Hyperbolic Geometry and the Lorentz Group

Norbert Dragon
Bad Honnef
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Motivation

Fermi-Walker transport

Wigner Rotation, Unitary Representations of the Poincaré Group

Hyperbolic Geometry

Poincaré Disk

Wigner Angle

Motivation

Just for fun, Curiosity

Not in the books, professional training

Uniformly accelerated worldlines (alternative mechanics, space flight)

Rotationfree transport

Thomas Precession

Quantum Mechanics

Nonassociativity (Gyro group, Ungar, Karzel)

Fermi-Walker Transport

Observer, vierbein, infinitesimal rigid body:

$$\Gamma : \tau \mapsto (x(\tau), e_1(\tau), e_2(\tau), e_3(\tau)) \ , \ e_0 = \frac{dx}{d\tau} \ , \ e_a \cdot e_b = \eta_{ab}$$

$$\Lambda = \begin{pmatrix} e_0, & e_1, & e_2, & e_3 \end{pmatrix} \ \in \text{Lorentzgroup}$$

$$SO(1,3) = H^3 \times SO(3) \ , \ \Lambda = L(\Lambda)O(\Lambda) \ , \ L = L^T \ , O^{-1} = O^T$$

$$\{e_0\} = H^3 = \{(q^0, q^1, q^2, q^3) : -(q^0)^2 + (q^1)^2 + (q^2)^2 + (q^3)^2 = -1\}$$

e_1, e_2, e_3 span tangent space at e_0

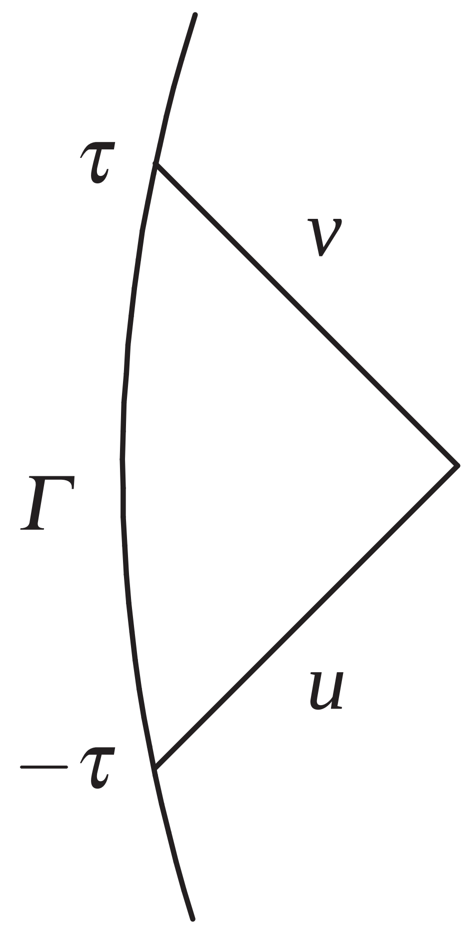
$$\frac{de_a}{d\tau} = e_c \omega^c{}_a \ , \ \omega_{ab} = -\omega_{ba}$$

$$\text{rotation free} \ \Leftrightarrow \omega_{ij} = 0 \ , \ \omega = \begin{pmatrix} 0 & b^T \\ b & 0 \end{pmatrix} \ ,$$

$$\frac{de_0}{d\tau} = b^1 e_1 + b^2 e_2 + b^3 e_3 \ , \ \frac{de_1}{d\tau} = b^1 e_0 \ , \ \frac{de_2}{d\tau} = b^2 e_0 \ , \ \frac{de_3}{d\tau} = b^3 e_0$$

Rotationfree (Fermi-Walker) Transport

Reflected light returns from the direction into which it was emitted



Directions of light from distant stars changed by aberration

Constant, rotation free acceleration

$$\Gamma : a \mapsto x(a) = \begin{pmatrix} \operatorname{sh} a \\ (\operatorname{ch} a - 1)n \end{pmatrix} + x(0) \ , \ \hat{\Gamma} : a \mapsto \frac{dx}{da} = \begin{pmatrix} \operatorname{ch} a \\ n \operatorname{sh} a \end{pmatrix}$$

Rapidity a: lenght of Γ and $\hat{\Gamma}$

Rotation free, linearly accelerated vierbein, initially at rest

$$\hat{\Gamma} : a \mapsto L_{a,n} = \begin{pmatrix} \operatorname{ch} a, & n^T \operatorname{sh} a \\ n \operatorname{sh} a, & 1 + (\operatorname{ch} a - 1) n n^T \end{pmatrix} = \exp a \begin{pmatrix} 0 & n^T \\ n & 0 \end{pmatrix}$$

with initial velocity $\vec{v} = \vec{n}_b \operatorname{th} b$, (read a to denote $(a, \vec{n}_a)$ as appropriate)

$$L_b \hat{\Gamma} : a \mapsto L_b L_a$$

L_a multiplies from the right, acceleration is constant in the observer frame

Relativistic Quantum Mechanics

Each unitary, irreducible representation of the Poincaré group is unitarily equivalent to the representation, which is induced by an irreducible, unitary representation R of the little group of a vector $\underline{p}$ on its mass shell. For $m^2 > 0$ and $p^0 > 0$ and with a Lorentz invariant integration measure it is given by

$$(U_\Lambda \Psi)(\Lambda p) = R(W(\Lambda, p))\Psi(p)$$

$$\text{Wigner Rotation: } W(\Lambda, p) = {L_{\Lambda p}}^{-1} \Lambda L_p \ , \ L_p \underline{p} = p$$

$$W(O, p) = O \ , \ L_{L_q p} W(L_q, p) = L_q L_p$$

$$L_p = \begin{pmatrix} \frac{p^0}{m} & \frac{\vec{p}^T}{m} \\ \frac{\vec{p}}{m} & 1 + \frac{\vec{p}\vec{p}^T}{m(p^0+m)} \end{pmatrix} \ , \ \frac{p}{m} = \begin{pmatrix} \operatorname{ch} a \\ n \operatorname{sh} a \end{pmatrix} \ , \ L_p = L_a \ , \ \text{independent of } m$$

Infinitesimal Transformations

infinitesimal transformation (antihermitean)

$$l_{ij} = -(p^i \frac{\partial}{\partial p^j} - p^j \frac{\partial}{\partial p^i}) + \Gamma_{ij} \ , \ \Gamma_{ij} = -\Gamma_{ji} = -(\Gamma_{ij})^\dagger$$

$$[\Gamma_{ij}, \Gamma_{kl}] = \delta_{ik}\Gamma_{jl} - \delta_{il}\Gamma_{jk} - \delta_{jk}\Gamma_{il} + \delta_{jk}\Gamma_{il}$$

$$l_{0i} = p^0 \frac{\partial}{\partial p^i} + \Gamma_{ik} \frac{p^k}{p^0 + m} \text{ independent of mass}$$

Hyperbolic Geometry

$$L_c^{-1}\, W(b,a) = L_b\, L_a$$

$$\begin{pmatrix} \operatorname{ch} c & * \\ * & * \end{pmatrix} = \begin{pmatrix} \operatorname{ch} c & * \\ * & * \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & W(b,a) \end{pmatrix} = \begin{pmatrix} \operatorname{ch} b & \operatorname{sh} b\, n_b^T \\ * & * \end{pmatrix} \begin{pmatrix} \operatorname{ch} a & * \\ \operatorname{sh} a\, n_a & * \end{pmatrix}$$

$$\operatorname{ch} c = \operatorname{ch} a \operatorname{ch} b - \operatorname{sh} a \operatorname{sh} b \cos\gamma \ , \gamma \text{ subtends } c$$

$$\operatorname{ch} c = \begin{pmatrix} \operatorname{ch} b \\ \operatorname{sh} b\, n_b \end{pmatrix} \cdot \begin{pmatrix} \operatorname{ch} a \\ \operatorname{sh} a\, n_a \end{pmatrix}$$

infinitesimally Euclidean $c^2 = a^2 + b^2 - 2\,a\,b\,\cos\gamma$ (γ subtends c)

Circumference and area of circle of radius r: $U = 2\pi \operatorname{sh} r$, $A = 2\pi(\operatorname{ch} r - 1)$

Thomas precession: deficit angle $\delta = 2\pi(\operatorname{ch} r - 1)$

Poincaré Disk $u^2 < 1$

$W(b, a)$ rotates the plane, spanned by a and b (SO(1,2) ~ SL(2,$\mathbb{R}^2$)), $\mathbb{R}^{1,2}$

Stereographic projection u of H^2 from the south pole $(-1, 0, 0)$

$$u^i = \frac{q^i}{1 + q^0} \ , \ \text{inverted: } q^0 = \frac{1 + u^2}{1 - u^2} \ , \ q^i = \frac{2u^i}{1 - u^2}$$

bijective, conformal: circles to circles, metric $g(u, du) = \frac{4}{(1-u^2)^2} du^i du^i$

$a \mapsto u(L_v L_a)$ circle in Poincaré disk $u^2 < 1$, which intersects $u^2 = 1$ orthogonally, straight lines

Wigner Rotation

$W(b,a)$ rotates the plane, spanned by a and b (SO(1,2) $\sim$ SL(2,$\mathbb{R}^2$))

$W(b,a) = \begin{pmatrix} 1 & & \\ & \cos\delta & \sin\delta \\ & -\sin\delta & \cos\delta \end{pmatrix}$ opposite to the rotation from a to b,

$W(a,b) = W(b,a)^{-1}$

Its angle δ is the hyperbolic area of the triangle a, b, c

δ in terms of the lengths of the sides: Heron's formula (some calculation)

$$\cos\frac{\delta}{2} = \frac{1 + \mathrm{ch}\,a + \mathrm{ch}\,b + \mathrm{ch}\,c}{4\,\mathrm{ch}\,\frac{a}{2}\,\mathrm{ch}\,\frac{b}{2}\,\mathrm{ch}\,\frac{c}{2}}$$

In terms a, b and the angle γ: $\mathrm{ch}\,c = \mathrm{ch}\,a\,\mathrm{ch}\,b - \mathrm{sh}\,a\,\mathrm{sh}\,b\cos\gamma$

$$\cos\delta = \frac{\left(1 + p^0 + q^0 + p^0 q^0 + \vec{p}\vec{q}\right)^2}{(1+p^0)(1+q^0)(1+p^0q^0+\vec{p}\vec{q})} - 1$$

Massless Representations

$$W(\Lambda, p) = L^{-1}_{\Lambda p} \Lambda L_p \ , L_p \underline{p} = p$$

$$p^0 = \sqrt{(p_z)^2 + \vec{p}^2} \ , \ \underline{p} = (1, 1, 0, \ldots 0) \text{ little group } \mathrm{ISO}(D-2)$$

$$L_p = D_p B_p$$

$$B_p = \begin{pmatrix} \frac{1}{2}(p^0 + \frac{1}{p^0}) & \frac{1}{2}(p^0 - \frac{1}{p^0}) & \\ \frac{1}{2}(p^0 - \frac{1}{p^0}) & \frac{1}{2}(p^0 + \frac{1}{p^0}) & \\ & & 1_{(D-2)\times(D-2)} \end{pmatrix}$$

$$D_p = \begin{pmatrix} 1 & \\ & \hat{D}_p \end{pmatrix} \ , \ \hat{D}_p = \begin{pmatrix} \frac{p_z}{p^0} & -\frac{\vec{p}^T}{p^0} \\ \frac{\vec{p}}{p^0} & 1 - \frac{\vec{p}\vec{p}^T}{p^0(p^0+p_z)} \end{pmatrix}$$

D_p shortest rotation from $(p^0, p^0, 0 \ldots)$ to $(p^0, p_z, \vec{p})$,

not defined for $(p^0, p_z, \vec{p}) = (p^0, -|p_z|, 0)$

Infinitesimal Transformations (antihermitean)

$$l_{ij} = -(p^i \frac{\partial}{\partial p^j} - p^j \frac{\partial}{\partial p^i}) + \gamma_{ij} \ , \ i, j \in \{2, \ldots D - 1\}$$

$$[\gamma_{ij}, \gamma_{kl}] = \delta_{ik}\gamma_{jl} - \delta_{il}\gamma_{jk} - \delta_{jk}\gamma_{il} + \delta_{jk}\gamma_{il}$$

$$l_{0i} = p^0 \frac{\partial}{\partial p^i} + \gamma_{ik} \frac{p^k}{p^0 + p_z} \ , \ p^0 + p_z = p_+$$

$$l_{zi} = -(p_z \frac{\partial}{\partial p^j} - p^j \frac{\partial}{\partial p_z}) + \gamma_{ik} \frac{p^k}{p^0 + p_z}$$

$$l_{oz} = p^0 \frac{\partial}{\partial p_z}$$

Singular on the negative z-axis.

Spin bundle with northern and southern hemisphere

Southern Chart

$L_p^S = D_p^S B_p^S D_{zx}$

D_{zx} rotation by π in zx-plane, $D_{zx}(1,1,0\ldots)^T = (1,-1,0\ldots)^T$

$$B_p^S = \begin{pmatrix} \frac{1}{2}(p^0 + \frac{1}{p^0}) & -\frac{1}{2}(p^0 - \frac{1}{p^0}) & \\ -\frac{1}{2}(p^0 - \frac{1}{p^0}) & \frac{1}{2}(p^0 + \frac{1}{p^0}) & \\ & & 1_{(D-2)\times(D-2)} \end{pmatrix}$$

$$D_p^S = \begin{pmatrix} 1 & \\ & \hat{D}_p^S \end{pmatrix} \;,\; \hat{D}_p^S = \begin{pmatrix} -\frac{p_z}{p^0} & \frac{\vec{p}^T}{p^0} \\ -\frac{\vec{p}}{p^0} & 1 - \frac{\vec{p}\vec{p}^T}{p^0(p^0-p_z)} \end{pmatrix}$$

D_p^S shortest rotation from $(p^0,-p^0,0\ldots)$ to $(p^0,p_z,\vec{p})$,

not defined for $(p^0,p_z,\vec{p}) = (p^0,|p_z|,0)$.

Southern Infinitesimal Transformations

$$l_{ij} = -(p^i \frac{\partial}{\partial p^j} - p^j \frac{\partial}{\partial p^i}) + \gamma_{ij} \ , \ i, j \in \{2, \dots D-1\}$$

$$l_{0i} = p^0 \frac{\partial}{\partial p^i} + \gamma_{ik} \frac{p^k}{p^0 - p_z}$$

$$l_{zi} = -(p_z \frac{\partial}{\partial p^j} - p^j \frac{\partial}{\partial p_z}) - \gamma_{ik} \frac{p^k}{p^0 - p_z}$$

$$l_{oz} = p^0 \frac{\partial}{\partial p_z}$$

Singular on the positive z-axis .

Transition Function

$$T_p = (L_p^S)^{-1} L_p^N = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 + 2\frac{p_x^2}{p^2} & 2\frac{p_x \vec{p}^T}{p^2} \\ & & -2\frac{p_x \vec{p}}{p^2} & 1 - 2\frac{\vec{p}\vec{p}^T}{p^2} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & C & S\vec{n}^T \\ & & -S\vec{n} & 1 + (C-1)\vec{n}\vec{n}^T \end{pmatrix}$$

$p^2 = p_x^2 + \vec{p}^2$ (no p_z^2) , $C = \cos 2\varphi$, $S = \sin 2\varphi$, $p \neq (p^0, p_z, 0, \ldots, 0)$

Transition well defined also for half-integer Spin(D-2).

Spherical Geometry

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$$

$$\cos c = \begin{pmatrix} \cos b \\ \sin b \, n_b \end{pmatrix} \cdot \begin{pmatrix} \cos a \\ \sin a \, n_a \end{pmatrix}$$

infinitesimally Euclidean $c^2 = a^2 + b^2 - 2\,a\,b\,\cos \gamma$