Resummation

Andrea Banfi

University of Sussex

Single-logarithmic resummation

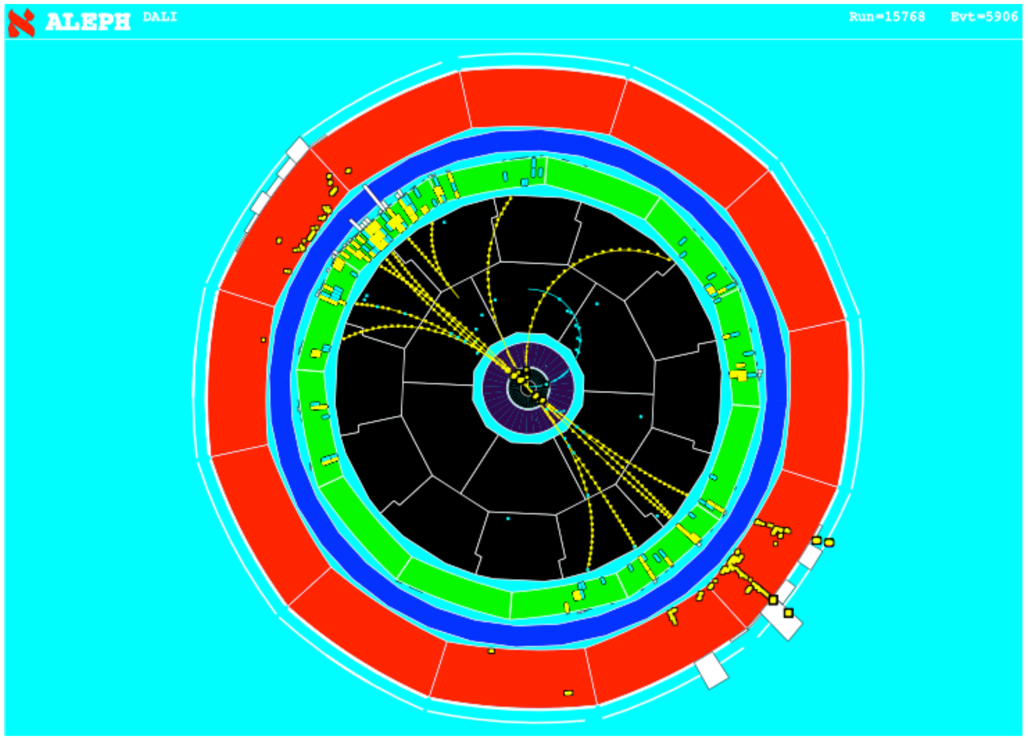
An example: event-shape variables

Event-shape variables $V(p_1, \ldots, p_n)$ are combinations of hadron final-state momenta into a number which gives insight on event geometry

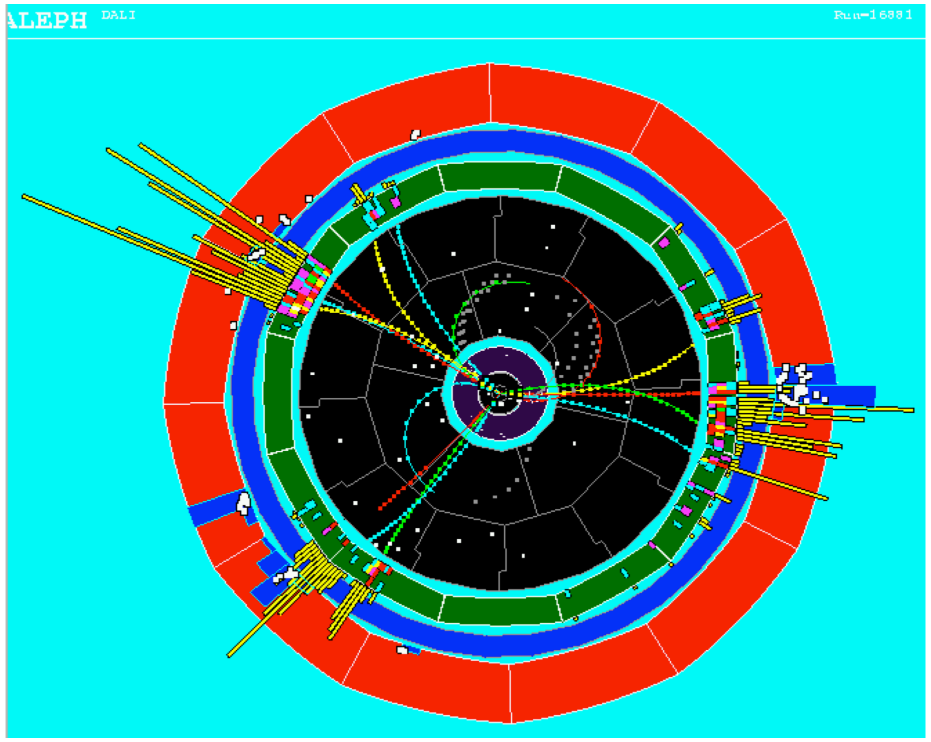
Example: Thrust, longitudinal particle alignment

$$T \equiv \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|} \qquad \tau \equiv 1 - T \qquad \sum_i |\vec{p}_i| \equiv Q = \sqrt{s}$$

Pencil-like event $T \lesssim 1$



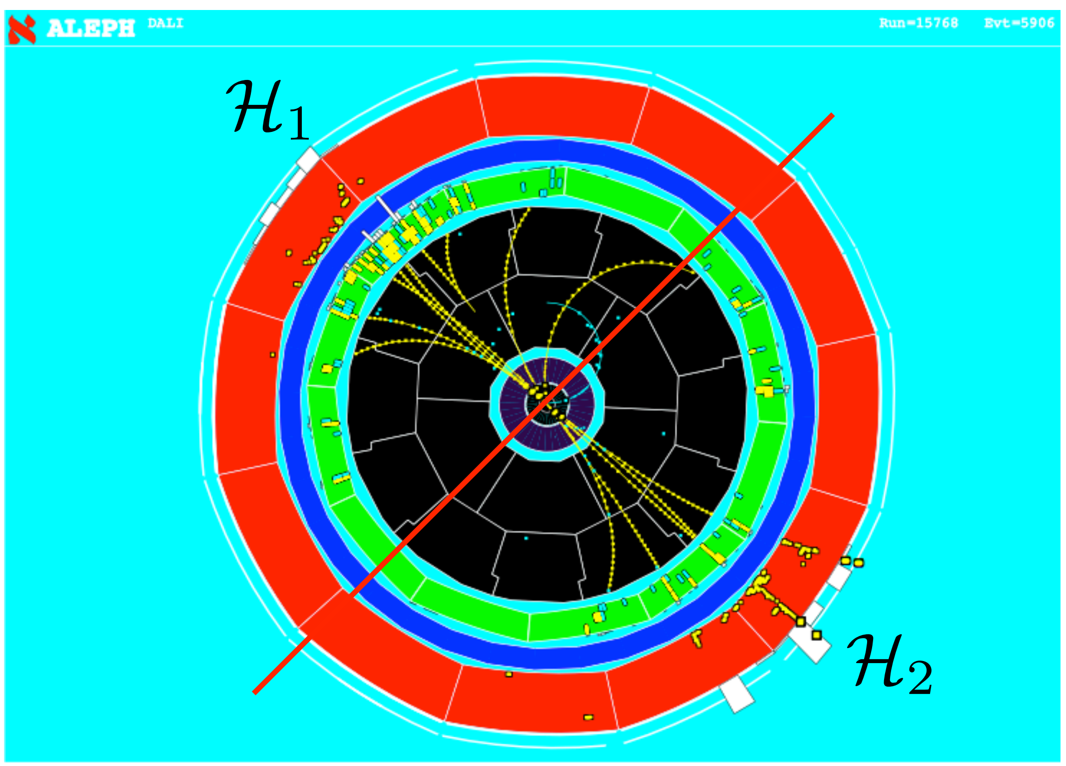
Planar event $T \gtrsim 2/3$

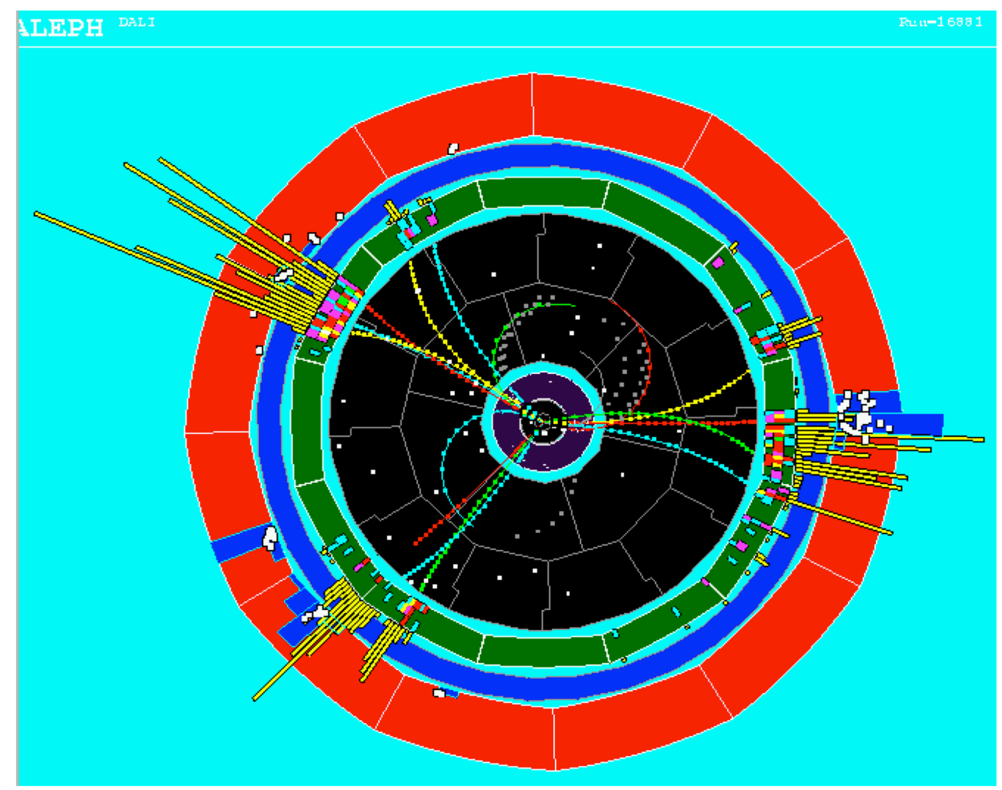


Final-state observables

Event-shapes are a class of final-state observables, continuous measures of the energy momentum flow of hadronic final states

$$\tau \to 0 \longleftrightarrow \tau \to 1/3$$



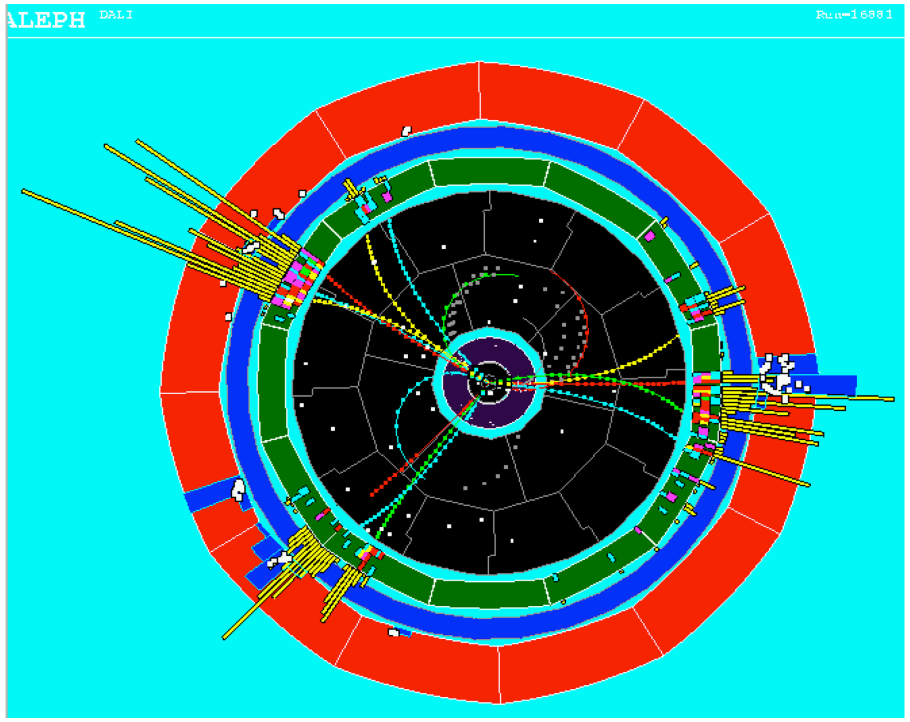


In the two-jet limit, one minus the thrust is the sum of the invariant masses of the two hemispheres in which an event is divided by the thrust axis

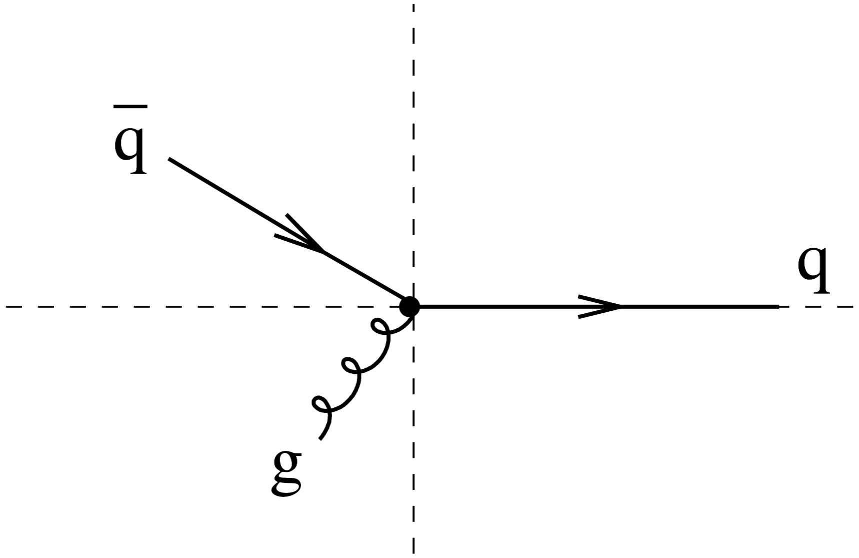
There are many final-state observables: jet broadenings, jet resolution parameters, etc.

Collinear and infrared safety

All final-state observables we consider are infrared and collinear (IRC) safe, so that we can safely compute their distributions using quark and gluon language



$\Longrightarrow$



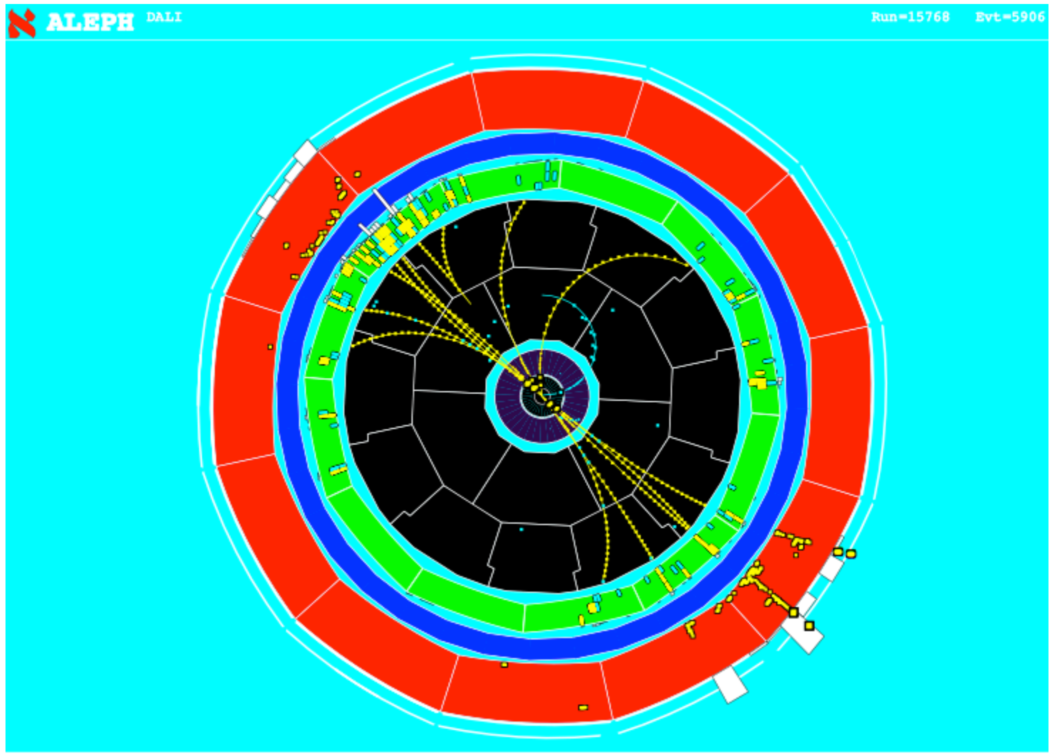
Example: jets obtained from parton momenta are close to those obtained from hadron momenta if they do not change after

- the addition of any number of soft partons (IR safety)
- any number of collinear splittings (collinear safety)

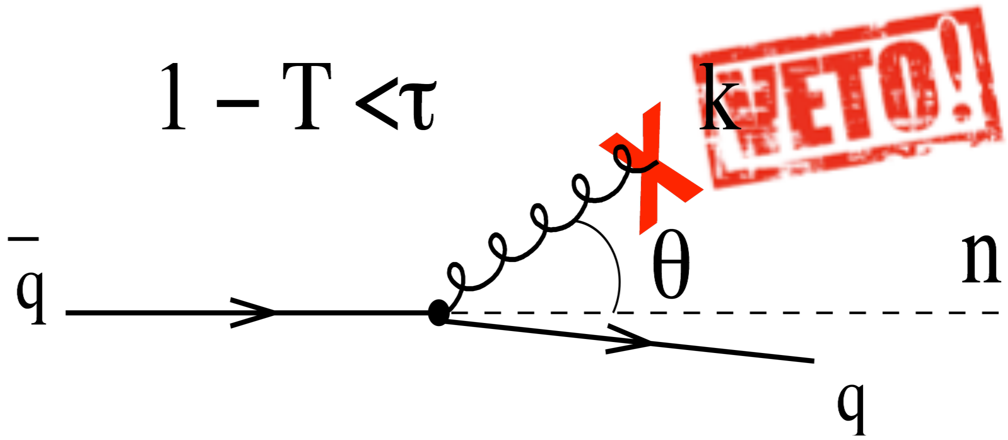
Departure from the Born limit

Final-state observables have the property that, for configurations close to the Born limit (e.g. a $q\bar{q}$ pair) their value is close to zero

Example: pencil-like events are selected by requiring that one minus the thrust is below a given threshold



The restriction $1 - T < \tau$ acts as a veto on gluon emissions

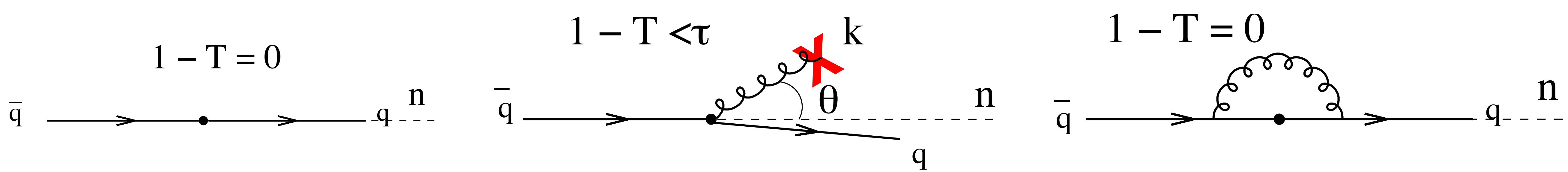


To quantify the departure from the Born limit we consider the cumulative distribution $\Sigma(v)$, the probability that $V(p_1, \ldots, p_n) < v$

One-gluon emission

Let us consider a single soft ($z = E_g/E_{\max} \to 0$) and collinear ($\theta \to 0$) emission, for which $1 - T(\{p\}, k) \simeq z\,\theta^2$

Imposing that we are close to the two-jet limit restricts the phase space for real emissions, but not for virtual corrections



soft collinear

$$1 \quad + \quad C_F \frac{\alpha_s}{\pi} \int \frac{dz}{z} \frac{d\theta^2}{\theta^2} \Theta(\tau - z\theta^2) \quad - \quad C_F \frac{\alpha_s}{\pi} \int \frac{dz}{z} \frac{d\theta^2}{\theta^2}$$

$$\boxed{\Sigma(\tau) \simeq 1 - C_F \frac{\alpha_s}{\pi} \ln^2 \frac{1}{\tau}}$$

Large logarithms

The cumulative thrust distribution contains logarithms that can become large in the two-jet limit $\tau \to 0$

$$\Sigma(\tau) \simeq 1 - C_F \frac{\alpha_s}{\pi} \ln^2 \frac{1}{\tau}$$

Typical two-scale problem: at small τ the invariant mass of each jet $\sim \sqrt{\tau} Q$ is much less than the energy of each jet $\sim Q$

breakdown of perturbation theory!

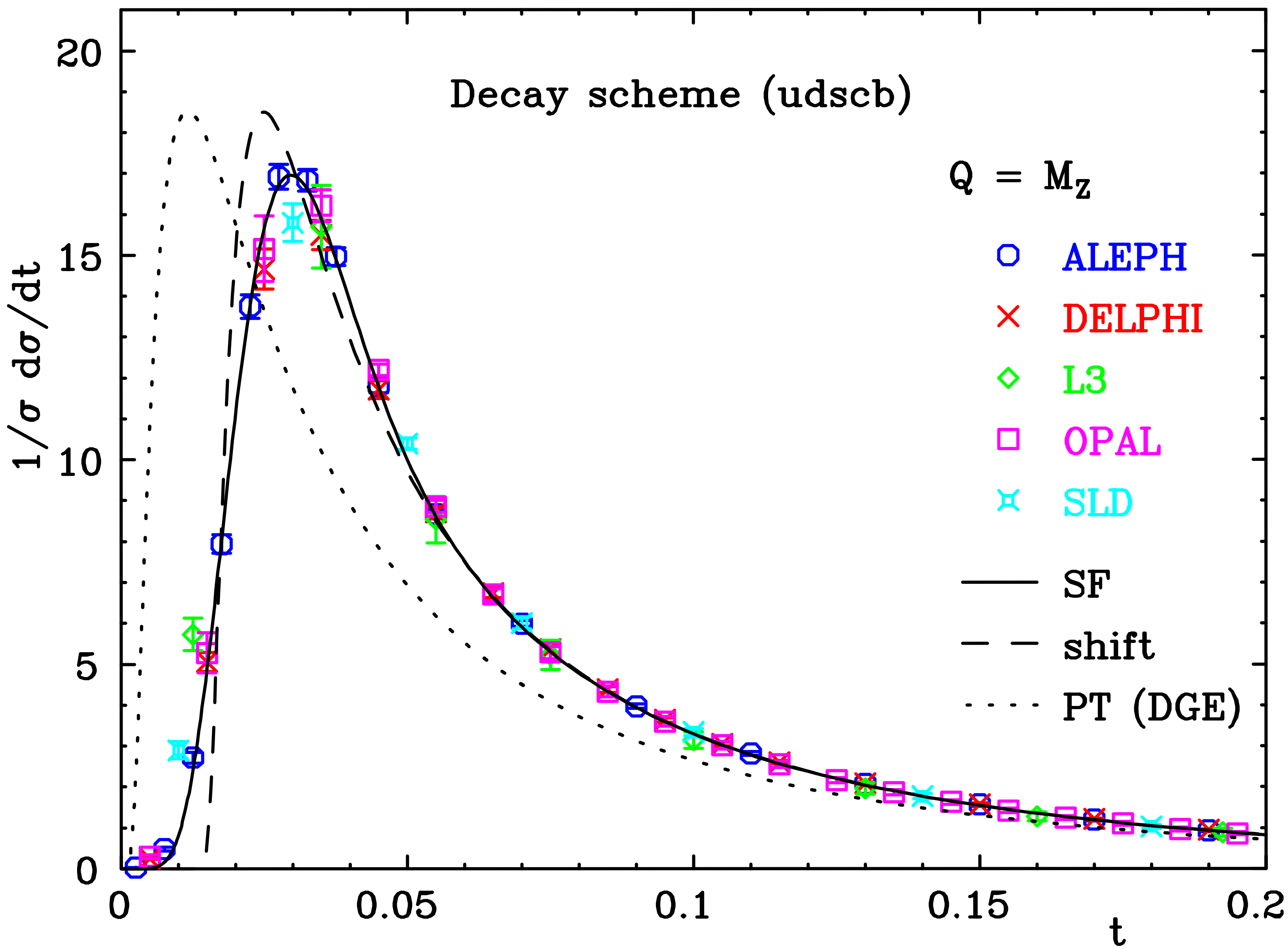
Resummation

In the region $\alpha_s L \sim 1$, where $L = \ln(1/v)$, we wish the cumulative distribution of any final-state observable to be written in the form

$$\Sigma(v) \simeq e^{\underbrace{Lg_1(\alpha_s L)}_{\text{LL}}} \times \left(\begin{array}{cccc} 1 & + & \alpha_s & + \dots \\ \underbrace{G_2(\alpha_s L)}_{\text{NLL}} & + & \underbrace{\alpha_s G_3(\alpha_s L)}_{\text{NNLL}} & + \dots \end{array} \right)$$

Why do we need to care about large logarithms?

Let us consider the thrust differential distribution



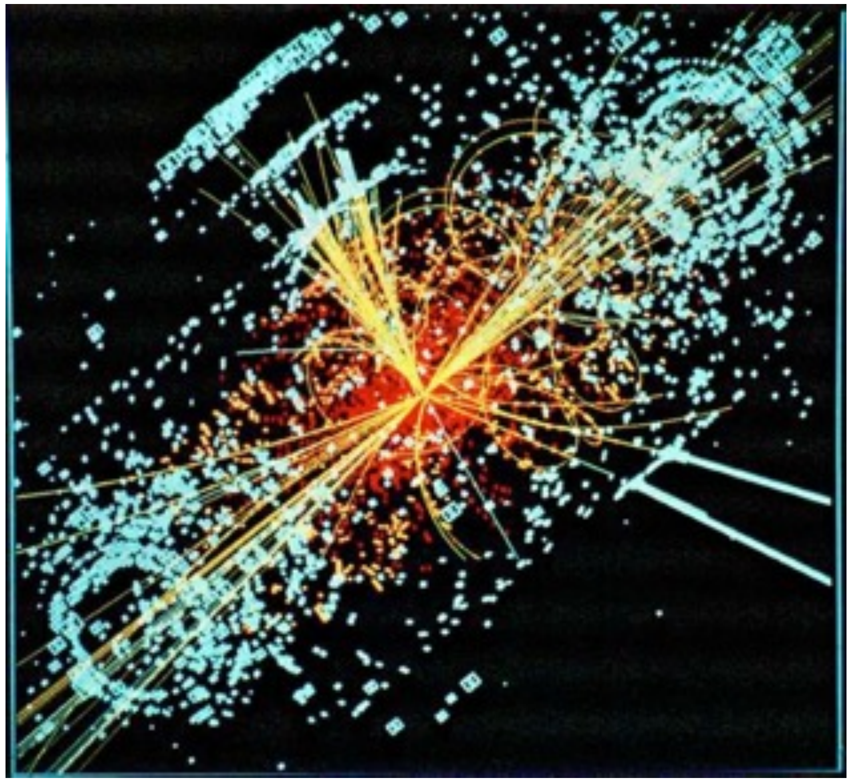
$$\frac{1}{\sigma}\frac{d\sigma}{dt} = \frac{d}{dt}\Sigma(t)$$

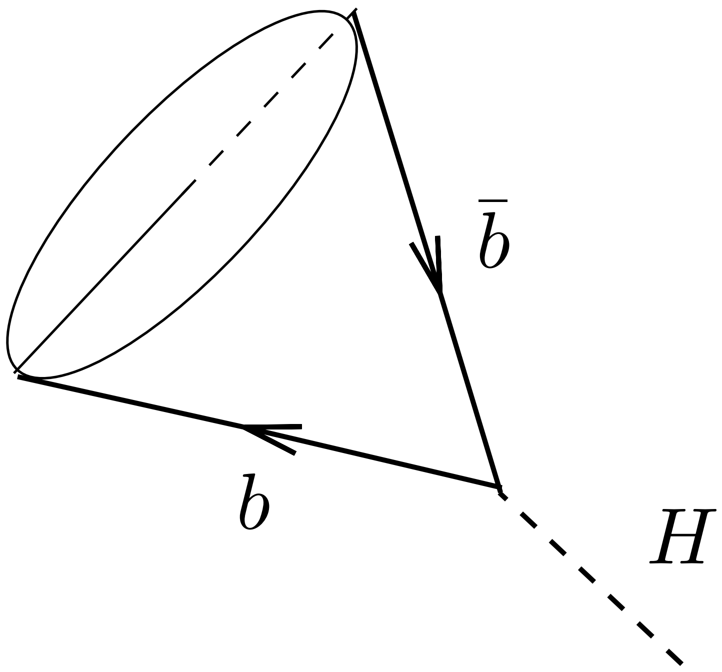
$\alpha_s \ln(1/t) \sim 1 \longleftrightarrow \alpha_s \ln(1/t) \ll 1$

Most events lie in the region in which logarithms are large!

A hadron-collider parallel

At the LHC it is possible to look for a boosted Higgs decaying into a $b\bar{b}$ pair





The decay products of the Higgs tend to fall into the same jet $\Rightarrow$ consider the invariant mass of fat jet and look for a peak for $m_{\mathrm{jet}} \sim m_H$

If $p_{\mathrm{t,jet}} \sim 1\,\mathrm{TeV}$, for background QCD jets $\alpha_s(p_{\mathrm{t,jet}})\ln(p_{\mathrm{t,jet}}/m_{\mathrm{jet}}) \sim 1$, so that we have again a two-scale problem when $m_{\mathrm{jet}} \ll p_{\mathrm{t,jet}}$

Resummation

In the region $\alpha_s L \sim 1$, where $L = \ln(1/v)$, we wish the cumulative distribution of any final-state observable to be written in the form

$$\Sigma(v) \simeq e^{\overbrace{Lg_1(\alpha_s L)}^{\text{LL}}} \times \Bigg(\underbrace{G_2(\alpha_s L)}_{\text{NLL}} + \underbrace{\alpha_s G_3(\alpha_s L)}_{\text{NNLL}} + \ldots \Bigg)$$

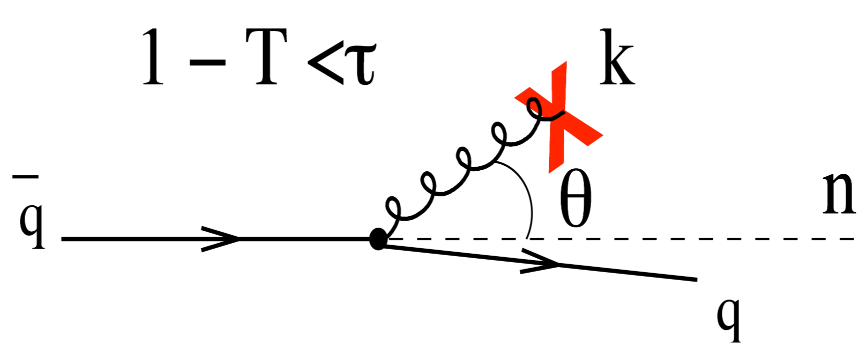
To achieve NLL accuracy we have to consider

- Double logarithms $\alpha_s L^2$: they come from soft *and* collinear contributions, and have to exponentiate
- Single logarithms $\alpha_s L$: they come from soft and/or collinear contributions, and have to factorise from double logarithms

One-gluon emission

We consider one-gluon emission k and compute the distribution $\Sigma(v)$

Example of kinematics: two light-like momenta along the thrust axis



$$P_1 = \frac{Q}{2}(1,0,0,1)$$

$$P_2 = \frac{Q}{2}(1,0,0,-1)$$

Sudakov decomposition of k along P_1 and P_2

$$k = z_1 P_1 + z_2 P_2 + k_t \qquad\qquad 0 = k^2 = z_1 z_2 Q^2 - k_t^2$$

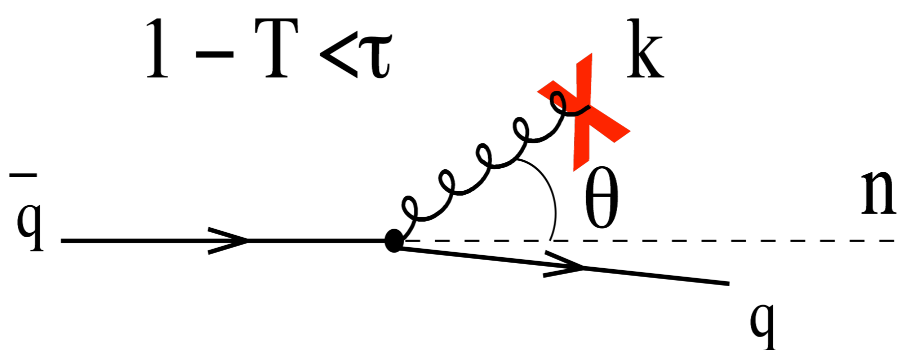
Phase space and matrix element squared in the soft-collinear limit

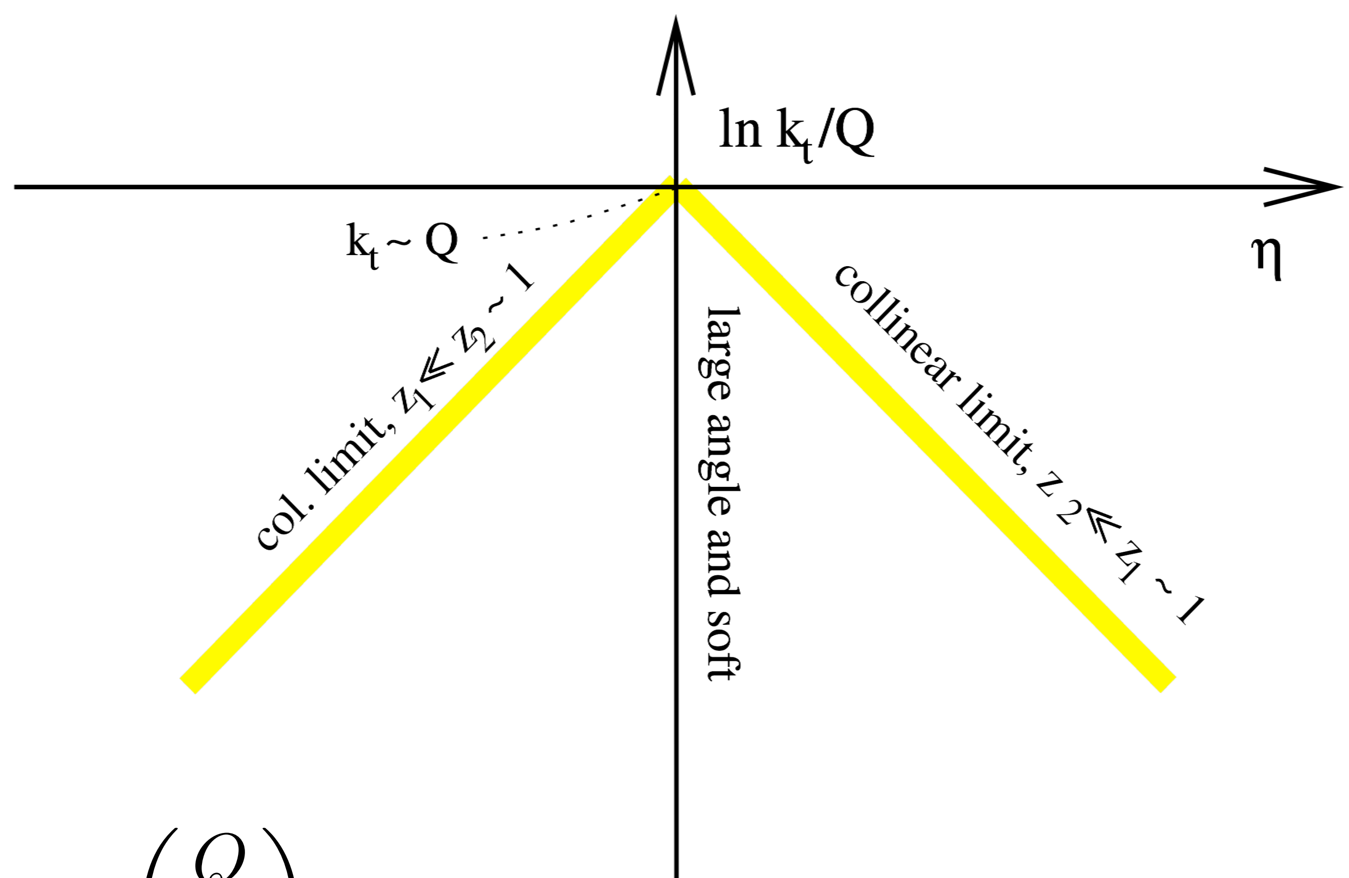
$$[dk] = Q^2 dz_1\, dz_2\, \frac{d\phi}{2\pi}\, dk_t^2\, \delta(z_1 z_2 Q^2 - k_t^2) \qquad M^2(k) = \frac{\alpha_s C_F}{4\pi}\frac{z_1\, p_{gq}(z_1)\cdot z_2\, p_{gq}(z_2)}{k_t^2}$$

The Lund plane

For resummation purposes, it is extremely useful to visualise soft and collinear emissions in the Lund plane. We need to introduce the emission rapidity

$$\eta = \frac{1}{2}\ln\left(\frac{z_1}{z_2}\right) \qquad \Rightarrow \qquad [dk]M^2(k) \simeq 2C_F \frac{\alpha_s}{\pi}\frac{dk_t}{k_t}d\eta$$





$$\eta \simeq \ln\frac{1}{\theta} \qquad z_1 \gg z_2$$

collinear limit: $z_1, z_2 < 1 \Rightarrow |\eta| < \ln\left(\frac{Q}{k_t}\right)$

The thrust in the Lund plane

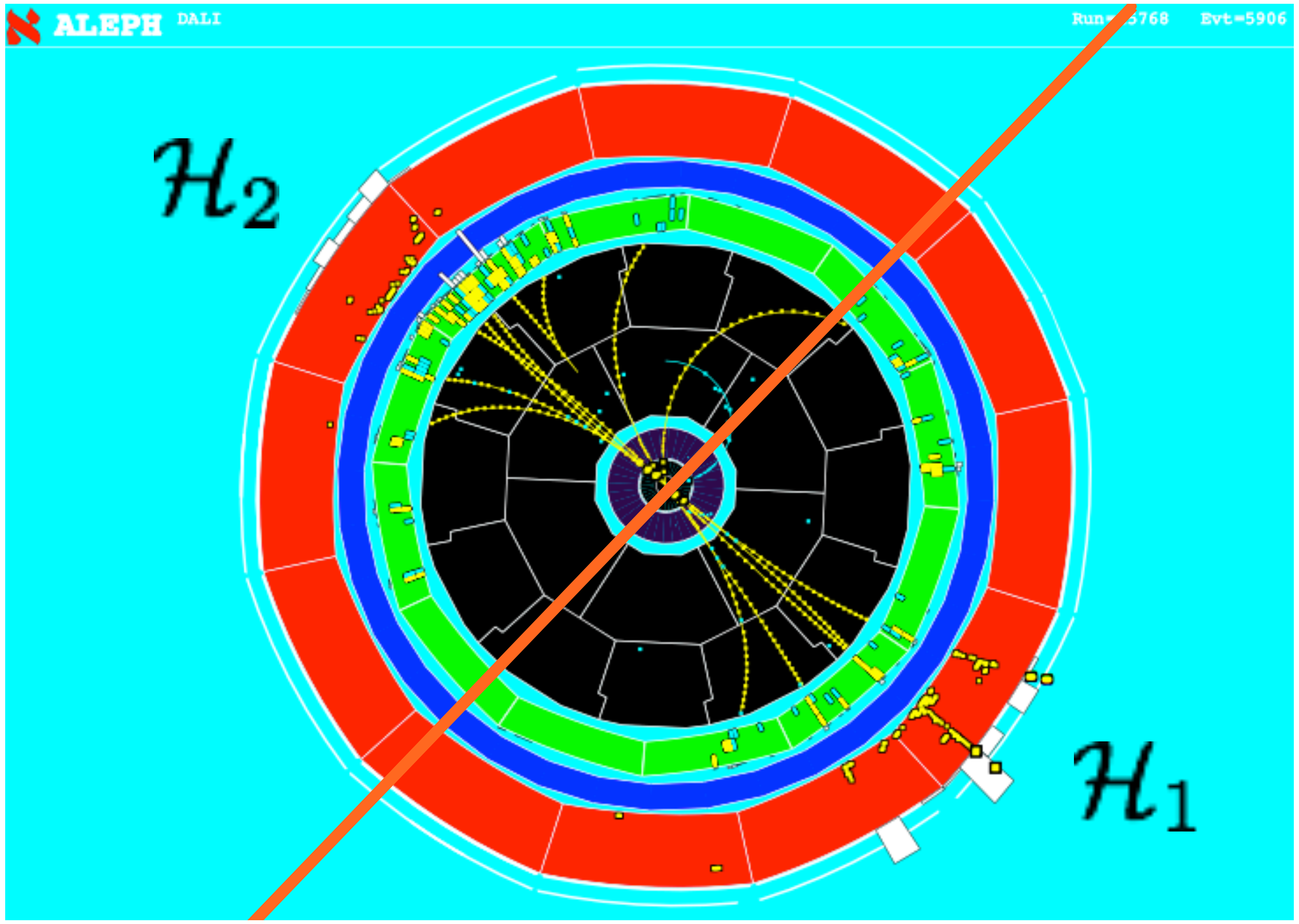
Exercise. Behaviour of $1 - T(\{\tilde{p}\}, k)$ in the soft-collinear region

$$1 - T(k_1, \ldots, k_n) = \sum_i \frac{k_{ti}}{Q} e^{-|\eta_i|} + \sum_{\ell=1,2} \frac{1}{Q^2} \frac{\left|\sum_{i \in \mathcal{H}_\ell} \vec{k}_{ti}\right|^2}{1 - \sum_{i \in \mathcal{H}_\ell} z_i}$$

- Soft and collinear

- Soft large-angle

- Hard collinear



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- Soft and collinear

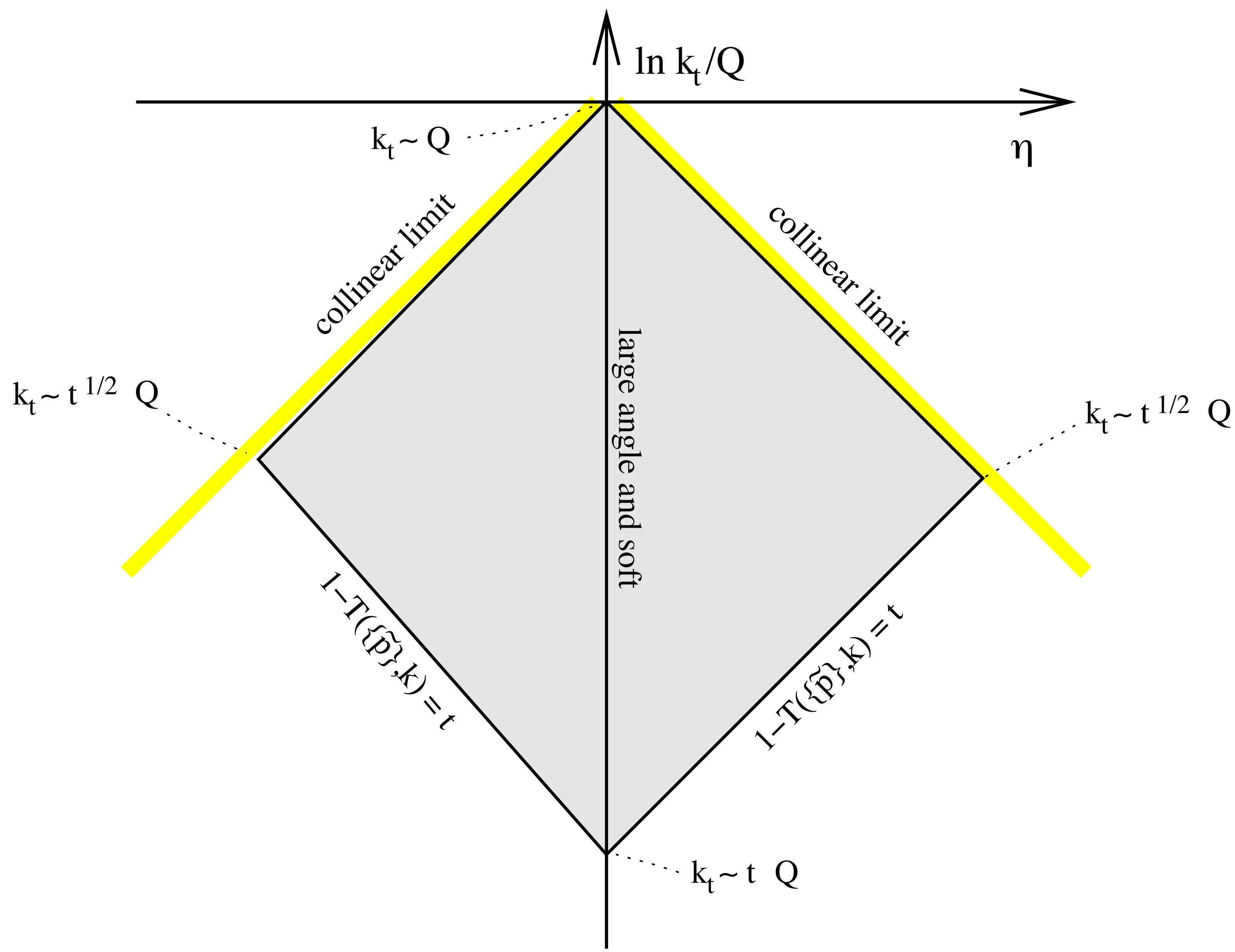
$$1 - T(\{\tilde{p}\}, k) \simeq \frac{k_t}{Q} e^{-|\eta|}$$

- Soft large-angle

$$1 - T(\{\tilde{p}\}, k) \sim k_t$$

- Hard collinear

$$1 - T(\{\tilde{p}\}, k) \sim k_t^2$$



An IRC safe final-state observable in the Lund plane

For a single soft-collinear emission k, an IRC safe final-state observable behaves as follows

- Soft and collinear to leg $\ell = 1, 2$

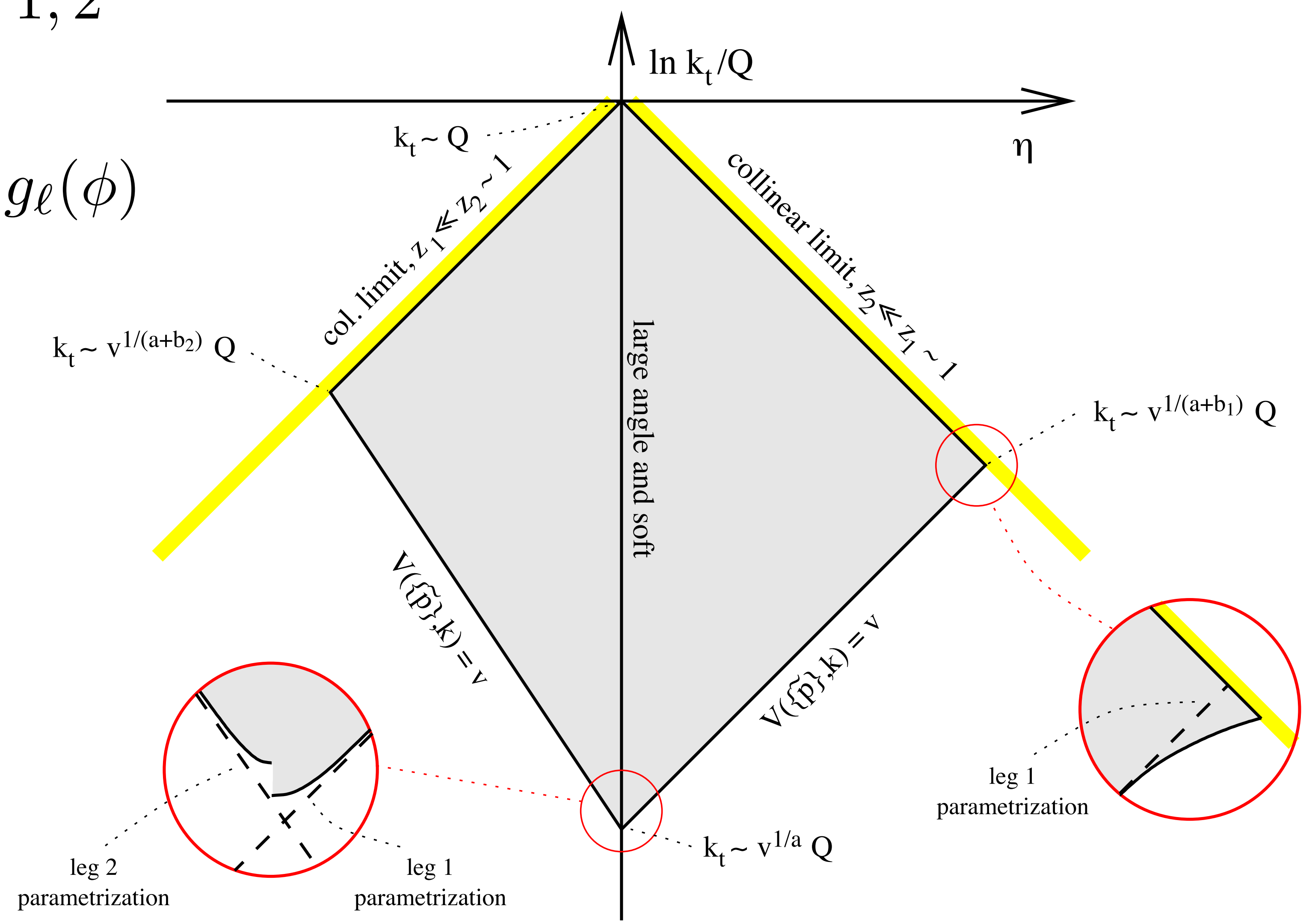
$$V(\{\tilde{p}\}, k) \simeq d_\ell \left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta^{(\ell)}} g_\ell(\phi)$$

- Soft large-angle

$$V(\{\tilde{p}\}, k) \sim k_t^a$$

- Hard and collinear to leg ℓ

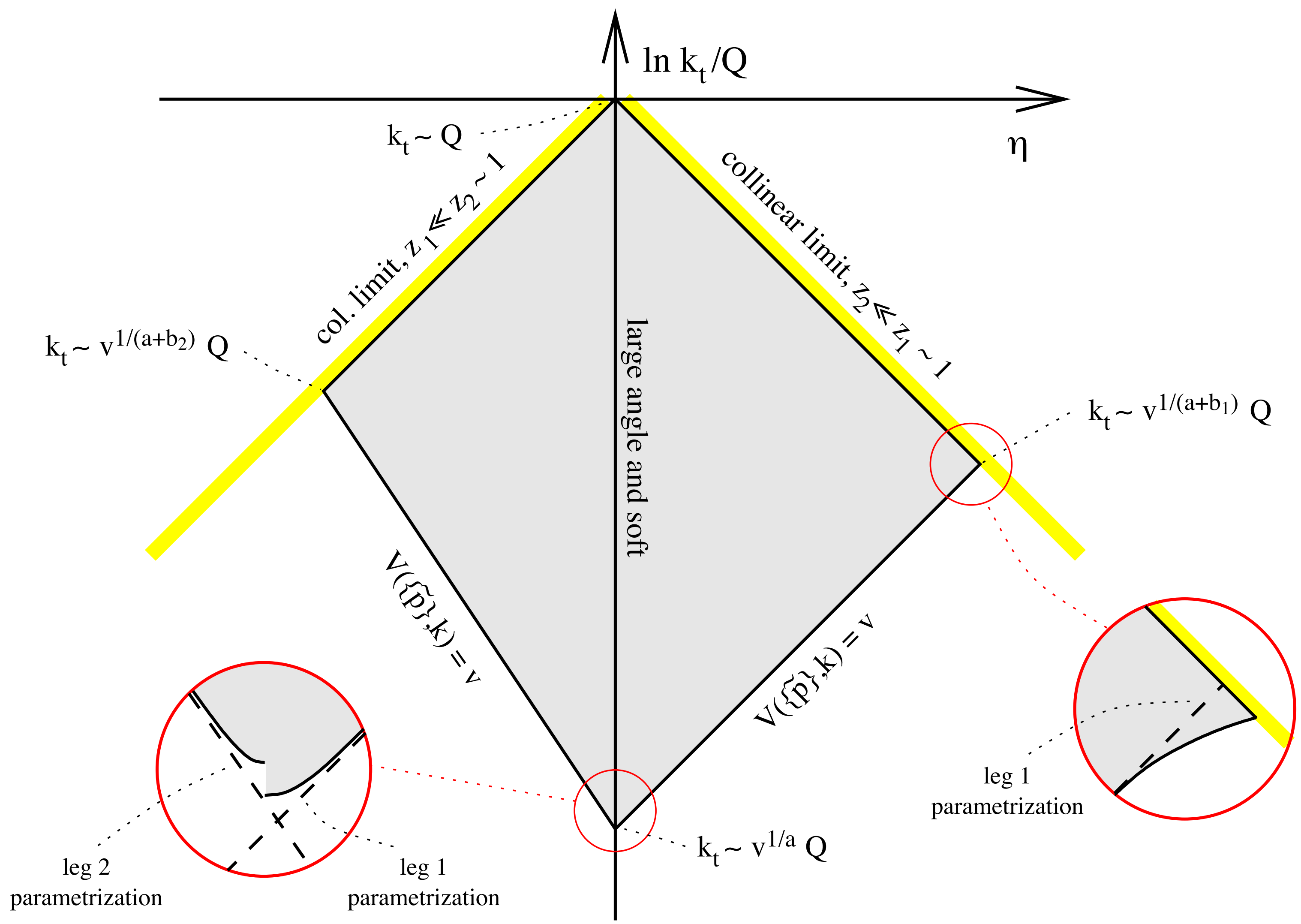
$$V(\{\tilde{p}\}, k) \sim k_t^{a+b_\ell}$$



Cumulative distributions in the Lund plane

For one emission, the cumulative distribution of any final-state observable is

$$\Sigma(v) = 1 - \int [dk] M^2(k)\, \Theta(V(\{\tilde{p}\}, k) - v)$$

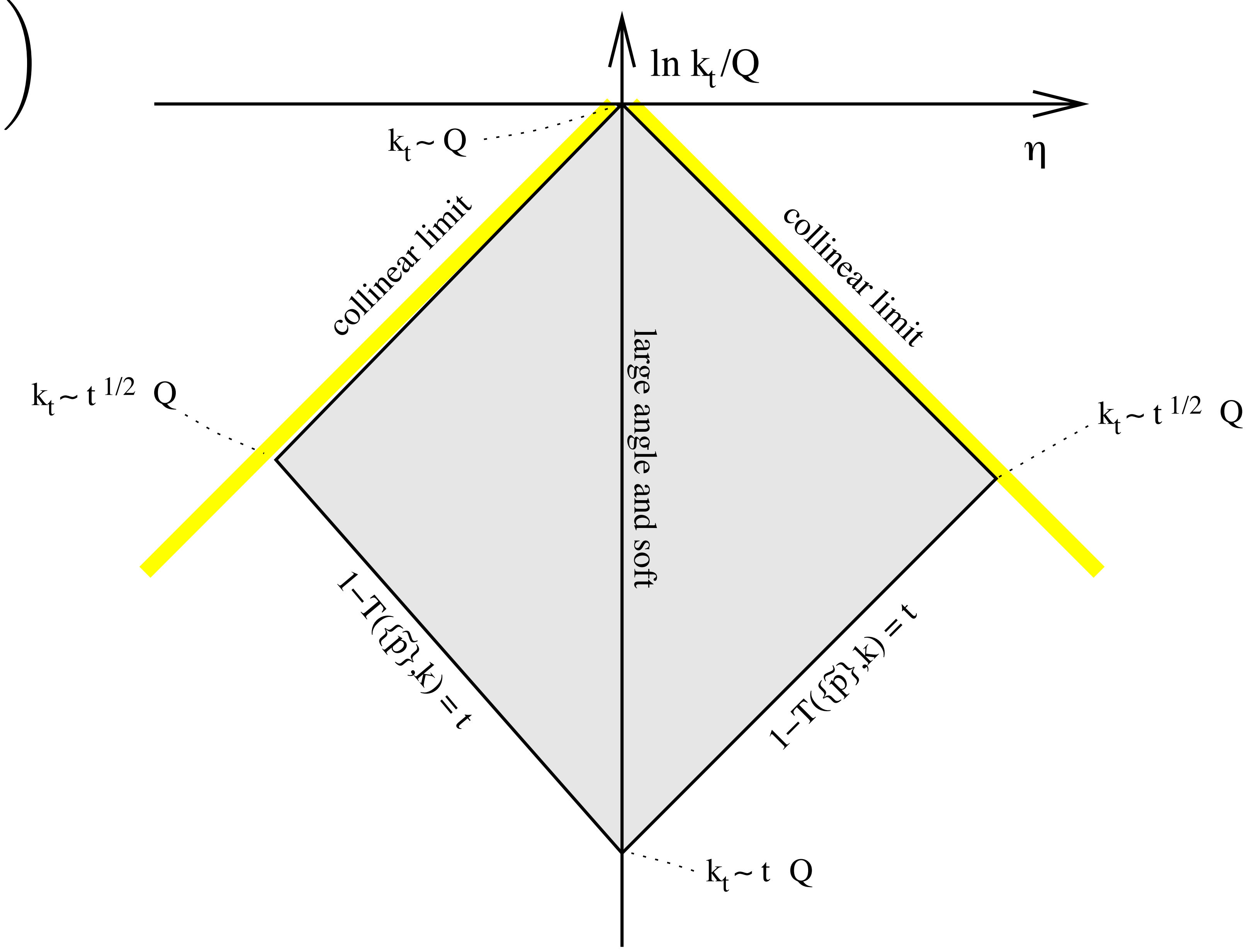


The first order contribution to $\Sigma(v)$ is just the area of the shaded region

Thrust distribution in the Lund plane

Exercise. Show that the thrust distribution is given by

$$\Sigma(\tau) = 1 - C_F \frac{\alpha_s}{\pi} \ln^2\left(\frac{1}{\tau}\right)$$



Thrust distribution in the Lund plane

Exercise. Show that the thrust distribution is given by

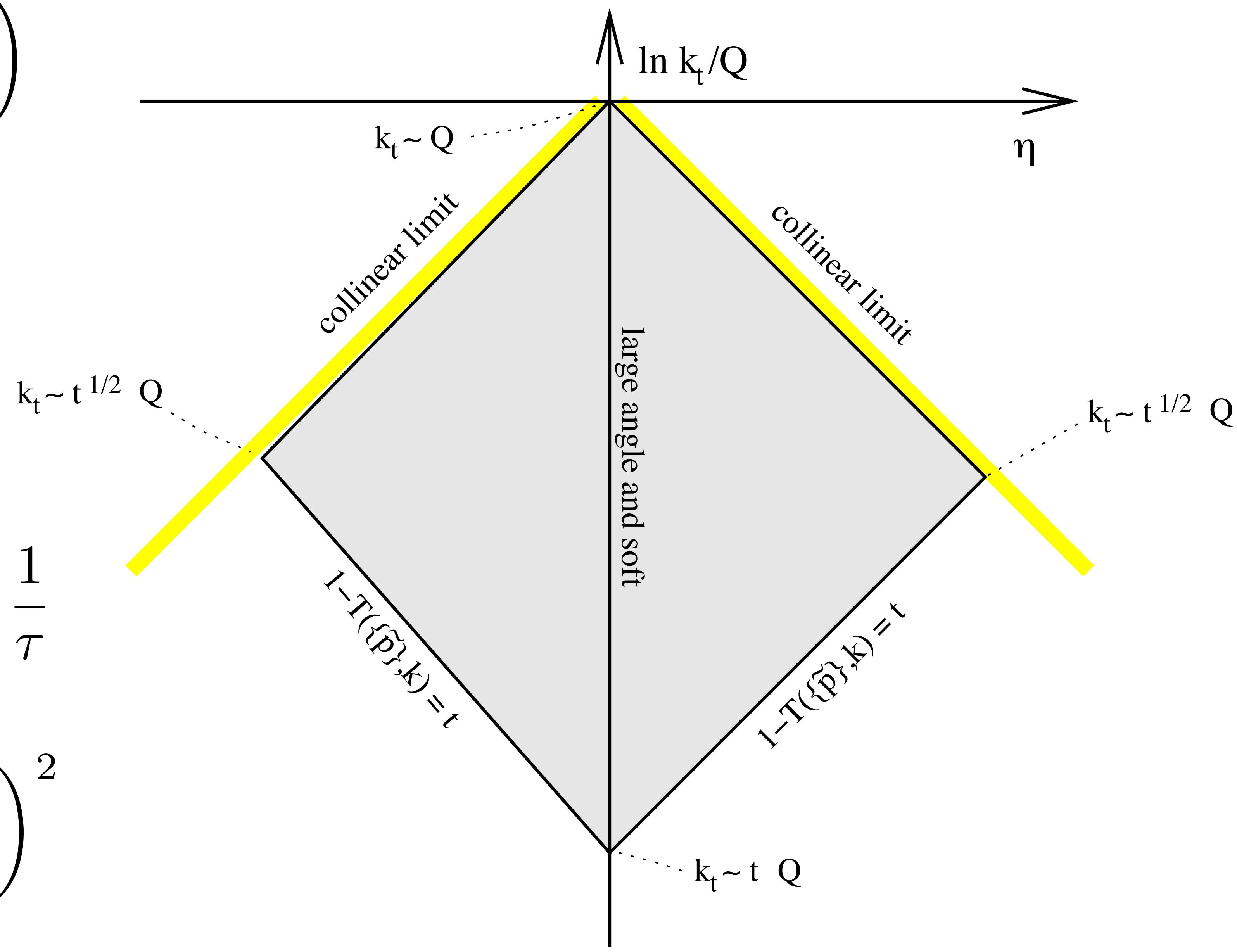
$$\Sigma(\tau) = 1 - C_F \frac{\alpha_s}{\pi} \ln^2 \left(\frac{1}{\tau}\right)$$

Solution.

$$[dk]M^2(k) \simeq 2C_F \frac{\alpha_s}{\pi} \frac{dk_t}{k_t} d\eta$$

$$\text{Area} = \left(\frac{1}{\sqrt{2}} \ln \frac{1}{\tau}\right)^2 = \frac{1}{2} \ln^2 \frac{1}{\tau}$$

$$\begin{aligned}\Sigma(\tau) &= 1 - 2\frac{\alpha_s}{\pi} C_F \left(\frac{1}{2} \ln^2 \frac{1}{\tau}\right)^2 \\ &= 1 - C_F \frac{\alpha_s}{\pi} \ln^2 \frac{1}{\tau}\end{aligned}$$



Radiator at single-logarithmic accuracy

The one-gluon contribution to $\Sigma(v)$ is conveniently written in terms of a radiator, in which one includes running coupling effects

$$\Sigma(v) = 1 - R(v) \quad \Rightarrow \quad R(v) = \int [dk] M^2(k) \Theta(V(\{\tilde{p}\}, k) - v)$$

For two quark legs, as in e^+e^- annihilation

$$R(v) = \sum_{\ell=1,2} C_F \int^{Q^2} \frac{dk_t^2}{k_t^2} \frac{\alpha_s^{\mathrm{CMW}}(k_t)}{2\pi} \int_0^{\ln(Q/k_t)} d\eta \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} z p_{gq}(z) \Theta\left(d_\ell \left(\frac{k_t}{Q}\right)^a e^{-b_\ell \eta} g_\ell(\phi) - v\right)$$

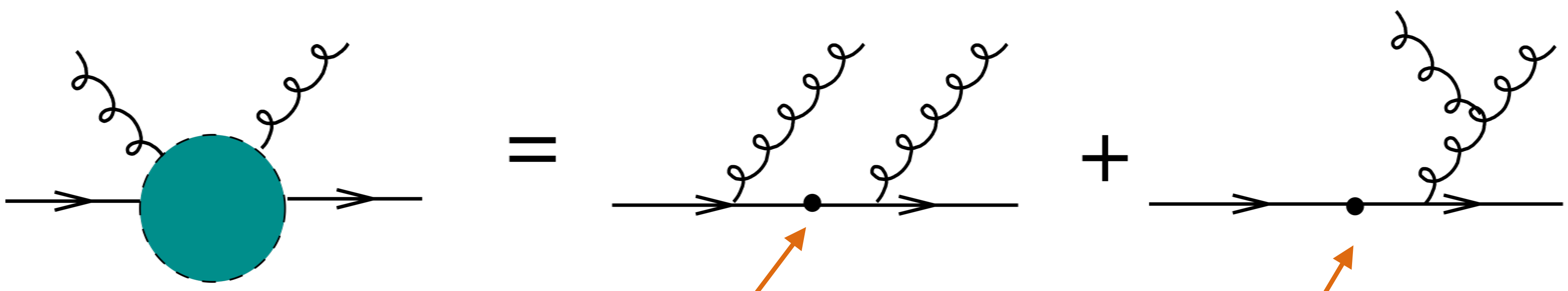
The coupling is to be evaluated in the Catani-Marchesini-Webber scheme

$$\alpha_s^{\mathrm{CMW}} = \alpha_s^{\overline{\mathrm{MS}}} \left(1 + K \frac{\alpha_s}{2\pi}\right) \qquad K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{5}{9} n_f$$

Exercise.* derive the expressions in section 2.1.3 of hep-ph/0407286

Double soft-collinear emission

Consider the most singular case of two soft gluons strongly ordered in energy



independent emission

correlated emission, singular only when the gluons are close in angle

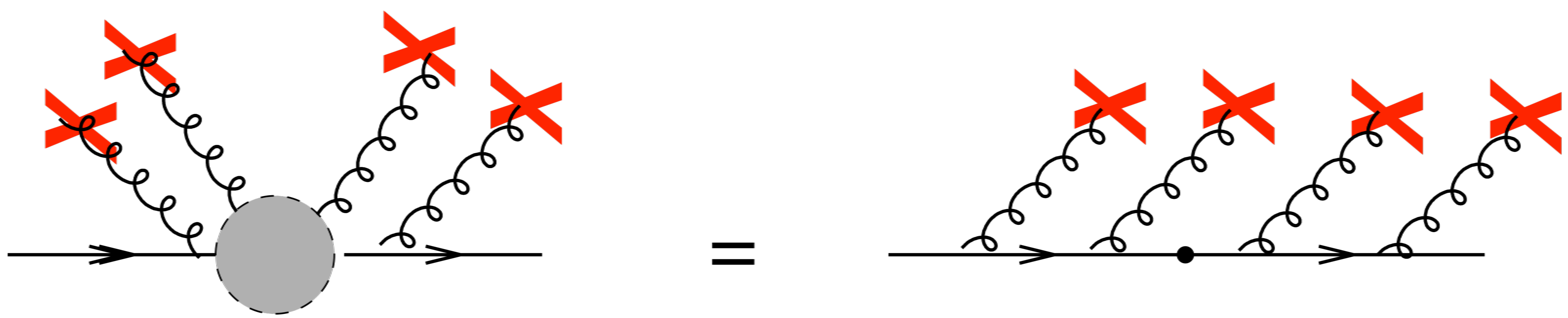
$$M^2(k_1,k_2) = M^2(k_1)M^2(k_2) \\ + C_A C_F \boxed{\left[\frac{(p_1p_2)}{(p_1k_1)(k_1k_2)(k_2p_2)} + \frac{(p_1p_2)}{(p_2k_1)(k_1k_2)(k_2p_1)} - \frac{(p_1p_2)}{(p_1k_1)(k_1p_2)}\frac{(p_1p_2)}{(p_1k_2)(k_2p_2)}\right]}$$

For gluons widely separated in angle, only independent emission survives

Multiple soft-collinear emissions

We first neglect correlated emission. Then the multi-gluon matrix element factorises into the product of single emission matrix elements

$$M^2(k_1, k_2, \ldots, k_n) \simeq M^2(k_1)M^2(k_2)\ldots M^2(k_n)$$



The all-order cumulative distribution $\Sigma(v)$ becomes

$$\Sigma(v) = e^{-\int [dk] M^2(k)} \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_i [dk_i] M^2(k_i)\, \Theta(v - V(\{\tilde{p}\}, k_1, \ldots, k_n))$$

virtual corrections, ensure that the inclusive sum over emissions gives one

Leading logarithmic resummation

Suppose that the emissions are strongly ordered

$$V(\{\tilde{p}\}, k_1) \gg V(\{\tilde{p}\}, k_2) \gg \cdots \gg V(\{\tilde{p}\}, k_n)$$

Assume also that the value of the observable is dominated by $V(\{\tilde{p}\}, k_1)$

$$V(\{\tilde{p}\}, k_1, \dots, k_n) \sim V(\{\tilde{p}\}, k_1) \quad \Rightarrow \quad \Theta(v - V(\{\tilde{p}\}, k_1, \dots, k_n)) = \prod_{i=1}^{n} \Theta(v - V(\{\tilde{p}\}, k_i))$$

$$\Sigma(v) = e^{-\int [dk] M^2(k)} \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_i [dk_i] M^2(k_i)\, \Theta(v - V(\{\tilde{p}\}, k_i)) = e^{-R(v)}$$

If the observable's value is dominated by the "hardest" emission, in the strongly ordered regime the cumulative distribution is obtained by the exponentiation of the contribution of a single gluon (Sudakov form factor)

Non-exponentiating leading logarithms

In the case of the JADE jet algorithm, double logarithms do not exponentiate

$$\Sigma(y_{\mathrm{cut}}) = 1 - \frac{C_F\alpha_s}{\pi}\ln^2\left(\frac{1}{y_{\mathrm{cut}}}\right) + \frac{1}{2!}\times\frac{5}{6}\times\left(\frac{C_F\alpha_s}{\pi}\ln^2\left(\frac{1}{y_{\mathrm{cut}}}\right)\right)^2$$

We try to identify what can go wrong

$$\Sigma(v) = \underbrace{e^{-R(v)}}_{\text{leading log exponentiation}}\left\{e^{-\int^v [dk]M^2(k)}\sum_{n=0}^{\infty}\frac{1}{n!}\int\prod_i \underbrace{[dk_i]M^2(k_i)\,\Theta(v-V(k_1,\ldots,k_n))}_{\text{corrections}}\right\}$$

We need to enforce that the corrections do not give double logarithms

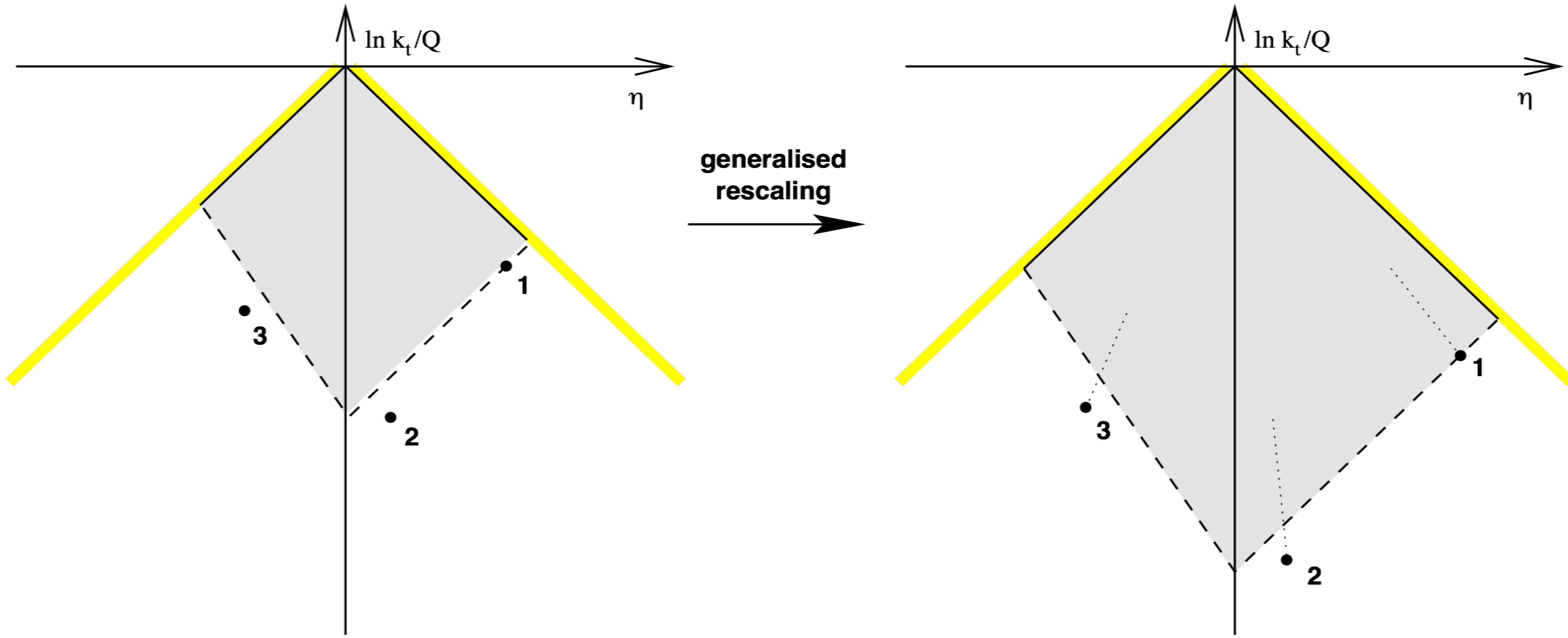
Recursive IRC safety: condition 1

Recursive infrared and collinear (rIRC) safety conditions are conditions we put on the observable, so that we have no surprises from multiple emissions

1. Let's scale all $V(\{\tilde{p}\}, k_i) \equiv v\zeta_i$,the observable should scale the same way

$$\lim_{v \to 0} \frac{V(\{\tilde{p}\}, k_1[v\zeta_1], \ldots, k_n[v\zeta_n])}{v} = \text{finite, and non-zero}$$

$$V(\{\tilde{p}\}, k_1) \sim V(\{\tilde{p}\}, k_2) \sim \cdots \sim V(\{\tilde{p}\}, k_n) \Rightarrow V(\{\tilde{p}\}, k_1, \ldots, k_n) \sim V(\{\tilde{p}\}, k_1)$$



Recursive IRC safety: condition 2a

Recursive infrared and collinear (rIRC) safety conditions are conditions we put on the observable, so that we have no surprises from multiple emissions

2a. Let's make emission k_{n+1} softer, or more collinear than the others

$$\zeta_{n+1} \ll \zeta_1 \sim \zeta_2 \sim \cdots \sim \zeta_n$$

$$\lim_{\zeta_{n+1}\to 0} \lim_{v\to 0} \frac{V(\{\tilde{p}\}, k_1[v\zeta_1], \ldots, k_n[v\zeta_n], k_{n+1}[v\zeta_{n+1}])}{v} = \lim_{v\to 0} \frac{V(\{\tilde{p}\}, k_1[v\zeta_1], \ldots, k_n[v\zeta_n])}{v}$$

$$V(\{\tilde{p}\}, k_1) \gg V(\{\tilde{p}\}, k_2) \gg \cdots \gg V(\{\tilde{p}\}, k_n) \Rightarrow V(\{\tilde{p}\}, k_1, \ldots, k_n) \sim V(\{\tilde{p}\}, k_1)$$

Exercise. With soft and collinear emissions, the thrust is

$$1 - T(\{\tilde{p}\}, k_1, \ldots, k_n) \simeq \sum_{i=1}^{n} \frac{k_{ti}}{Q} e^{-|\eta_i|}$$

show that the thrust satisfies both condition 1 and condition 2a

Exercise.* Show that the JADE three-jet resolution fails either 1 or 2a

Recursive IRC safety conditions 1 and 2a

Exercise. With soft and collinear emissions, the thrust is

$$1 - T(\{\tilde{p}\}, k_1, \ldots, k_n) \simeq \sum_{i=1}^{n} \frac{k_{ti}}{Q} e^{-|\eta_i|}$$

show that the thrust satisfies both condition 1 and condition 2a

Solution. The thrust is an additive observable

$$1 - T(\{\tilde{p}\}, k_1, \ldots, k_n) \simeq \sum_{i=1}^{n} \frac{k_{ti}}{Q} e^{-|\eta_i|} = \sum_{i=1}^{n} [1 - T(\{\tilde{p}\}, k_i)] = \sum_{i=1}^{n} \tau \zeta_i$$

Check of condition 1

$$\lim_{\tau \to 0} \frac{1 - T(\{\tilde{p}\}, k_1, \ldots, k_n)}{\tau} = \lim_{\tau \to 0} \frac{\sum_{i=1}^{n} \tau \zeta_i}{\tau} = \sum_{i=1}^{n} \zeta_i$$

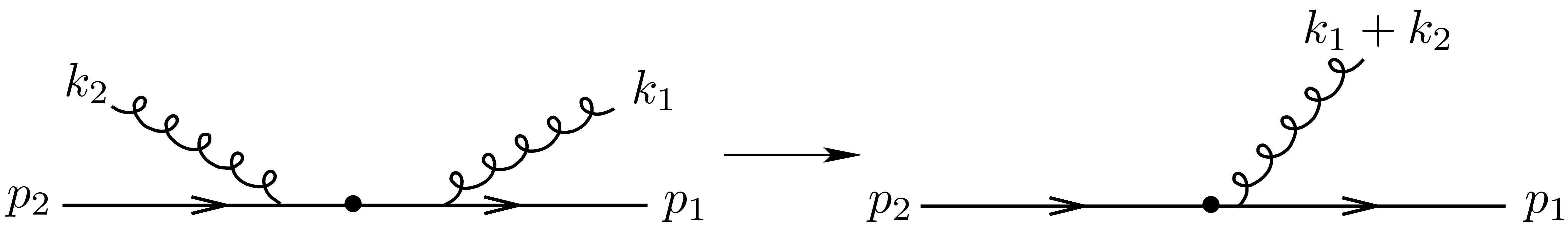
Check of condition 2a

$$\lim_{\zeta_{n+1} \to 0} \lim_{\tau \to 0} \frac{1 - T(\{\tilde{p}\}, k_1, \ldots, k_n, k_{n+1})}{\tau} = \lim_{\zeta_{n+1} \to 0} \sum_{i=1}^{n+1} \zeta_i = \sum_{i=1}^{n} \zeta_i$$

Failure of recursive IRC condition 1

Exercise.* Show that the JADE three-jet resolution fails either 1 or 2a

Solution. The distance measure of the JADE algorithm is the invariant mass of two partons. The JADE can cluster together two soft and collinear gluons belonging to different hemispheres



$$y_{k_1 p_1} = y_3(\{\tilde{p}\}, k_1) = \frac{(k_1 + p_1)^2}{Q^2} \simeq \frac{k_{t1}}{Q} e^{-\eta_1} = y_{\rm cut}$$

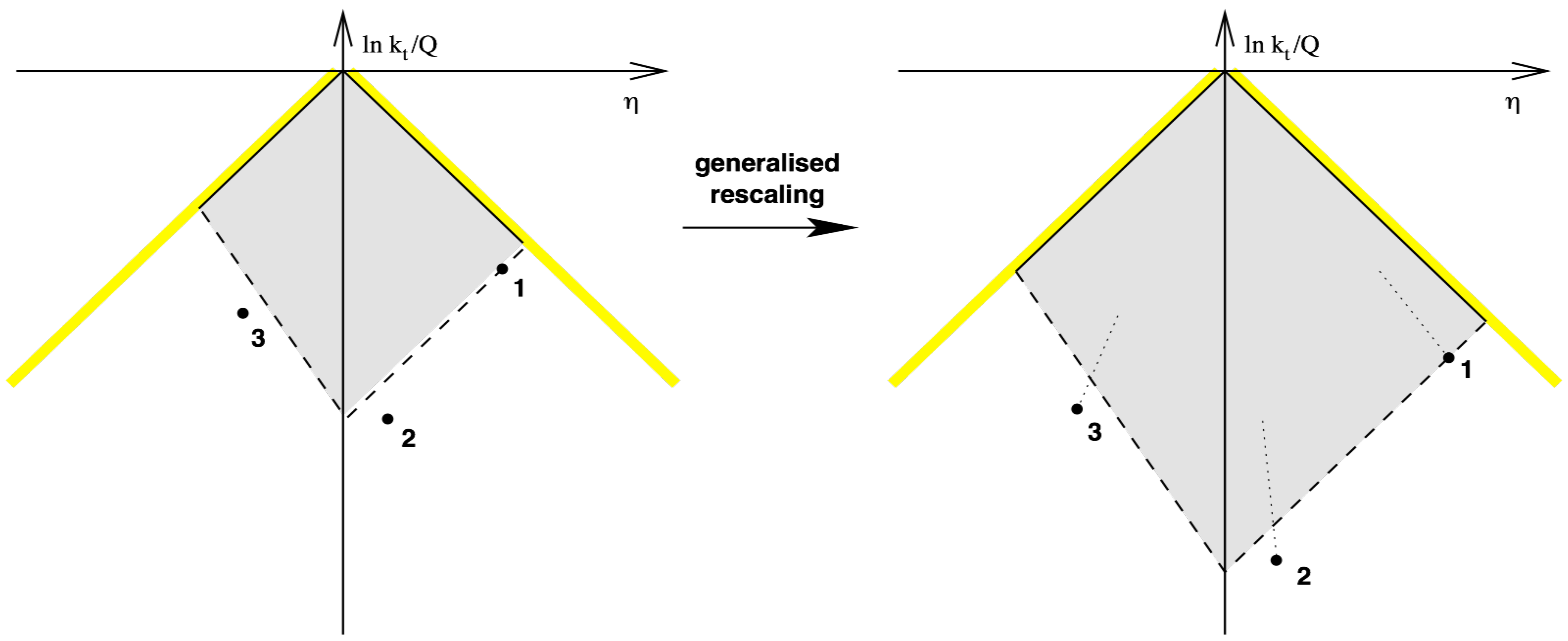
$$y_{k_2 p_2} = y_3(\{\tilde{p}\}, k_2) = \frac{(k_2 + p_2)^2}{Q^2} \simeq \frac{k_{t2}}{Q} e^{+\eta_2} = y_{\rm cut}$$

$$y_{k_1 k_2} = \frac{(k_1 + k_2)^2}{Q^2} \simeq \frac{k_{t1} k_{t2}}{Q^2} \left[\cosh(\eta_1 - \eta_2) - \cos(\phi_1 - \phi_2)\right] \simeq \frac{k_{t1} k_{t2}}{Q^2} e^{\eta_1 - \eta_2}$$

Failure of recursive IRC condition 1

Exercise.* Show that the JADE three-jet resolution fails either 1 or 2a

Solution. Subtlety: when performing the rescaling, the rapidity fraction of each emission with respect to the total available rapidity stays fixed!



$$\eta_1 = \xi_1 \eta_{\max} = \xi_1 \ln \frac{1}{\sqrt{y_{\rm cut}}}$$

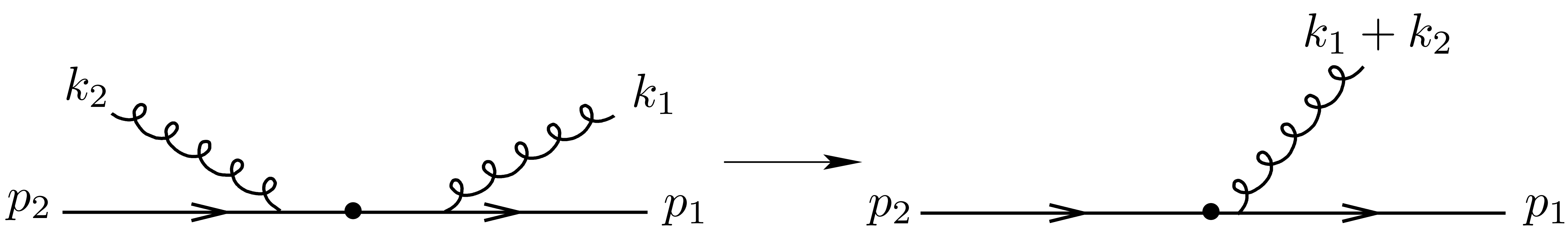
$$\Rightarrow \qquad e^{\eta_1 - \eta_2} = \left(\sqrt{y_{\rm cut}}\right)^{\xi_1 + \xi_2}$$

$$\eta_2 = -\xi_2 \eta_{\max} = -\xi_2 \ln \frac{1}{\sqrt{y_{\rm cut}}}$$

Failure of recursive IRC condition 1

Exercise.* Show that the JADE three-jet resolution fails either 1 or 2a

Solution. Recombination occurs if $y_{k_1k_2} < y_{k_1p_1} = y_{k_2p_2} = y_{\mathrm{cut}}$



$$y_{k_1k_2} \simeq \frac{k_{t1}k_{t2}}{Q^2} e^{\eta_1-\eta_2} = \frac{k_{t1}e^{-\eta_1}}{Q} \frac{k_{t1}e^{\eta_2}}{Q} e^{2(\eta_1-\eta_2)} = y_{\mathrm{cut}}^{2-\xi_1-\xi_2}$$

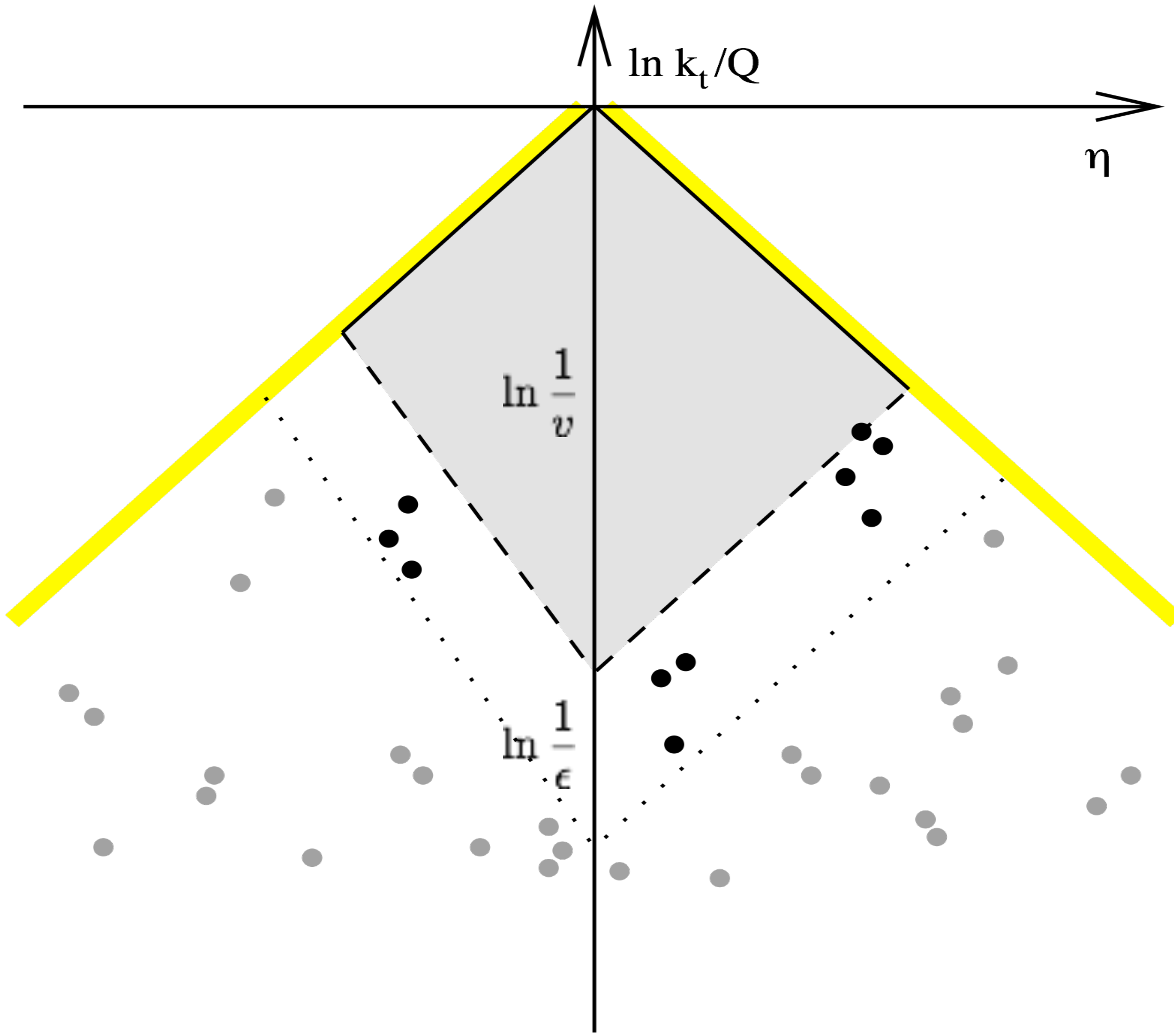
$$y_{k_1k_2} < y_{k_1p_1} = y_{k_2p_2} = y_{\mathrm{cut}} \quad \Leftrightarrow \quad \xi_1 + \xi_2 < 1$$

If recombination occurs, rIRC safety condition 1 fails

$$\lim_{y_{\mathrm{cut}}\to 0} \frac{y_3(\{\tilde{p}\}, k_1, k_2)}{y_{\mathrm{cut}}} = \lim_{y_{\mathrm{cut}}\to 0} y_{\mathrm{cut}}^{1-\xi_1-\xi_2} = 0$$

rIRC safety in the Lund plane

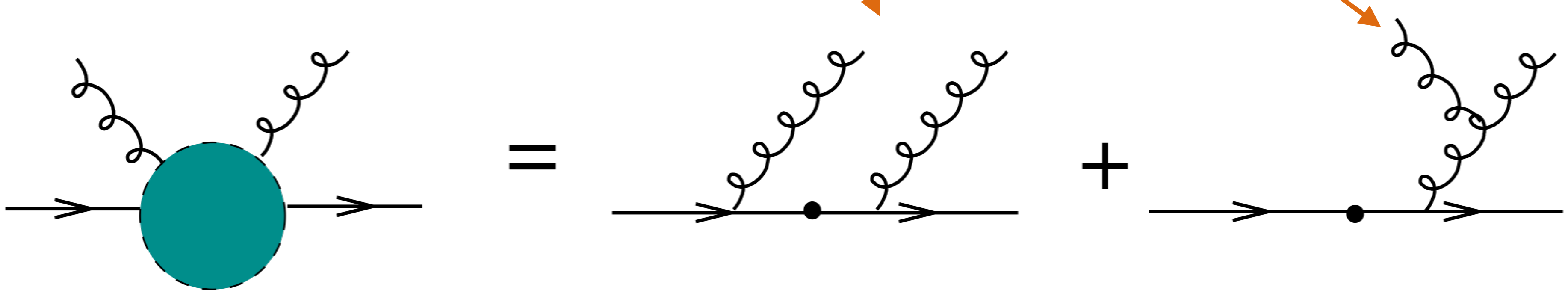
An immediate consequence of the rIRC safety conditions is that we can neglect all emissions with $V(\{\tilde{p}\}, k_i) < \epsilon v$,where $\epsilon \gg v$



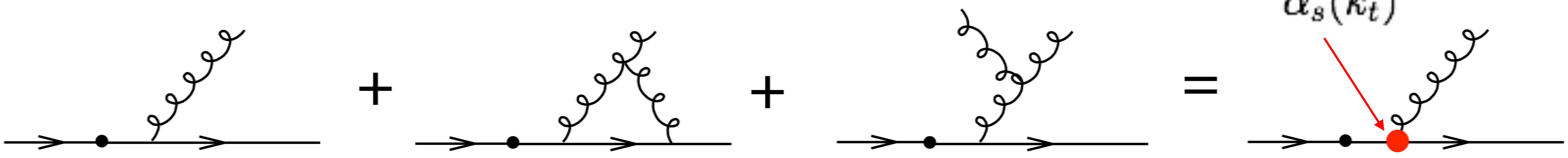
Two-gluon correlated emission

The two-gluon matrix element can be always written as the sum of an independent and correlated emission part

$$M^2(k_1,k_2) = M^2(k_1)M^2(k_2) + \tilde{M}^2(k_1,k_2)$$



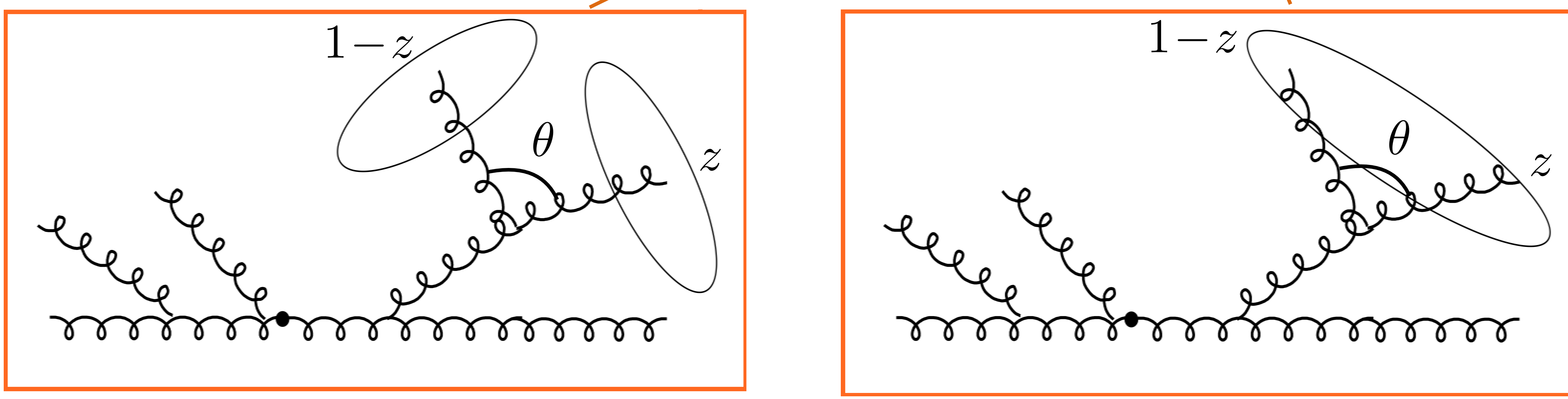
The correlated emission part, if integrated inclusively, is combined with the one-loop one-gluon matrix element to give the running coupling



Two-gluon correlated emission

The remainder after the extraction of the running coupling is

$$\int_{\epsilon v}[dk_1]\int_{\epsilon v}[dk_2]\tilde{M}^2(k_1,k_2)\left[\Theta(v-V(\{\tilde{p}\},k_1,k_2))-\Theta(v-V(\{\tilde{p}\},k_1+k_2))\right]$$



Example: in a jet-rate, the two gluons are clustered in different jets

$$\int_{\epsilon v}[dk]M^2(k)\times\left[C_A\int\frac{d\theta^2}{\theta^2}\int\frac{dz}{z(1-z)}\frac{\alpha_s[z(1-z)\theta k_t]}{2\pi}\right]$$

potential source of single logarithms (why not double?)

Recursive IRC safety: condition 2b

We require a further condition with respect to collinear splittings

2b Consider the splitting $k \to k_a k_b$ with

$$\mu^2 \equiv \frac{(k_a + k_b)^2}{k_t^2} \sim z(1-z)\theta^2 \qquad \lim_{\mu \to 0} \{k_a, k_b\}[v\zeta, \mu] = k[v\zeta]$$

$$\lim_{\mu \to 0} \lim_{v \to 0} \frac{V(\{\tilde{p}\}, \{k_a, k_b\}[v\zeta, \mu], k_1, \ldots, k_n)}{v} = \lim_{v \to 0} \frac{V(\{\tilde{p}\}, k[v\zeta], k_1, \ldots, k_n)}{v}$$

Notice the order of the limits, first you rescale the observable, and then you take the collinear limit. If you take the limit in reverse order, the result is trivial because the observables we consider are all collinear safe.

Exercise. With two collinear emissions only, show that the thrust satisfies rIRC safety condition 2b

Recursive IRC safety: condition 2b

Exercise. With two collinear emissions only, show that the thrust satisfies rIRC safety condition 2b

Solution. The collinear emissions are in the same hemisphere

$$1 - T(\{\tilde{p}\}, k_1, k_2) = \frac{(k_a + k_b)^2}{Q^2} \to \frac{k_t}{Q} e^{-|\eta|} = \tau\zeta \qquad \mu^2 \to 0$$

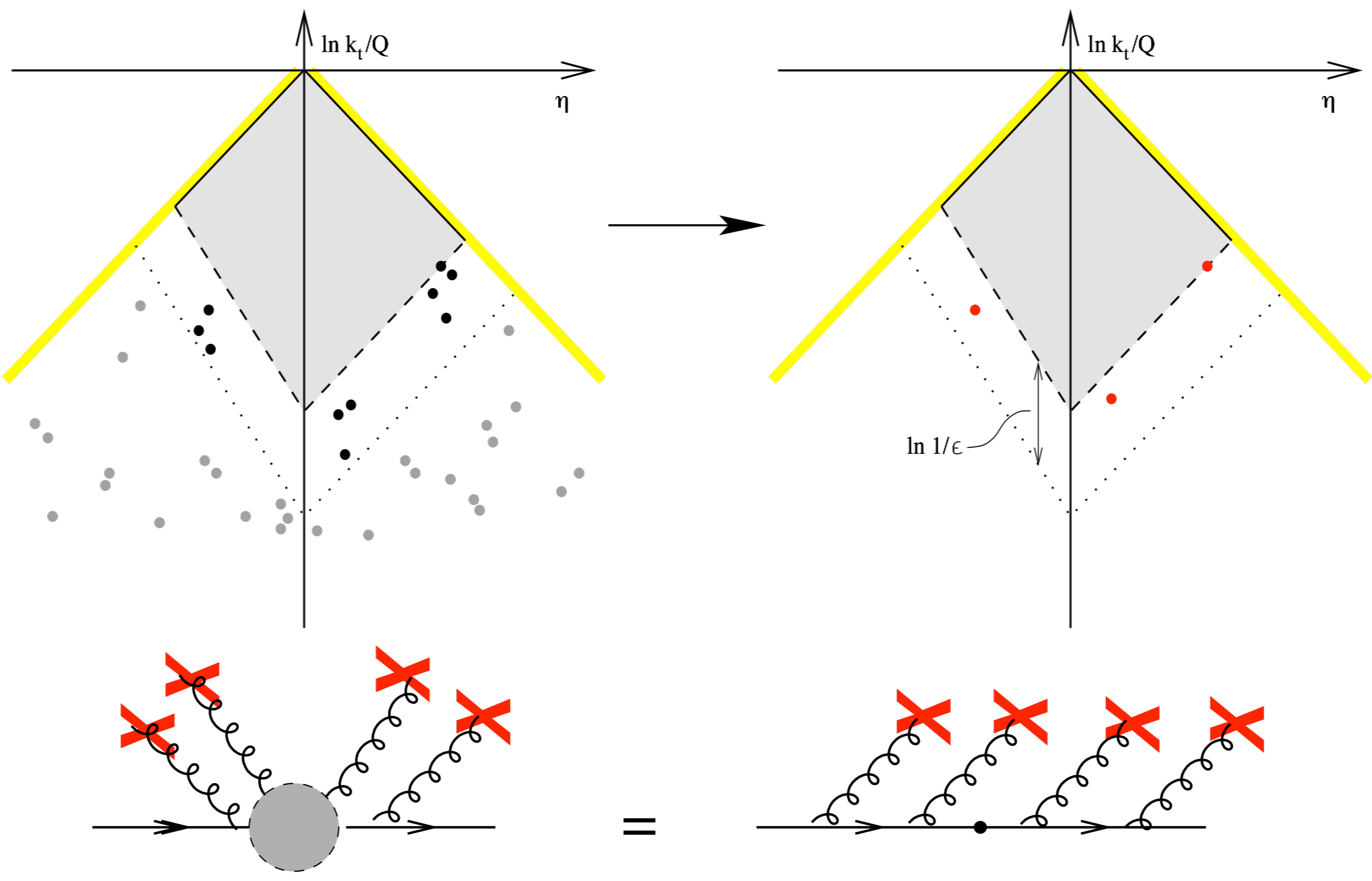
The parent gluon k has a mass squared $m^2 = (k_a + k_b)^2 = \mu^2 k_t^2$

$$1 - T(\{\tilde{p}\}, k_a, k_b) \simeq \frac{\sqrt{k_t^2 + m^2}}{Q} e^{-|\eta|} = \frac{k_t}{Q} e^{-|\eta|} \sqrt{1 + \mu^2} = \tau\zeta\sqrt{1 + \mu^2}$$

$$\lim_{\mu^2 \to 0} \lim_{\tau \to 0} \frac{1 - T(\{\tilde{p}\}, k_a, k_b)}{\tau} = \lim_{\mu^2 \to 0} \zeta\sqrt{1 + \mu^2} = \zeta = \lim_{\tau \to 0} \frac{1 - T(\{\tilde{p}\}, k)}{\tau}$$

rIRC safety 2b in the Lund plane

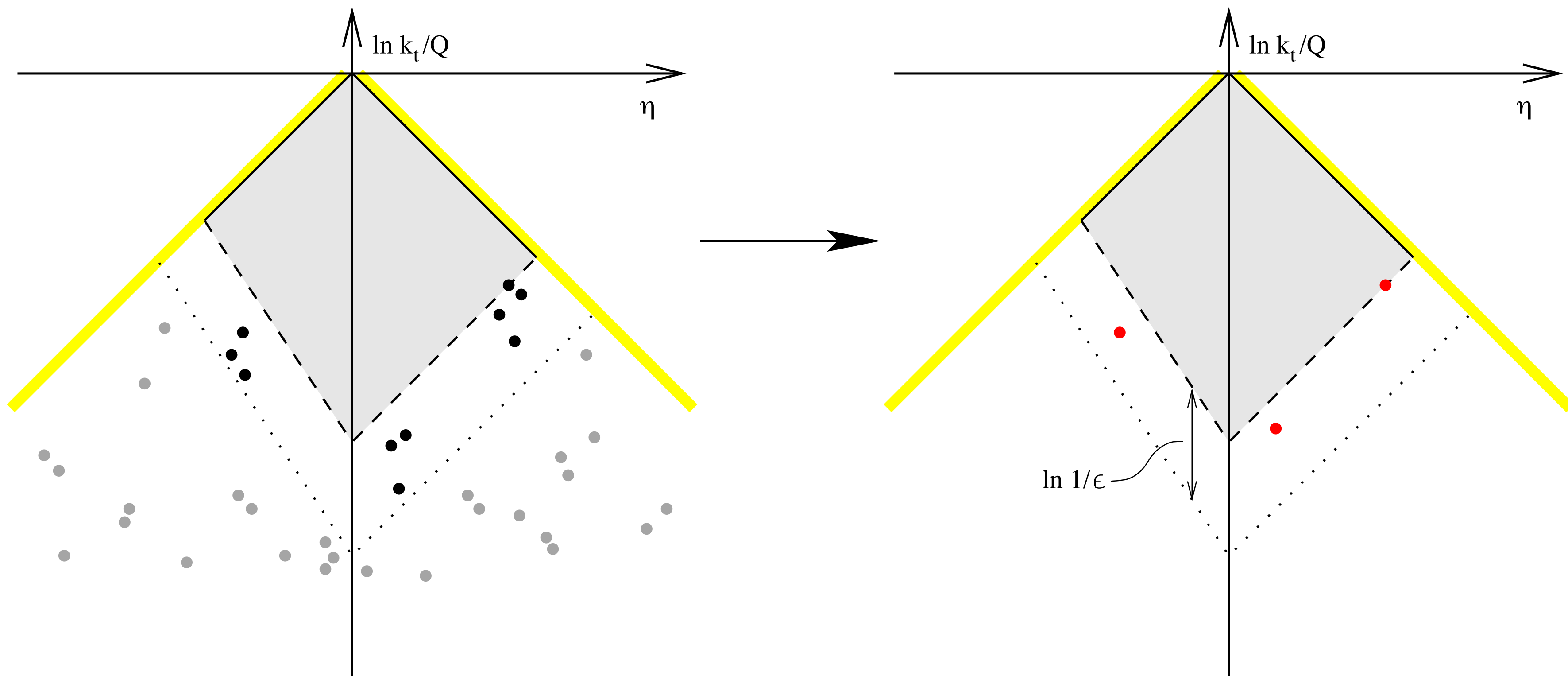
Clustering emissions close in rapidity does not produce extra logarithms

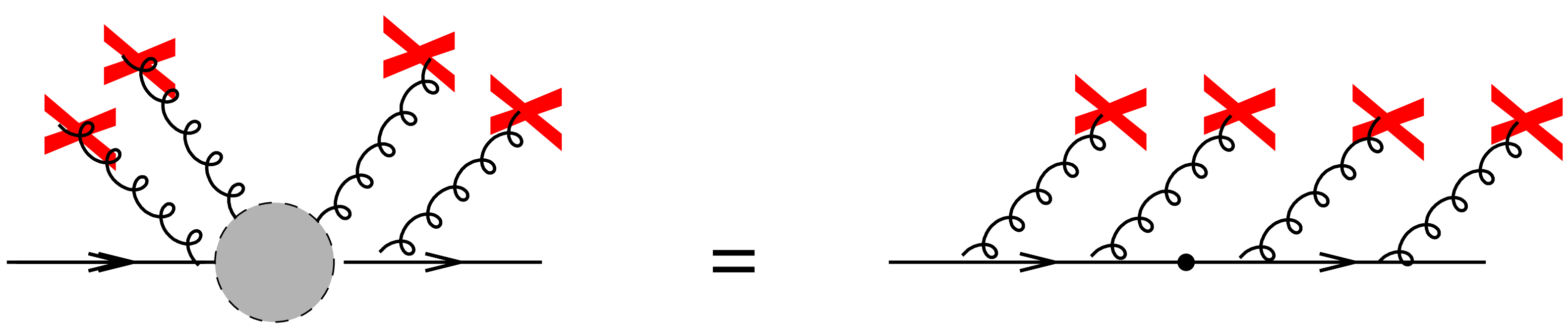


The relevant emissions are soft and collinear, widely separated in angle, and in a strip of size $\ln v \times \ln \epsilon$: this is a line, i.e. a single logarithmic contribution

NLL resummation for rIRC safe observables

At last, we can write the NLL formula for a rIRC safe final-state observable



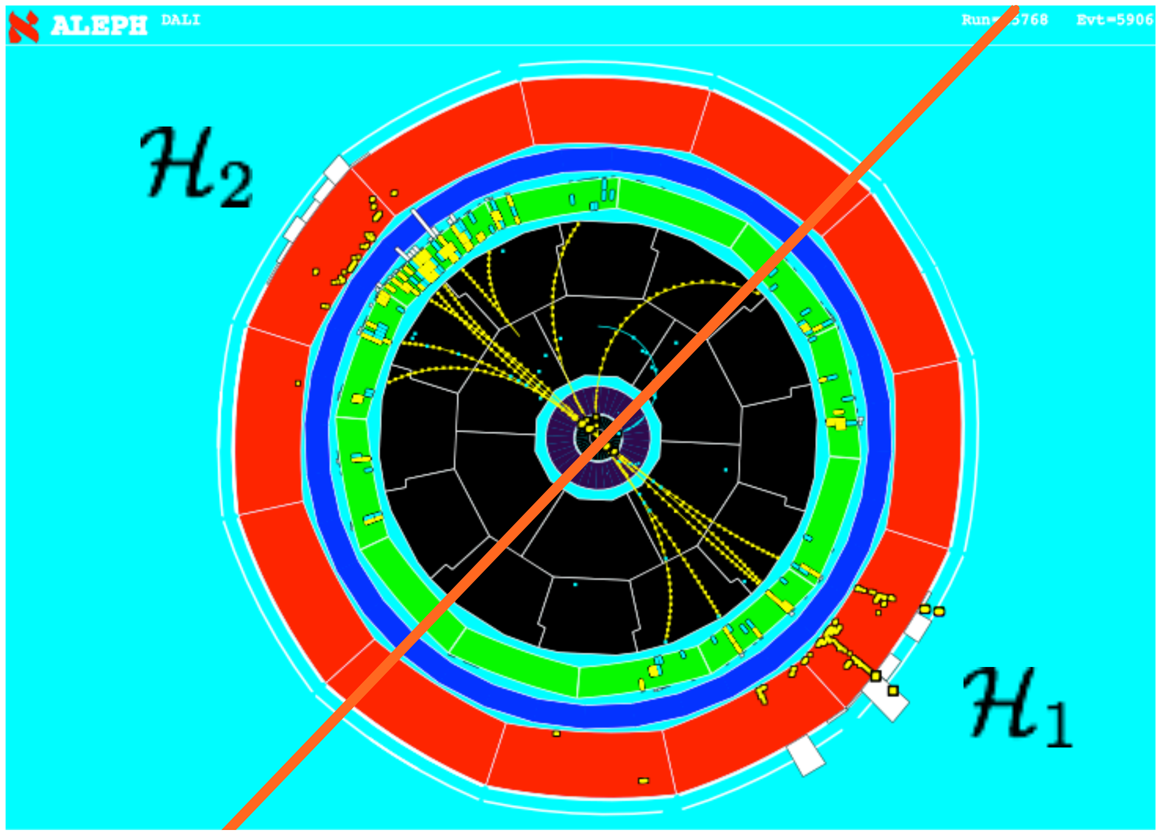


$$R'(v) = -v\frac{dR(v)}{dv}$$

$$\Sigma(v) = e^{-R(v)} \underbrace{\left\{ e^{-\int_{\epsilon v}^{v}[dk]M^2(k)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\epsilon v} \prod_i [dk_i] M^2(k_i)\, \Theta(v - V(\{\tilde{p}\}, k_1, \dots, k_n)) \right\}}_{\text{single-logarithmic correction } \mathcal{F}(R')}$$

The thrust at NLL accuracy

For soft and/or collinear emissions, the thrust can be written as



$$1 - T(\{\tilde{p}\}, k_1, \ldots, k_n) = \sum_i \frac{k_{ti}}{Q} e^{-|\eta_i|}$$

$$+ \sum_{\ell=1,2} \frac{1}{Q^2} \frac{\left|\sum_{i\in\mathcal{H}_\ell} \vec{k}_{ti}\right|^2}{1 - \sum_{i\in\mathcal{H}_\ell} z_i}$$

Exercise. Show that the multiple emission function for the thrust is given by

$$\mathcal{F}(R') = \frac{e^{-\gamma_E R'}}{\Gamma(1+R')}$$

Thrust at NLL accuracy

Exercise. Show that the multiple emission function for the thrust is given by

$$\mathcal{F}(R') = \frac{e^{-\gamma_E R'}}{\Gamma(1+R')}$$

Analytic solution. Change variable from k to $1 - T(\{\tilde{p}\}, k) = \tau\zeta$

$$\int [dk] M^2(k) \tau \delta \left(1 - T(\{\tilde{p}\}, k) - \tau\right) = -\tau \frac{dR(\tau)}{d\tau} = R'(\tau)$$

$$\int_{\epsilon\tau} [dk] M^2(k) \to \int_{\epsilon}^{\infty} \frac{d\zeta}{\zeta} R'(\zeta\tau) \simeq R'(\tau) \int_{\epsilon}^{\infty} \frac{d\zeta}{\zeta}$$

$$\mathcal{F}(R') = e^{-\int_{\epsilon\tau}^{\tau} [dk] M^2(k)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{i=1}^{n} \int_{\epsilon\tau} [dk_i] M^2(k_i) \right) \Theta(\tau - 1 + T(\{\tilde{p}\}, k_1, \dots, k_n))$$
$$= \epsilon^{R'} \sum_{n=0}^{\infty} \frac{(R')^n}{n!} \left(\prod_{i=1}^{n} \int_{\epsilon}^{\infty} \frac{d\zeta_i}{\zeta_i} \right) \Theta\left(1 - \sum_{i=1}^{n} \zeta_i\right)$$

Thrust at NLL accuracy

Exercise. Show that the multiple emission function for the thrust is given by

$$\mathcal{F}(R') = \frac{e^{-\gamma_E R'}}{\Gamma(1+R')}$$

Analytic solution. Factorise the observable constraint

$$\Theta\left(1 - \sum_{i=1}^{n} \zeta_i\right) = \int \frac{d\nu}{2\pi i \nu} e^{\nu\left(1-\sum_{i=1}^{n}\zeta_i\right)} = \int \frac{d\nu}{2\pi i \nu} e^{\nu} \prod_{i=1}^{n} e^{-\nu\zeta_i}$$

$$\mathcal{F}(\mathcal{R}') = \int \frac{d\nu}{2\pi i \nu} e^{\nu}\, \epsilon^{R'} \sum_{n=0}^{\infty} \frac{(R')^n}{n!} \left(\prod_{i=1}^{n} \int_{\epsilon}^{\infty} \frac{d\zeta_i}{\zeta_i} e^{-\nu\zeta_i}\right)$$

$$= \int \frac{d\nu}{2\pi i \nu} e^{\nu}\, \epsilon^{R'} \exp\left[R' \int_{\epsilon}^{\infty} \frac{d\zeta}{\zeta} e^{-\nu\zeta}\right] = \frac{e^{-\gamma_E R'}}{\Gamma(1+R')}$$

The thrust distribution exponentiates in Laplace space!

Thrust at NLL accuracy

Exercise. Show that the multiple emission function for the thrust is given by

$$\mathcal{F}(R') = \frac{e^{-\gamma_E R'}}{\Gamma(1+R')}$$

Monte Carlo solution. Isolate the emission with largest value of ζ and resurrect all emissions with $\epsilon\zeta < \zeta_i < \epsilon$

$$\mathcal{F}(R') = R' \int_0^\infty \frac{d\zeta}{\zeta} \zeta^{R'} \epsilon^{R'} \sum_{n=0}^{\infty} \frac{(R')^n}{n!} \left(\prod_{i=1}^{n} \int_{\epsilon\zeta}^{\zeta} \frac{d\zeta_i}{\zeta_i} \right) \Theta\left(1 - \zeta - \sum_{i=1}^{n} \zeta_i\right)$$

Rescale $\zeta_i = \zeta \tilde{\zeta}_i$ and do the ζ integration analytically

$$\mathcal{F}(R') = \epsilon^{R'} \sum_{n=0}^{\infty} \frac{(R')^n}{n!} \left(\prod_{i=1}^{n} \int_{\epsilon}^{1} \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \right) R' \int_0^\infty \frac{d\zeta}{\zeta} \zeta^{R'} \Theta\left(\frac{1}{1+\sum_{i=1}^{n} \tilde{\zeta}_i} - \zeta \right)$$

$$= \epsilon^{R'} \sum_{n=0}^{\infty} \frac{(R')^n}{n!} \left(\prod_{i=1}^{n} \int_{\epsilon}^{1} \frac{d\tilde{\zeta}_i}{\tilde{\zeta}_i} \right) \left(1 + \sum_{i=1}^{n} \tilde{\zeta}_i\right)^{-R'}$$

This expression can be numerically implemented as a shower event generator

Summary

In this lecture we have learnt

1. what is the origin of large logarithm in final-state observable distributions
2. how to compute double logarithms fast using Lund diagrams
3. there are rIRC safety conditions you have to impose on observables to that leading logarithms exponentiate
4. for rIRC safe observables, NLL accuracy forces all real emissions to be soft, collinear and widely separated in rapidity
5. multiple emission contributions give at most a single-logarithmic function