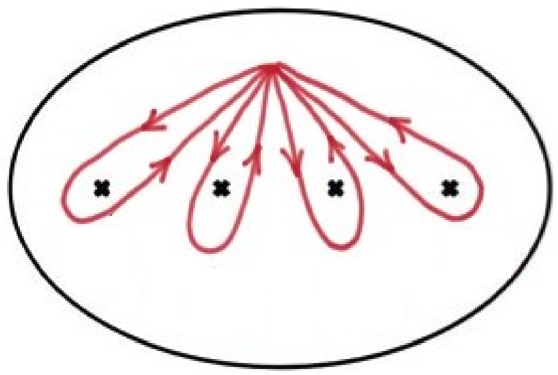
ω_{spin}

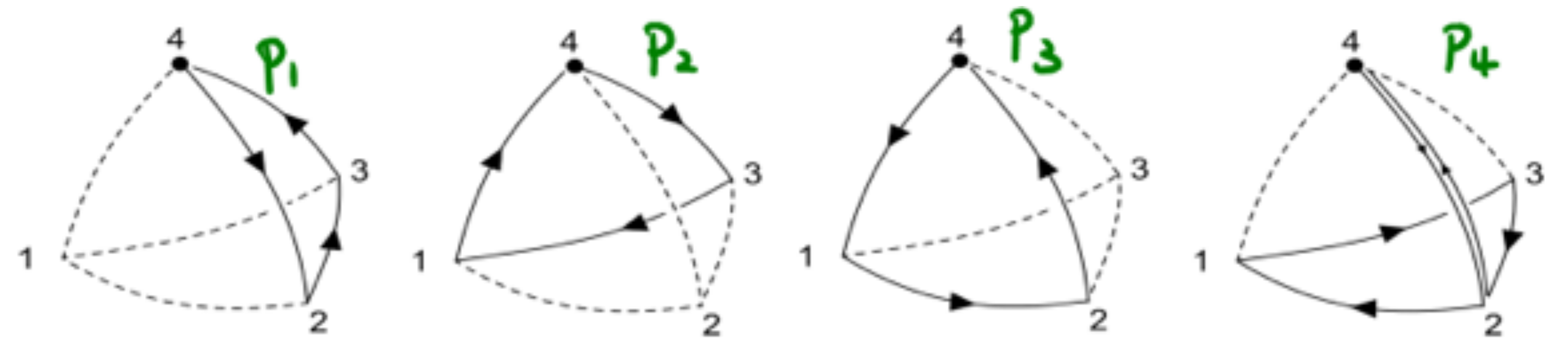
$\mathrm{SO}(d)$ or $\mathrm{SO}(d-1,1)$

modulo gauge

$$\omega_{spin} = \omega_{flat} \circ S$$

4-holed sphere *v.s.* tetrahedron



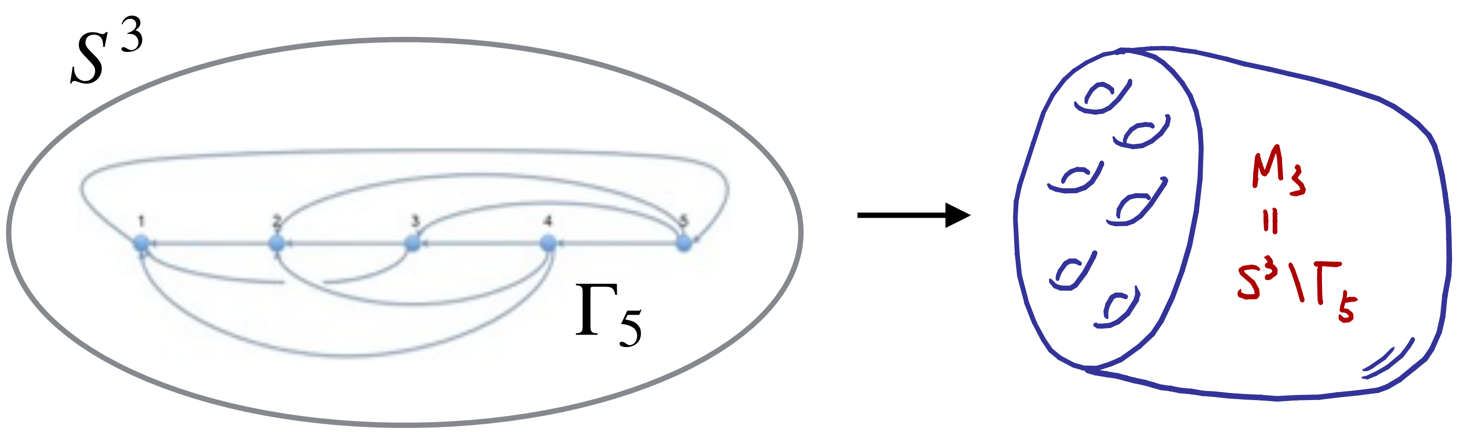


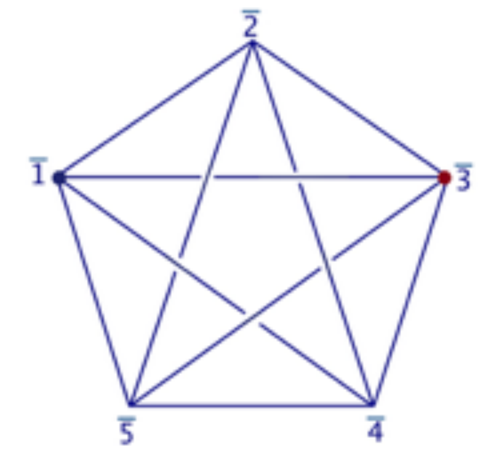
$$\pi_1(\text{4-holed sphere}) = \left\langle \mathrm{l}_1, \cdots, \mathrm{l}_4 \middle| \mathrm{l}_4\mathrm{l}_3\mathrm{l}_2\mathrm{l}_1 = \mathrm{e} \right\rangle \quad \overset{S}{\simeq} \quad \pi_1(\mathrm{sk(Tetra)}) = \left\langle p_1, \cdots, p_4 \middle| p_4p_3p_2p_1 = e \right\rangle$$

ω_{flat} $\searrow$ $\swarrow$ ω_{spin}

$$\left\langle H_1, \cdots, H_4 \in \mathrm{SO}(3) \middle| H_4H_3H_2H_1 = 1 \right\rangle / \text{conjugation}$$

Γ_5 graph complement *v.s.* 4-simplex





$$
\begin{aligned}
\text{vertex 1}: &\quad \mathfrak{l}_{14}\mathfrak{l}_{13}^{(1)}\mathfrak{l}_{12}\mathfrak{l}_{15} = 1,\\
\text{vertex 2}: &\quad \mathfrak{l}_{12}^{-1}\mathfrak{l}_{24}\mathfrak{l}_{23}\mathfrak{l}_{25} = 1,\\
\text{vertex 3}: &\quad \mathfrak{l}_{23}^{-1}(\mathfrak{l}_{13}^{(2)})^{-1}\mathfrak{l}_{34}\mathfrak{l}_{35} = 1,\\
\text{vertex 4}: &\quad \mathfrak{l}_{34}^{-1}\mathfrak{l}_{24}^{-1}\mathfrak{l}_{14}^{-1}\mathfrak{l}_{45} = 1,\\
\text{vertex 5}: &\quad \mathfrak{l}_{25}^{-1}\mathfrak{l}_{35}^{-1}\mathfrak{l}_{45}^{-1}\mathfrak{l}_{15}^{-1} = 1,\\
\text{crossing}: &\quad \mathfrak{l}_{13}^{(1)} = \mathfrak{l}_{24}\mathfrak{l}_{13}^{(2)}\mathfrak{l}_{24}^{-1}.
\end{aligned}
$$

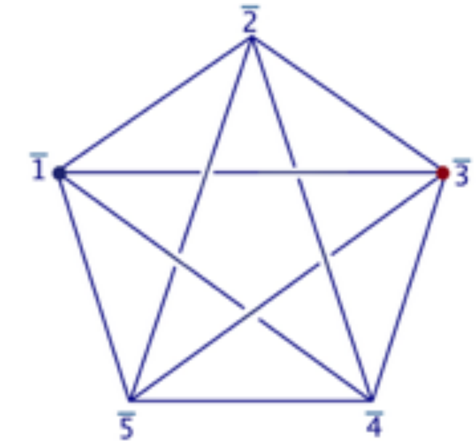
$$\overset{S}{\simeq}$$

$$
\begin{aligned}
\text{tetra 1}: &\quad p_{14}p_{13}^{(1)}p_{12}p_{15} = 1,\\
\text{tetra 2}: &\quad p_{12}^{-1}p_{24}p_{23}p_{25} = 1,\\
\text{tetra 3}: &\quad p_{23}^{-1}(p_{13}^{(2)})^{-1}p_{34}p_{35} = 1,\\
\text{tetra 4}: &\quad p_{34}^{-1}p_{24}^{-1}p_{14}^{-1}p_{45} = 1,\\
\text{tetra 5}: &\quad p_{25}^{-1}p_{35}^{-1}p_{45}^{-1}p_{15}^{-1} = 1,\\
\text{“crossing”}: &\quad p_{13}^{(1)} = p_{24}p_{13}^{(2)}p_{24}^{-1}.
\end{aligned}
$$

ω_{flat} $\qquad$ ω_{spin}

$$\left\langle H_{ab} \in \mathrm{SO}(3,1) \,\middle|\, \quad \cdots \quad \right\rangle / \text{conjugation}$$

$$\omega_{spin} = \omega_{flat} \circ S$$



are a set of holonomies along closed paths on 1-skeleton

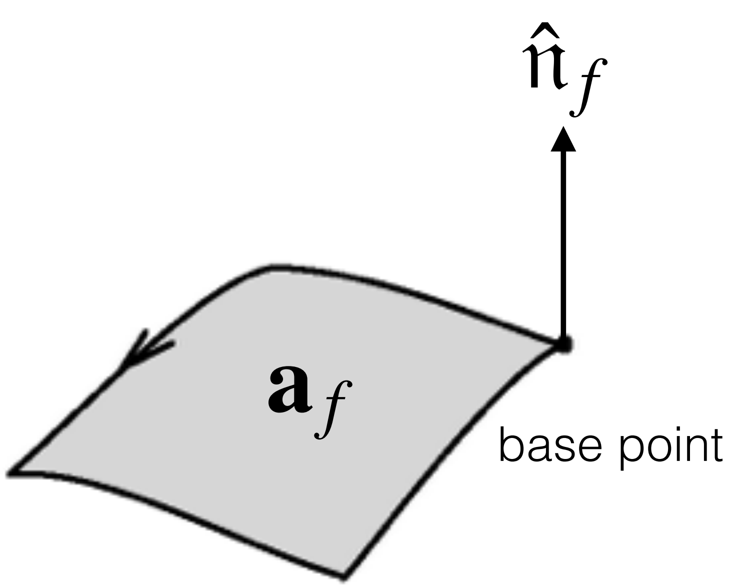
For constant curvature simplex, whose 2-faces are flatly embedded surfaces:

<u>Lemma</u>: Given 2-surface flatly embedded (K=0) in constant curvature space, the holonomy of spin connection along the boundary of surface:

$$h_{\partial f}(\omega_{spin}) = \exp\left[-i\frac{\Lambda}{6}\mathbf{a}_f \hat{\mathrm{n}}_f \cdot \vec{\sigma}\right]$$

in 3d space

replaced by normal bivector in 4d spacetime



Area and normal data determine the simplex geometry

Quantum Theory

Flat connection on $S^3 \setminus \Gamma_5$ **= 4-simplex geometry**

Quantum flat connection on $S^3 \setminus \Gamma_5$ **= Quantum 4-simplex geometry**

Quantization of 4d geometry

↓

Quantization of flat connection on 3-manifold

Quantization of flat connections on 3-manifold

$$\mathcal{M} = \mathcal{M}_{flat}(\Sigma_g, \mathrm{SL}(2,\mathbb{C})) \qquad \omega = \sum_c \mathrm{d}\ln y_c \wedge \mathrm{d}\ln x_c$$

Holomorphic symplectic coordinates $u_c = \ln x_c, \quad v_c = \ln y_c$

$$\hat{u}_c f(u) = u_c f(u), \quad \hat{v}_c f(u) = -i\hbar\partial_{u_c} f(u)$$

Quantization $\mathcal{M}_{flat}(M_3, \mathrm{SL}(2,\mathbb{C})) \simeq \mathcal{L}_A \hookrightarrow \mathcal{M}_{flat}(\Sigma_g, \mathrm{SL}(2,\mathbb{C}))$ $\qquad \mathbf{A}_m(x_c, y_c) = 0$

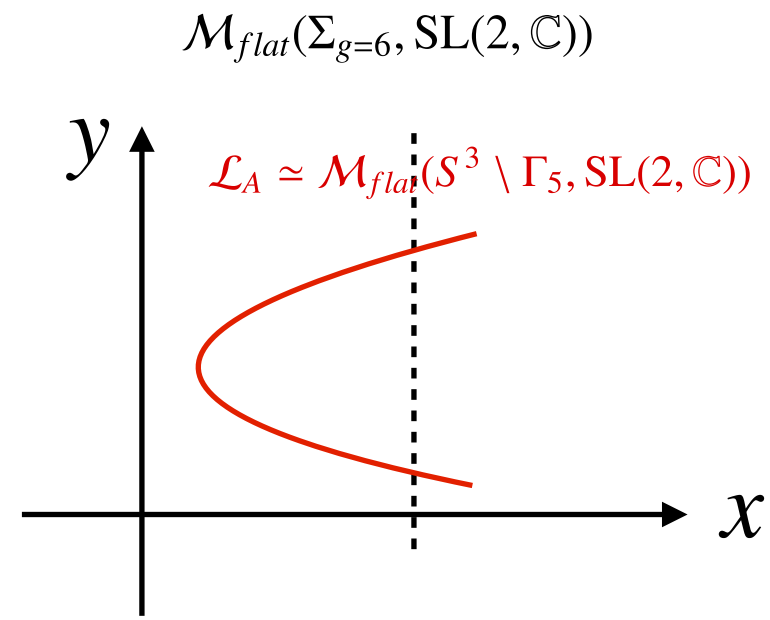
$$\hat{\mathbf{A}}_m(e^{\hat{u}}, e^{\hat{v}}, \hbar)\, Z(u) = 0, \quad m = 1, \cdots, 3g-3$$ **(Quantum Curve)**

The holomorphic solutions $Z(u)$ are the physical states for quantum flat connections on 3-manifold, which quantizes SL(2,C) Chern-Simons theory on 3-manifold.

Dimofte, Gukov, Lenells, Zagier 2009
Dimofte 2011
Gukov, Sułkowski 2011
Gukov, Saberi 2012

$$\mathbf{A}_m(x_c, y_c) = 0, \quad m = 1, \cdots, 3g-3$$

$$\hat{\mathbf{A}}_m(e^{\hat{u}}, e^{\hat{v}}, \hbar)\, Z(u) = 0, \quad m = 1, \cdots, 3g-3$$



Solution: holomorphic 3d block Witten 2010

$$Z^{(\alpha)}\left(M_3 \middle| u\right) = \int_{\mathcal{J}_\alpha} e^{\frac{i}{\hbar}\int_{M_3} A dA + \frac{2}{3}A^3}\, DA$$

Analytic continued CS along the integration cycle (Lefschetz thimble) labeled by α

Semiclassical expansion Dimofte, Gukov, Lenells, Zagier 2009

$$Z^{(\alpha)}\left(M_3 \middle| u\right) = \exp\left[\frac{i}{\hbar}\int_{\substack{(u_0,v_0)\\ \mathfrak{C}\subset\mathcal{L}_{\mathbf{A}}}}^{(u,v^{(\alpha)})} \vartheta + o(\log\hbar)\right]$$

$$\omega = \mathrm{d}\vartheta$$

↑ Liouville 1-form

α labels the branch of Lagrangian submanifold.

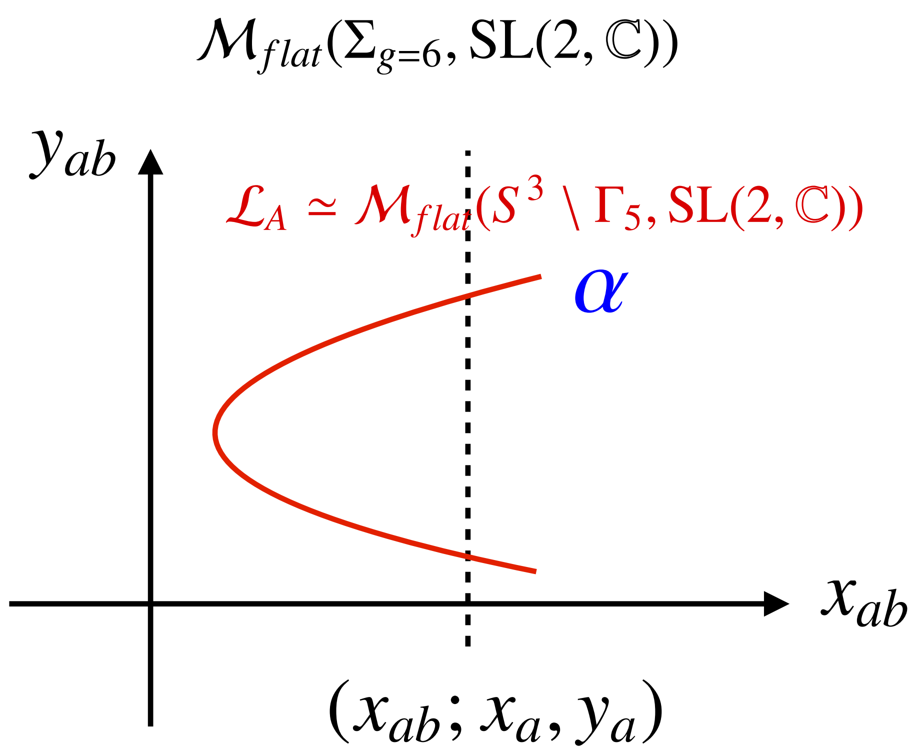
Wave function of 4-geometry

Recall:

A class of $\mathrm{SL}(2,\mathbb{C})$ flat connetion on $S^3 \setminus \Gamma_5$	=	Lorentzian 4-simplex geometries with constant curvature Λ

(A subset of branches in Lagrangian submanifold)

- **Quantize** $\mathcal{L}_A \simeq \mathcal{M}_{flat}(S^3 \setminus \Gamma_5, \mathrm{SL}(2,\mathbb{C}))$
- **pick the branch α corresponding to constant curvature 4-simplex geometries**



We propose:

Holomorphic block at the branch α is a state for 4d quantum simplicial geometry

Holomorphic block at the branch α is a state for 4d quantum simplicial geometry

Semiclassical limit:

$$Z^{(\alpha)}\left(S^3 \setminus \Gamma_5 \middle| u\right) \sim \exp\left[\frac{i}{\hbar} S^{(\alpha)}(u) + o(\log \hbar)\right]$$

classical action for 4-dimensional geometry

(dynamics of 4-dimensional geometry)

Holomorphic block at the branch α is a state for 4d quantum simplicial geometry

Semiclassical limit:

$$Z^{(\alpha)}\left(S^3\setminus\Gamma_5\middle|\, u\right) \sim \exp\left[\frac{i}{\hbar}S^{(\alpha)}(u) + o(\log\hbar)\right]$$

classical action for 4-dimensional geometry

(dynamics of 4-dimensional geometry)

$$S^{(\alpha)}(u) \;=\; S_{\mathrm{EH}}[g_{\mu\nu},\Lambda]$$

4-dimensional Einstein-Hilbert action on a constant curvature 4-simplex:

$$S_{\mathrm{EH}}[g_{\mu\nu},\Lambda] = \sum_{a<b}\mathbf{a}_{ab}\Theta^{\Lambda}_{ab} - \Lambda\mathrm{Vol}^{\Lambda}_{4}$$

also known as Regge action.

$Z^{(\alpha)}\left(S^3\setminus\Gamma_5\middle|\, u\right)$ **is a wave function for 4d Einstein gravity**

Equivalence of Semiclassical Wave Functions

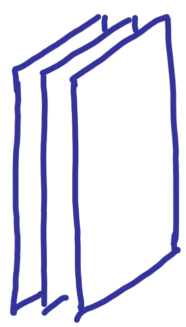
$$Z^{(\text{4-geometry})}_{\text{Simplicial Gravity}}\left(M_4 \middle| h_{ab}\right) = Z^{(\alpha)}\left(M_3 \middle| u\right)$$

Simplicial gravity on 4-manifold

SL(2,C) CS on 3-manifold

3d-3d correspondence

M-theory in 11d:



N

M5-brane $\longrightarrow$ **IR dynamics: 6d (2,0) SCFT with gauge group G**

Compactify M5 on $M_3 \times S^3_b$ 3d ellipsoid

(or $S^2 \times_q S^1$ or $\mathbb{R}^2 \times_q S^1$)

$\mathrm{G}_{\mathbb{C}}$ CS on M_3 $\Leftrightarrow$ 3d $\mathcal{N}=2$ SUSY gauge theory T_{M_3} (SCFT with 4 Q's)

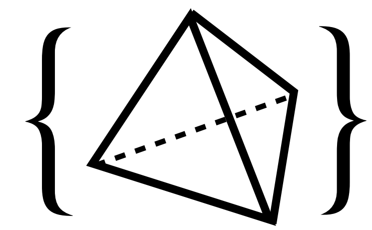
- $Z_{CS}(M_3) = Z^{\mathcal{N}=2}_{T_{M_3}}(S^3_b)$
- $\mathcal{M}_{flat}(M_3, \mathrm{G}_{\mathbb{C}}) \simeq \mathcal{M}_{SUSY}(T_{M_3})$
- $Z^{(\alpha)}(M_3) = Z^{\mathcal{N}=2}_{T_{M_3}}(\mathbb{R}^2 \times_q S^1)$ with boundary SUSY ground state α

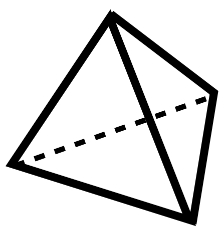
Dimofte, Gaiotto, Gukov 2011
C. Beem, T. Dimofte, S. Pasquetti 2012
Cordova, Jafferis 2013
Lee, Yamazaki 2013
Chung, Dimofte, Gukov, Sułkowski 2014

Dimofte-Gaiotto-Gukov (DGG) Construction

T_{DGG,M_3} **3d $\mathcal{N}$= 2 SCFT with Abelian gauge group** $\mathrm{U}(1)^n$ Dimofte, Gaiotto, Gukov 2011

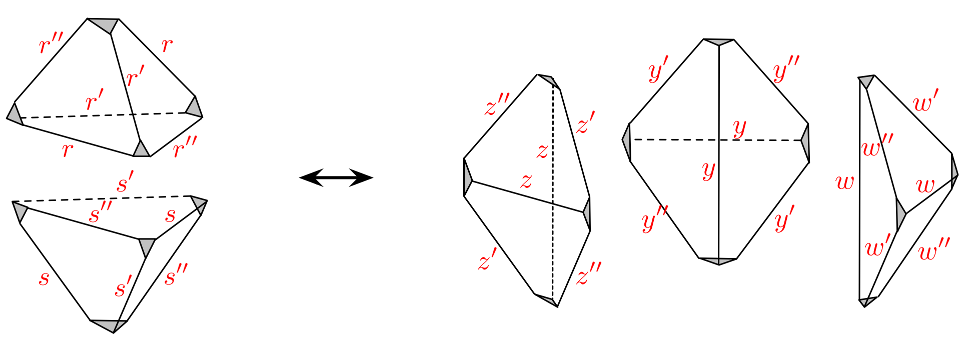
(Gauge theories labelled by 3-manifolds)

M_3 —Ideal triangulation→ 



$T_\Delta =$ **3d $\mathcal{N}$= 2 chiral multiplet ; gluing → gauging + superpotential**

→ **Pachner move = 3d mirror symmetry**



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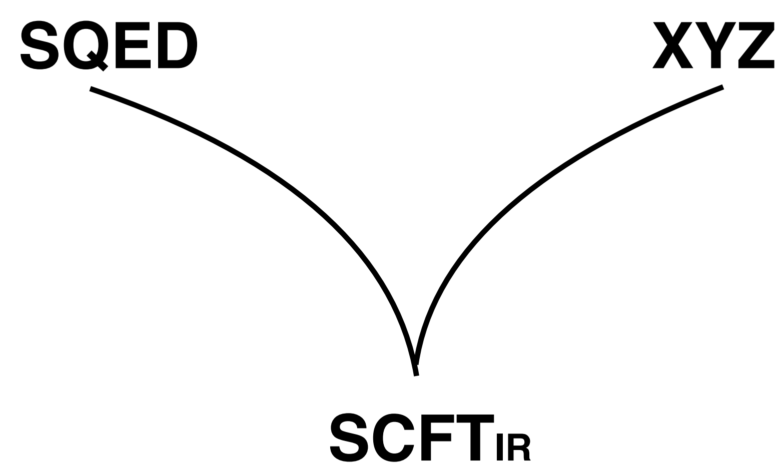

- $\mathcal{M}_{flat}(M_3, \mathrm{SL}(2,\mathbb{C})) \hookleftarrow \mathcal{M}_{SUSY}(T_{DGG,M_3})$
- $Z'_{CS}(M_3) = Z_{DGG,M_3}(S^3_b)$
- $Z^{(\alpha)}(M_3) = Z_{DGG,M_3}(\mathbb{R}^2 \times_q S^1)$ with boundary SUSY ground state α

Dimofte, Gaiotto, Gukov 2011
C. Beem, T. Dimofte, S. Pasquetti 2012

4d Simplicial Gravity and 3d SCFT

$$Z^{(4\text{-geometry})}_{\text{Simplicial Gravity}}(M_4) = Z^{(\alpha)}(M_3) = Z^{\mathcal{N}=2}_{T_{M_3}}\left(\mathbb{R}^2 \times_q S^1\right)$$

Simplicial gravity on 4-manifold

SL(2,C) CS on 3-manifold

3d SCFT

$$\text{4d simplicial geometries} \hookrightarrow \mathcal{M}_{SUSY}$$

Effective twisted superpotential $\widetilde{\mathcal{W}}_{\text{eff}}(\sigma)$ = Einstein-Hilbert action in 4d

4d Simplicial Gravity and 3d SCFT

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The end

Thanks for your attention !