Lectures on Renormalization?

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Outline

Rant about QFT:

(Alexander Voigt)

- “physicists” don’t care about serious maths
- manipulate undefined, divergent integrals in arbitrary ways
- provocative proposal:
  - “regularize” by defining all divergent integrals:=0

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- divergent! $F(p)$ undefined!
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- Let's press on in a sloppy "physicist's way"

Formal manipulation 1 (out of 2)

- mathematically well-defined expression
- determines $F(p)$ up to a constant

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$$F(p) - F(p_0) = \int_0^\infty dk \left(\frac{1}{k-p} - \frac{1}{k-p_0} \right)$$

- mathematically well-defined expression
- determines $F(p)$ up to a constant

Difference:

$$F(p) - F(p_0)$$

$$= \int_0^\infty dk \frac{p - p_0}{k^2 - k(p + p_0) + pp_0}$$

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Formal manipulation 2 (out of 2)

- requires $F(p)$ must be a constant!

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$$F(ap) = \int_0^\infty dk \frac{1}{k - ap}$$

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Summary of three relations

$$0: \quad F(p) := \int_0^\infty dk \frac{1}{k-p}$$

$$1: \quad F(p) - F(p_0) = -\log\left(\frac{-p}{-p_0}\right)$$

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Can derive inconsistent results ("0=1") if we start from mathematically ill-defined expression

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Let's assume: fundamental physics requires equation 1

Turn around the logic

Starting point (from fundamental physics requirements):

$$F(p) - F(p_0) = -\log\left(\frac{-p}{-p_0}\right)$$

Consequence: formal manipulation 2 (scale invariance) is wrong:

$$F(p) \neq \text{const.}$$

Divergent integral can be viewed as a convenient “abbreviation”:

$$“F(p) = \int_0^\infty dk \frac{1}{k-p}”$$

which is meaningful only if it is applied to differences etc.

Outline

1. Renormalization — main theorems and their logical connections
   - Fundamental physics requirements and divergences
   - Regularization, Counterterms — Renormalization in Practice
   - Renormalization group

Fundamental physics requirements define S-matrix up to certain constants…

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Unitarity and Causality

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Plan: explain this with the help of one example!

Framework: Perturbation theory

For QED: interaction strength e

- all quantities = power series in e, equations hold order by order

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- S-operator then can be written as

$$\begin{aligned} S &= 1 \\ &+ \int d^4x \; e(x) \; S_1(x) \\ &+ \frac{1}{2} \int d^4x \, d^4y \; e(x) \, e(y) \; S_2(x,y) + \ldots \end{aligned}$$

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- other quantities similar

Often, usual “derivation” (which leads to divergences):

$$S = T\exp\left(i\int d^4x\mathcal{L}_{\text{int}}(x)\right), \qquad \mathcal{L}_{\text{int}} = -e\bar{\psi}\gamma^\mu\psi A_\mu$$

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Question:

Implications of Unitarity and Causality on higher orders?

Implication of Unitarity $S^\dagger S = 1$ at $\mathcal{O}(e^2)$

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This implies at $\mathcal{O}(e^2)$

$$
S_2^\dagger(x,y) + S_2(x,y) = -2S_1^\dagger(x)S_1(y)
$$

Imaginary part of loop contributions completely fixed/predicted by unitarity

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Mathematical formulation: two switching-on functions $e_1(x)$, $e_2(x)$ where supp(e_2) is later than supp(e_1). Then:

$$S(e_1 + e_2) = S(e_2)S(e_1)$$

S-operator factorizes!

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Note: if four-vectors $x_1 \neq x_2$, there is always a reference frame in which either $x_2^0 > x_1^0$ or $x_1^0 > x_2^0$ — so such a factorization must always hold unless $x_1 = x_2$

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Non-local part of loop contributions completely fixed/predicted by causality

What are these local/non-local terms?

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Summary

- Causality fixes the loop contributions up to local terms (=polynomials in external momenta)
- Unitarity fixes the imaginary part of loop contributions

(analogous at all orders)

Example

generate finite loop integral by combining

$$\Pi^{\mu\nu}_{\text{fin}}(p) := \left(\Pi^{\mu\nu}(p) - \Pi^{\mu\nu}(0) - \frac{p^\rho p^\sigma}{2}\frac{\partial^2}{\partial p^\rho \partial p^\sigma}\Pi^{\mu\nu}(0)\right)_{\text{combine integrands}}$$

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Then, the most general result for the photon self energy allowed by unitarity and causality is given by

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[ignoring gauge invariance]

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Then, the most general result for the photon self energy allowed by unitarity and causality is given by

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[ignoring gauge invariance]

$$\text{like } F(p) - F(p_0) = -\log\left(\frac{-p}{-p_0}\right)$$

Theorem 1:

- Write down usual Feynman diagrams and loop integrals
- Apply R-operation (subtraction of polynomial in external momenta on integrand level, recursively applied also on subdiagrams)
- In this way, obtain finite S-matrix/Green functions which are in agreement with unitarity and causality

Theorem 2:

- The remaining arbitrary real, local terms are in one-to-one correspondence with terms arising from adding

$$\mathcal{L}_{\text{counterterm}}$$

a local, hermitian counterterm Lagrangian

[Bogoliubov, Parasiuk, Hepp, Zimmermann]

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The correct logic

Starting point (from fundamental physics requirements):

(loop contributions fixed up to real polynomial in external momenta)

$$F(p) - F(p_0) = -\log\left(\frac{-p}{-p_0}\right)$$

Divergent integral can be viewed as a convenient “abbreviation”:

$$\text{“}F(p) = \int_0^\infty dk \frac{1}{k-p}\text{''}$$

which is meaningful only if it is applied to differences etc.

This justifies regularization

Regularization := modification of ill-defined integral

$$F(p;\epsilon) := \left(\int_0^\infty dk \frac{1}{k-p}\right)_{\text{reg. }\epsilon}$$

which satisfies the fundamental physics requirement

$$F(p;\epsilon) - F(p_0;\epsilon) = -\log\left(\frac{-p}{-p_0}\right) + \mathcal{O}(\epsilon)$$

Not every ϵ-dependent modification satisfies this!!!

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Renormalized, counterterm, bare contributions

The most general allowed $F(p)$ can be obtained as

($\lim_{\epsilon\to 0}$ understood)

$$F(p) = \delta(\epsilon) + F(p;\epsilon)$$

renormalized result

Renormalized, counterterm, bare contributions

The most general allowed $F(p)$ can be obtained as

($\lim_{\epsilon\to 0}$ understood)

$$F(p) = \delta(\epsilon) + F(p;\epsilon)$$ renormalized result

- $\delta(\epsilon)$ cancels the divergence — counterterm
- and contains an arbitrary constant — renormalization scheme

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$$= e + \delta e(\epsilon) \qquad + e^3(\epsilon) F(p; \epsilon)$$

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\begin{aligned}
&= e + \delta e(\epsilon) && + e^3(\epsilon) F(p;\epsilon) \\
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Lessons:

- can use regularization and counterterms to obtain correct result

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Lessons:

- here: arbitrary constant is no new parameter
- theory only depends on bare parameter (for fixed regularization)

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Lessons:

- equivalence:

$$e_{\text{scheme 1}} + \delta e_{\text{scheme 1}} = e_{\text{scheme 2}} + \delta e_{\text{scheme 2}}$$

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Theorem 3:

dimensional regularization, dimensional reduction, Pauli-Villars ok

['t Hooft, Veltman; Breitenlohner, Maison; Jack, Jones, Roberts; DS]

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Absorb by adding counterterms

$$\begin{aligned}\mathcal{L}_{\text{cl}} + \mathcal{L}_{\text{ct}} = & \ldots - e\,\bar{\psi}\gamma^{\mu}\psi A_{\mu} \\ & + \ldots - \delta e(\epsilon)\bar{\psi}\gamma^{\mu}\psi A_{\mu} \\ & + \ldots + \delta g_6 \bar{\psi}\psi\bar{\psi}\psi\end{aligned}$$

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Theorem 2b:

start with dim$\leq$ 4 only $\Rightarrow$ dim$\leq$ 4 remains sufficient

requires gauge theories for spin 1 particles

in general: all terms needed which are not forbidden

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- If finite number of terms is sufficient: “renormalizable”
- Renormalizable theory: $\leftrightarrow$ finite number of bare parameters
- Choose renormalization scheme to define split $e_{\text{bare}}(\epsilon) = e + \delta e(\epsilon)$

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broken by non-local terms $\leftrightarrow$ unitarity/causality — "Anomaly"
"Anomaly" is a physical effect, no regularization-artifact!!

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Momentum dependence in fixed, renormalized theory?

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$$= \quad \beta \, \partial_e \left[e + e^3 F(p)\right]$$

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$$= -e^3 \partial_e \left[e + e^3 F(p)\right] + \mathcal{O}(\text{2loop})$$

$$= \beta\, \partial_e \left[e + e^3 F(p)\right]$$

$\rightarrow$ Callan-Symanzik equations, universal β for all observables

Renormalization scale μ dependence in DREG/$\overline{\text{MS}}$?

$$\overline{\text{MS}}: \qquad e + \delta e(\epsilon) \qquad\qquad + e^3 F(p;\epsilon)$$

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$$= e - e^3 \frac{1}{\epsilon} \qquad + e^3 \left(\frac{1}{\epsilon} - \log\left(\frac{-p}{\mu}\right) + \mathcal{O}(\epsilon) \right)$$

$$\overline{\text{MS}}: \quad \bar{e}_{(\mu)} + \delta\bar{e}_{(\mu)}(\epsilon) \quad + \bar{e}_{(\mu)}{}^3 F(p;\epsilon)$$

$$= \bar{e}_{(\mu)} - \bar{e}_{(\mu)}{}^3 \frac{1}{\epsilon} \quad + \bar{e}_{(\mu)}{}^3 \left(\frac{1}{\epsilon} - \log\left(\frac{-p}{\mu}\right) + \mathcal{O}(\epsilon) \right)$$

$$\overline{\text{MS}}: \qquad \Big[e + e^3 F(p)\Big] = e + e^3 \left(-\log\left(\frac{-p}{\mu}\right)\right)$$

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running coupling $\bar{e}_{(\mu)} = e$ for $\mu \approx p$ tracks physical p-dependence!

$$\overline{\text{MS}}: \qquad \left[e + e^3 F(p)\right] = e + e^3 \left(-\log\left(\frac{-p}{\mu}\right)\right)$$

Renormalization scale μ dependence in DREG/$\overline{\text{MS}}$?

$$0 = \mu \frac{d}{d\mu}\left[e + e^3 F(p)\right]$$
$$= \mu \frac{de}{d\mu} + e^3 + \mathcal{O}(\text{2loop})$$

$\rightarrow$ renormalization group equations, running coupling (universal!)

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$\rightarrow$ renormalization group equations, running coupling (universal!)

Outline

1. Renormalization — main theorems and their logical connections
2. More on regularizations
3. Renormalizability of gauge theories — QCD
4. Operator renormalization in $gg \to H$
5. Additional topics

Outline

2. More on regularizations
   - Criteria for possible regularizations
   - Regularized quantum action principle
   - QCD gauge invariance of dimensional regularization
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   - FDH/DRED and infrared structure

Remarks on QFT regularizations: Necessary property

- suppose, theory has been defined up to n-loop level
- any correct regularization must satisfy at the $(n+1)$-loop level:
  - it may differ from BPHZ only by real, local terms

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Counter example: set all divergent integrals = 0 — yields finite theory that violates causality and unitarity

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Counter example 2: DREG with anticommuting γ_5 — some loops will be incorrectly set to zero!!

In practice, check correctness of your calculation!

e.g. 2-loop muon decay [Freitas,Hollik,Walter,Weiglein '02], 2-loop $g-2$ [Heinemeyer,DS,Weiglein '04]

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Dimensional regularization [Breitenlohner, Maison '77],
Dimensional reduction [Jack, Jones, Roberts '93; DS '05],
Pauli-Villars... are ok

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. . . can go into more details later

Further remarks: optional properties

- gauge invariance/SUSY/other symmetries
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Further remarks: optional properties

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$$\int \frac{k^2}{k^2 - m^2} \stackrel{?}{=} \int 1 + \int \frac{m^2}{k^2 - m^2}$$

- unambiguous also if diagrams appear as subdiagrams?

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Symmetry transformations of Green functions — formally

$$\phi_i(x) \to \phi_i(x) + \delta\phi_i(x), \qquad \mathcal{L}(x) \to \mathcal{L}(x) + \delta\mathcal{L}(x)$$

How do Green functions behave?

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Path integral:

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(measure invariant)

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$$= \int \mathcal{D}\phi \; (1 + i\int \delta\mathcal{L} + J\delta\phi) e^{i\int \mathcal{L}+J\phi}$$

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formal “derivation” shows

$$\langle(\delta\phi_1)\phi_2\ldots\rangle + \langle\phi_1(\delta\phi_2)\ldots\rangle + \ldots = -i\langle\phi_1\phi_2\ldots(\int\delta\mathcal{L})\rangle$$

Symmetry transformations of Green functions — really

Regularized quantum action principle

$$\langle(\delta\phi_1)\phi_2\ldots\rangle + \langle\phi_1(\delta\phi_2)\ldots\rangle + \ldots = -i\langle\phi_1\phi_2\ldots(\textstyle\int\delta\mathcal{L})\rangle$$

Interpret this as an identity between regularized Feynman diagrams

- becomes a property of regularization scheme, does not necessarily hold (no fundamental QFT requirement)
- if desired, must be proven for each regularization
- valid in BPHZ: [Lowenstein et al '71], DREG: [Breitenlohner, Maison '77], DRED: [DS '05]

Symmetry transformations of Green functions — really

Regularized quantum action principle

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Interpret this as an identity between regularized Feynman diagrams

Idea of proof in DREG/DRED: look at possible Wick contractions

- $\delta\mathcal{L} = \delta\mathcal{L}_{\text{quadratic}} + \delta\mathcal{L}_{\text{int}}, \qquad \delta\mathcal{L}_{\text{quadratic}} = (\delta\phi_i)D_{ij}\phi_j$
- Use properties of DREG/DRED: *D* is inverse propagator even on regularized level, scaleless integrals vanish
- then, combinatorics leads to above identity

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DREG is "gauge invariant" (in QCD)

- Lagrangian is gauge invariant even in D dimensions (without γ_5)
- it is even BRS invariant (see later) and satisfies the Slavnov-Taylor identity in D dimensions
- Hence, the appropriate $\delta\mathcal{L} = 0$
- Therefore, all regularized Green functions satisfy the appropriate Slavnov-Taylor identities at all orders (for $D \neq 4$)

$$\langle(\delta\phi_1)\phi_2\ldots\rangle + \langle\phi_1(\delta\phi_2)\ldots\rangle + \ldots = 0$$

Further examples of regularization schemes and symmetries

- DREG breaks gauge invariance in EWSM because of γ_5

  take this into account in renormalizability proof [BRS '75. . . Kraus '97, Grassi '98]

  need symmetry-restoring counterterms, e.g. [Martin, Sanchez-Ruiz 2000]

- DREG breaks scale invariance because of μ

  physical breaking by non-local terms, required by theory, cannot be repaired

- DREG breaks SUSY

  need SUSY-restoring counterterms, e.g. [Martin,Vaughn '93][Mihaila '09][DS,Varso '11]

- DRED preserves SUSY to large extent

  . . . [Hollik, DS '05][Harlander,Kant,Mihaila,Steinhauser'07]

  but not completely [Avdeev, Chochia, Vladimirov '81][DS '05]

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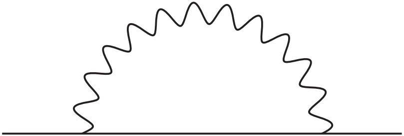
Common Regularization Schemes for gauge theories/SUSY

- Dimensional Regularization (DREG) ['t Hooft, Veltman '72]
- Dimensional Reduction (DRED)/Four-dimensional helicity scheme (FDH) [Siegel '79]

What do we need to define?



- *D*-dimensional Integral

Dim. Regularization (DREG)

D dimensions
D Gluon/photon-components
4 Gluino/photino-components

Dim. Reduction (DRED)

D dimensions
4 Gluon/photon-components
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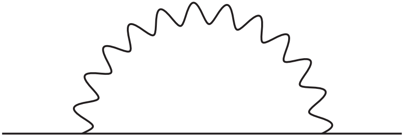

- *D*-dimensional Integral
- *D*-dim covariants
  $\gamma^\mu (p_\rho \gamma^\rho) \gamma_\mu = (2 - D)(p_\rho \gamma^\rho)$
- *D*-,4-dim covariants
  $\gamma^\mu (p_\rho \gamma^\rho) \gamma_\mu = (2 - 4)(p_\rho \gamma^\rho)$

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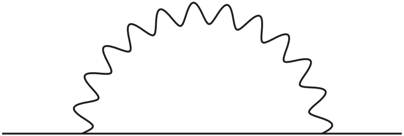

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Definition of external gluons

do not have to regularize external/observed gluons!

	CDR	DRED
“internal” gluon	$\hat{g}^{\mu\nu}$	$g^{\mu\nu}$
“external” gluon	$\hat{g}^{\mu\nu}$	$g^{\mu\nu}$

Definition of external gluons

do not have to regularize external/observed gluons!

3 spaces: 4S $\bar{g}^{\mu\nu}$ $\subset$ QDS $\hat{g}^{\mu\nu}$ $\subset$ Q4S $g^{\mu\nu} = \hat{g}^{\mu\nu} + \tilde{g}^{\mu\nu}$

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4S		QDS		Q4S
$\bar{g}^{\mu\nu}$	$\subset$	$\hat{g}^{\mu\nu}$	$\subset$	$g^{\mu\nu} = \hat{g}^{\mu\nu} + \tilde{g}^{\mu\nu}$

	CDR	HV	FDH	DRED
"internal" gluon	$\hat{g}^{\mu\nu}$	$\hat{g}^{\mu\nu}$	$g^{\mu\nu}$	$g^{\mu\nu}$
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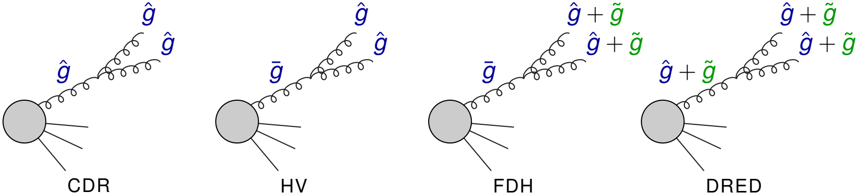

DREG: How does it work?

- "*D*-dimensional space" $\{k^\mu\}$ can be consistently defined as a truly ∞-dimensional space with some *D*-dim characteristics:

  [Wilson'73],[Collins]

- *D*-dimensional Integral: linear mapping

- $g^{(D)\mu\nu}$: bilinear form (γ-matrices similar)

explicit construction $\Rightarrow$ no contradictions possible

DREG: How does it work?

- "D-dimensional space" $\{k^\mu\}$ can be consistently defined as a truly ∞-dimensional space with some D-dim characteristics:

[Wilson'73],[Collins]

- D-dimensional Integral: linear mapping

$$\int d^D k e^{-k^2} = \pi^{D/2}$$

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- $g^{(D)\mu\nu}$: bilinear form (γ-matrices similar)

$$\mu = 0, 1, 2, \ldots \infty, \quad g^{(D)\mu}{}_{\mu} = D$$

explicit construction ⇒ no contradictions possible

Dimensional Reduction: We need more!

- also 4-dim space

- algebraic identities

$$g^{(4)}{}_{\mu\nu} g^{(D)}{}_{\rho}{}^{\nu} = g^{(D)}{}_{\mu}{}^{\rho}$$

Dim. Reduction (DRED)

D dimensions "$D < 4$"
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Dim. Reduction (DRED)

D dimensions "$D < 4$"
4 Gluon/photon-components
4 Gluino/photino-components

$\Rightarrow$ Replace ordinary 4-dim space by yet another ∞-dimensional space with some 4-dim characteristics $\rightarrow$ "quasi-4-dim space"

$$D\text{-dim space} \subset \text{quasi-4-dim space}$$

$$g^{(D)\mu}{}_{\mu} = D, \quad g^{(4)\mu}{}_{\mu} = 4, \quad \mu = 0, 1, 2, \ldots \infty$$

$\Rightarrow$ proof: DRED is mathematically consistent, too! [DS 2005]

Practical consequences

- In practice one can forget that the "*D*-dim" and quasi-4-dim spaces are in reality ∞-dimensional
- Algebraic id. for $g^{(D)\mu\nu}$, $g^{(4)\mu\nu}$ as desired
- Only exception: one cannot rely on 4-dim identities like index counting or Fierz identities
    - For many SUSY loop calculations, this doesn't make a difference

Definition of DREG and DRED: The computational rules based on these constructions will never lead to inconsistent results

How do we avoid Siegel's inconsistency?

Siegel: "With

$$\epsilon^{(4)}_{\mu_1\mu_2\mu_3\mu_4}\epsilon^{(4)}_{\nu_1\nu_2\nu_3\nu_4} \propto \det((g^{(4)}_{\mu_i\nu_j}))$$

calculate

$$\epsilon^{(D)\mu\nu\rho\sigma}\,\epsilon^{(\epsilon)}{}_{\alpha\beta\gamma\delta}\,\epsilon^{(D)}{}_{\mu\nu\rho\sigma}\,\epsilon^{(\epsilon)\alpha\beta\gamma\delta}$$

in two different ways

$$\Rightarrow 0 = D(D-1)^2(D-2)^2(D-3)^2(D-4)$$

different calculational steps lead to different results,

mathematical inconsistency!!!" [Siegel'80]

Don't allow explicit index counting (step one) any more, because $g^{(4)}{}_{\mu\nu} \in$quasi-4-dim space!

Basic properties and practical consequences

- Consistent definitions exist (⇒ no contradictions arise) ['t Hooft, Veltman '72] [Wilson '73] [Breitenlohner, Maison '77] [Collins][DS '05]
- No strictly 4-dim. index counting/Fierz identities possible (doesn't make a difference in many applications)
- regularized quantum action principle valid [Breitenlohner, Maison '77][DS '05]
  $\Delta = S(\Gamma_{\text{cl}}^{\text{DRED,DREG}}) \neq 0$ in both cases!
- Many highly nontrivial multi-loop calculations performed [Harlander, Kant, Mihaila, Steinhauser, et al]
- Renormalization: treat $(4 - D)$-dim. gluons as additional matter fields (not gauge fields!) $\rightarrow$ ϵ-scalars with independent couplings and masses. [Jack, Jones, Roberts '94] [Harlander, Kant, Mihaila, Steinhauser, et al]

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Interesting Cases

- Does DREG preserve gauge invariance?
- Does DRED preserve SUSY?

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How do we know? STI combines all identities of the form

$$0 = \delta_{\text{sym}} \langle T\phi_1 \ldots \phi_n \rangle$$

complicated equation between many Green's functions

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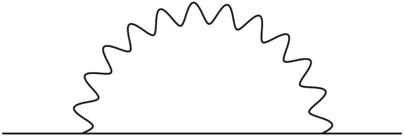
complicated equation between many Green's functions

- check identities explicitly or use regularized quantum action principle

Properties of DREG/DRED



SUSY?

Consider SUSY-relation

$$m_e = m_{\tilde{e}}$$

at 1-loop: $m^2(1L) = m^2 - \Sigma(p^2 = m^2)$

Properties of DREG/DRED

DREG:

$$m_e(1L) = m_e\left[1 + \frac{\alpha}{4\pi}(2B_0 - 1)\right]$$
$$m_{\tilde{e}}(1L) = m_e\left[1 + \frac{\alpha}{4\pi}\left(2B_0 + \frac{2}{3}\right)\right]$$

- DREG breaks SUSY!

DRED:

$$m_e(1L) = m_{\tilde{e}}(1L)$$

- DRED preserves SUSY in this case!

Properties of DREG/DRED

DREG:

$$m_e(1L) = m_e\left[1 + \frac{\alpha}{4\pi}(2B_0 - 1)\right]$$
$$m_{\tilde{e}}(1L) = m_e\left[1 + \frac{\alpha}{4\pi}\left(2B_0 + \frac{2}{3}\right)\right] - \frac{\alpha}{4\pi} m_e \frac{5}{3}$$

- DREG breaks SUSY! SUSY-restoring counterterm $\delta m_{\tilde{e}}^{\rm rest}$

DRED:

finally: $m_e(1L, ren) = m_{\tilde{e}}(1L, ren)$

$$m_e(1L) = m_{\tilde{e}}(1L)$$

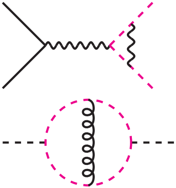
- DRED preserves SUSY in this case!

DRED current status: has passed many tests

- $\Delta = S(\Gamma_{\text{cl}}^{\text{DRED}}) \neq 0$ in quantum action principle (because of Fierz identities)
- Can check SUSY either directly or by using the quantum action principle ($\Delta \Rightarrow$ Feynman rules):

  1-Loop Ward identities [Capper,Jones,van Nieuvenhuizen'80] β-functions [Martin, Vaughn '93] [Jack, Jones, North '96] 1-Loop S-matrix relation [Beenakker,Höpker,Zerwas'96] 1-Loop Slavnov-Taylor identities [Hollik,Kraus,DS'99] [Hollik,DS'01] [Fischer,Hollik,Roth,DS'03] Higher order Ward and Slavnov-Taylor identities [DS, Hollik, DS '05][Harlander,Kant,Mihaila,Steinhauser'07]

  - sufficient for many SUSY processes
    $\Rightarrow$ multiplicative renormalization o.k.
    $\Rightarrow$ no SUSY-restoring counterterms



Transition between DREG and DRED

- difference $\Gamma^{\text{DRED}} - \Gamma^{\text{DREG}}$ can be compensated by counterterms

$$\Gamma^{\text{DRED}} = \Gamma^{\text{DREG}} + \Gamma_{\text{ct}}^{\text{transition}}$$

- can be computed once and for all
  - 1-loop couplings [Martin, Vaughn ’93], 2-loop SUSY-QCD couplings [Mihaila ’09]
  - 1-loop complete MSSM FeynArts model file [Varso ’11]
    (UV transition rules, complementary to IR ones)

- transition c.t.s act as SUSY-restoring counterterms for DREG
- realize $\overline{DR}$-scheme in context of DREG
- infrared regularization by DREG, UV reg. by DRED

Outline

2. More on regularizations
   - Criteria for possible regularizations
   - Regularized quantum action principle
   - QCD gauge invariance of dimensional regularization
   - More details on DREG, DRED, FDH: Consistent definitions
   - Symmetries in DREG and DRED
   - Renormalization of ϵ-scalars in FDH/DRED
   - FDH/DRED and infrared structure

Why are ϵ-scalar couplings independent?

- because

$$A_\mu^{(4)} = A_\mu^{(D)} + A_\mu^{(\epsilon)}$$

- only $A_\mu^{(D)}$ is a D-dimensional gauge field in D_μ
- but $A_\mu^{(\epsilon)}$ transforms like a scalar field ("ϵ-scalars")
- general renormalization theory applies: all gauge invariant terms can (and will) appear as independent counterterms

$$\mathcal{L}_{\text{ct}} = \delta g_s \bar{\psi} \gamma^\mu A_\mu^{(D)} \psi + \delta g_e \bar{\psi} \gamma^\mu A_\mu^{(\epsilon)} \psi$$

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$$\mathcal{L}_{\rm ct} = \delta g_s \bar{\psi}\gamma^\mu A_\mu^{(D)}\psi + \delta g_e \bar{\psi}\gamma^\mu A_\mu^{(\epsilon)}\psi$$

Consequence:

treat δg_e independently, may not set $\delta g_e = \delta g_s$ or $\beta_e = \beta_s$
(otherwise loss of unitarity, finiteness — has appeared in literature)

[Jack,Jones,Roberts '93][Harlander,Kant,Mihaila,Steinhauser'07] [Kilgore '11]

Outline

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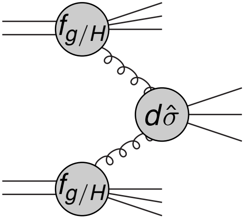
Hadronic Processes and infrared properties of DREG/DRED

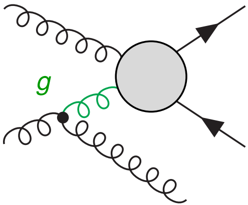


$$d\sigma_{\rm RS}^{\rm real} \sim P_{g\to gg}\ d\sigma_{\rm RS}^{\rm Born}$$

$$d\sigma_{\rm RS}^{\rm virt} \sim \gamma_g\ d\sigma_{\rm RS}^{\rm Born}$$

- IR singularities should factorize

Factorization problem and its solution



$$\sigma^{\text{DRED}}(gg \to t\bar{t}g) \xrightarrow{2\|3} ?$$

- Apparent factorization problem

  [Beenakker, Kuijf, van Neerven, Smith '88] [van Neerven, Smith '04]

  [Beenakker, Höpker, Spira, Zerwas '96]

$$\sim \frac{1}{k_2 k_3} P_{g\to gg}\, \sigma^{\text{DRED}}(gg \to t\bar{t}) + \frac{1}{k_2 k_3} K_g\, \sigma^{\text{puzzle}}$$

- Reconcile DRED with factorization by decomposing gluon [Signer, DS '05,'08]

$$\sim P_{g\to\hat{g}g}\, \sigma_{g\hat{g}} + P_{g\to\tilde{g}g}\, \sigma_{g\tilde{g}}$$

DRED and the gluon

4-component Gluon in DRED g	=	D-component gauge field $\hat{g}$	+	ϵ-scalars $\tilde{g}$

- Simple kinematics: e.g. $gg \to q\bar{q}$ (massless) $\quad \sigma_{gg\to q\bar{q}} = \sigma_{g\hat{g}\to q\bar{q}} = \sigma_{g\tilde{g}\to q\bar{q}}$
- in general / here: $gg \to t\bar{t}$ (massive) $\quad \sigma_{gg\to q\bar{q}} \neq \sigma_{g\hat{g}\to q\bar{q}} \neq \sigma_{g\tilde{g}\to q\bar{q}}$

$\hat{g}$ and $\tilde{g}$ have to be treated as seperate partons!

Definition of external/observed gluons

Beware of different versions of DRED/FDH!

3 spaces: 4S $\bar{g}^{\mu\nu}$ $\subset$ QDS $\hat{g}^{\mu\nu}$ $\subset$ Q4S $g^{\mu\nu} = \hat{g}^{\mu\nu} + \tilde{g}^{\mu\nu}$

	CDR	DRED
"unobserved" gluon	$\hat{g}^{\mu\nu}$	$g^{\mu\nu}$
"observed" gluon	$\hat{g}^{\mu\nu}$	$g^{\mu\nu}$

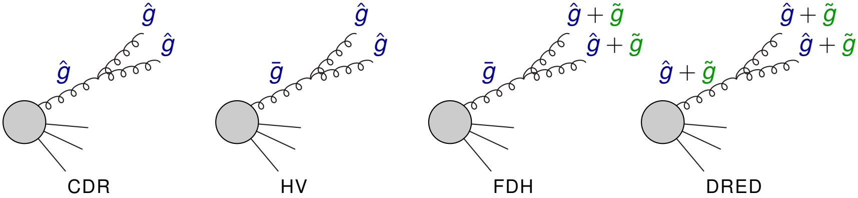
Definition of external/observed gluons

Beware of different versions of DRED/FDH!

3 spaces: 4S $\bar{g}^{\mu\nu}$ $\subset$ QDS $\hat{g}^{\mu\nu}$ $\subset$ Q4S $g^{\mu\nu} = \hat{g}^{\mu\nu} + \tilde{g}^{\mu\nu}$

	CDR	HV	FDH	DRED
"unobserved" gluon	$\hat{g}^{\mu\nu}$	$\hat{g}^{\mu\nu}$	$g^{\mu\nu}$	$g^{\mu\nu}$
"observed" gluon	$\hat{g}^{\mu\nu}$	$\bar{g}^{\mu\nu}$	$\bar{g}^{\mu\nu}$	$g^{\mu\nu}$

Main Results



CDR: $\gamma(\hat{g}) = \int P_{\hat{g}\to\hat{g}\hat{g}}$

HV: $\ldots$

FDH: $\ldots$

$+\int P_{\hat{g}\to\tilde{g}\tilde{g}}$

[Catani, S., T. '97]

[Kunszt, S., T. '94]

DRED: $\ldots$

$\ldots$

$\gamma(\tilde{g}) = \int P_{\tilde{g}\to\hat{g}\tilde{g}} + \int P_{\tilde{g}\to\tilde{g}\hat{g}}$

[Signer, DS '08]

- RS dependence:

  HV$\to$FDH: additional final state $\tilde{g}$: *value* of $\gamma(\hat{g})$ changes
  FDH$\to$DRED: additional splitting $\tilde{g}$: additional $\gamma(\tilde{g})$
  only in DRED: split $g = \hat{g} + \tilde{g}$ required to understand factorization

Consequences

- Factorization: detailed understanding in CDR, HV, FDH, DRED
  - 1-loop differences described by different γ's
  - DRED: split $g = \hat{g} + \tilde{g}$ required to understand factorization
- IR translation rules between RSs
- i.e. compute in DRED, then switch to DREG to use e.g. $\overline{\text{MS}}$ PDFs
- no PDF for ϵ-scalars $\tilde{g}$ required (of $\mathcal{O}(\epsilon)$ and contributes only at $\mathcal{O}(\epsilon)$)

$$f_{\hat{g}/H} \otimes d\hat{\sigma}_{\text{FS}}(\hat{g}_1 \ldots) + f_{\tilde{g}/H} \otimes d\hat{\sigma}_{\text{FS}}(\tilde{g}_1 \ldots)$$

Two further remarks

1. Outlook 2-loop: Becher/Neubert formula for q/g form factor: what changes for FDH, DRED? [Gnendiger]

$$F^{(2)}_{q/g}|_{\text{pole}} = \frac{1}{\epsilon^3}\left(-\frac{3C_{q/g}\gamma^{(0)}_{\text{cusp}}\beta_0}{8}\right) + \frac{1}{\epsilon^2}\left(-\frac{\beta_0\gamma^{(0)}_{\text{cusp}}}{2} - \frac{3C_{q/g}\gamma^{(1)}_{\text{cusp}}}{8}\right) + \frac{1}{\epsilon}\left(\frac{\gamma^{(1)}_{q/g}}{2}\right) + \frac{1}{2}(F^{(1)}_{q/g})^2$$

2. FDH as a renormalization scheme
   - often: no seperate ϵ-scalar renormalization ($\alpha_s^{\text{bare}} = \alpha_e^{\text{bare}}$)
   - inconsistent, leads to incorrect/non-unitary/divergent results

     [Jack, Jones, Roberts '94][Harlander, Kant, Mihaila, Steinhauser '06][Kilgore '11]
   - should renormalize like in DRED

Status DRED and its relation to DREG

- Both DREG and DRED formulated consistently, quantum action principle valid
- Renormalization in DRED understood
- SUSY of DRED established at 1-loop, in many 2-, 3-loop cases
- Factorization holds in both schemes
- UV and IR transition rules $\Rightarrow$ both schemes can be mixed

Outline

1. Renormalization — main theorems and their logical connections
2. More on regularizations
3. Renormalizability of gauge theories — QCD
4. Operator renormalization in $gg \to H$
5. Additional topics

Outline

3. Renormalizability of gauge theories — QCD
   - Reminder and overview
   - Definition and proof of renormalizability
   - Outlook: algebraic renormalization
   - Two small but important applications

QCD — classical definition

SU(3) gauge theory, massless matter fermion ψ

$$\mathcal{L}_{\text{QCD,g.inv.}} = \bar{\psi} i \gamma^\mu D_\mu \psi - \frac{1}{4} F_a^{\mu\nu} F_{a\mu\nu}$$

$$D^\mu = \partial^\mu + igT^a A_a^\mu$$

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How to define the quantum theory?

QCD — classical definition

SU(3) gauge theory, massless matter fermion ψ

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Traditional:
$\mathcal{L}_{\text{QCD}}$ $\rightarrow$gauge fix$\rightarrow$Faddeev-Popov$\rightarrow$BRS$\rightarrow$Slavnov-Taylor

QCD — Questions

$$\mathcal{L}_{\text{to be quantized}} = \mathcal{L}_{\mathrm{QCD,g.inv.}} + \mathcal{L}_{\mathrm{fix,gh}}$$

1. Finiteness at all orders:
   - multiplicative renormalization of coupling and fields possible?
   - only 4 ren. constants sufficient to cancel all divergences?
2. Phys. meaning of theory:
   - def. of physical states with positive norm?
   - phys. S-matrix: unitary, gauge independent?

QCD — Answers

['t Hooft, ..., BRS, ... Textbooks Weinberg, Kugo, Böhm/Denner/Joos, ...]

Main tool: STI $S(\Gamma) = 0$ defining theory in regularization-independent way, describing BRS-invariance of qu. theory

1. DREG "is gauge invariant (in QCD)" $\qquad S(\Gamma^{\mathrm{DREG}}) = 0$
2. Div.s at n-loop are "BRS-invariant" $\qquad s_{\Gamma_{\mathrm{cl}}} \Gamma^{\mathrm{div,n}} = 0$
3. can be cancelled by multiplicative renormalization $\Rightarrow$ finiteness
   multiplicative counterterms generally symmetric
4. counterterms also BRS-invariant $\qquad \Rightarrow S(\Gamma^{\mathrm{renorm.}}) = 0$
5. $S(\Gamma^{\mathrm{renorm.}}) = 0 \Rightarrow$ phys. states, S-matrix can be defined and shown to be unitary, gauge-indep.

QCD — $\mathcal{L}_{\text{QCD}}$ →gauge fixing→Faddeev-Popov

Need gauge fixing and ghosts (Faddeev Popov or BRST)

$$\mathcal{L}_{\text{fix,gh}} = B_a(\partial_\mu A_a^\mu) + \frac{\xi}{2}B_a^2 - \bar{c}_a\partial_\mu(D^\mu c)_a$$
$$= s[\bar{c}_a((\partial_\mu A_a^\mu) + \frac{\xi}{2}B_a)]$$

Full theory to be quantized

$$\mathcal{L}_{\text{cl}} = \mathcal{L}_{\text{QCD,g.inv.}} + \mathcal{L}_{\text{fix,gh}}$$

QCD — Faddeev-Popov→BRS→Slavnov-Taylor

- Ghosts for all generators $\longrightarrow$ BRS:

$$s\varphi = c_a \delta_{\text{gauge},a}\varphi$$

- BRS transformations of ghosts $\leftrightarrow$ $s^2 = 0$:

$$sc_a = \frac{1}{2} g f_{abc} c_b c_c$$

- Slavnov–Taylor operator

$$S(\Gamma) = \int d^4x \quad \underbrace{\langle s\varphi_i(x)\rangle}_{\neq\, s\varphi_i(x)} \quad \frac{\delta\Gamma}{\delta\varphi_i(x)}$$

at the quantum level if non-linear

- Add sources $\mathcal{L}_{\text{ext}} = Y_{\varphi_i} s\varphi_i$

$$S(\Gamma) = \int d^4x \frac{\delta\Gamma}{\delta Y_{\varphi_i}(x)} \frac{\delta\Gamma}{\delta\varphi_i(x)}$$

Outline

3. Renormalizability of gauge theories — QCD
   - Reminder and overview
   - Definition and proof of renormalizability
   - Outlook: algebraic renormalization
   - Two small but important applications

Definition of QCD

- field content (SU(3) indices suppressed)

- Slavnov-Taylor identity

Definition of QCD

- field content (SU(3) indices suppressed)

	physical		$N_{gh} > 0$	aux.	$N_{gh} < 0$			
	A^μ	ψ	c	B	$\bar{c}$	Y_{A^μ}	Y_ψ	Y_c
N_{gh}	0	0	1	0	-1	-1	-1	-2
dimension	1	3/2	0	2	2	3	5/2	4

- Slavnov-Taylor identity

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- Slavnov-Taylor identity

$$S(\Gamma) \equiv \int d^4x \frac{\delta\Gamma}{\delta Y_{\varphi_i}(x)} \frac{\delta\Gamma}{\delta\varphi_i(x)} + B \frac{\delta\Gamma}{\delta\bar{c}(x)} = 0$$

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- can require further constraints

most general classical solution

- (only dim$\leq$ 4-terms)
- 2 steps: Y-terms, rest

General classical solution — step 1a: Y_c-part

	physical		$N_{gh} > 0$	aux.	$N_{gh} < 0$			
	A^μ	ψ	c	B	$\bar{c}$	Y_{A^μ}	Y_ψ	Y_c
N_{gh}	0	0	1	0	-1	-1	-1	-2
dimension	1	3/2	0	2	2	3	5/2	4

- Only possible ansatz:

$$\Gamma_{\text{cl}} = \int d^4x Y_{c_a} \frac{1}{2} g_0 z F_{abc} c_b c_c + \ldots$$

- STI requires

$$0 = S(\Gamma_{\text{cl}})|_{Y_c\text{-terms}} = \int d^4x \frac{\delta\Gamma}{\delta Y_{c_a}} \frac{\delta\Gamma}{\delta c_a} + \ldots \propto F_{abc} F_{dae} c_b c_c c_e$$

- Jacobi id.$\Rightarrow F_{abc}$ must be structure constants of some Lie algebra!

General classical solution — step 1b: $Y_{A^\mu,\psi}$-part

	physical		$N_{gh} > 0$	aux.	$N_{gh} < 0$			
	A^μ	ψ	c	B	$\bar{c}$	Y_{A^μ}	Y_ψ	Y_c
N_{gh}	0	0	1	0	-1	-1	-1	-2
dimension	1	3/2	0	2	2	3	5/2	4

- Only possible ansatz:

$$\Gamma_{\rm cl} = \int d^4x Y_{A_a^\mu} \underbrace{z(\partial^\mu c_a + g_1 F'_{abc} c_b A_c^\mu)}_{\tilde{s}A_a^\mu} + Y_{\psi_i} \underbrace{(-ig_2 T^a_{ij} c_a \psi_j)}_{\tilde{s}\psi_i} + \ldots$$

- STI requires

$$0 = \tilde{s}\tilde{s}A_a^\mu = \tilde{s}\tilde{s}\psi_i \Rightarrow [T^a, T^b] = iF_{abc}T^c$$

- T^a is representation of Lie algebra, $F'_{abc} = F_{abc}$
- universality $g_0 = g_1 = g_2$
- $\tilde{s}$ is normal BRS transformation; it contains the ordinary gauge transformation

General classical solution — step 2a: no-Y-part

	physical		$N_{gh}>0$	aux.	$N_{gh}<0$			
	A^μ	ψ	c	B	$\bar{c}$	Y_{A^μ}	Y_ψ	Y_c
N_{gh}	0	0	1	0	-1	-1	-1	-2
dimension	1	3/2	0	2	2	3	5/2	4

- We already have

$$\Gamma_{\rm cl} = \int d^4x Y_{\varphi_i} \tilde{s}\varphi_i + \Gamma_{\rm rest}$$

$$\tilde{s} = \text{ordinary BRS transformation (up to } z)$$

$$\tilde{s}^2 = 0$$

- STI requires

$$0 = S(\Gamma_{\rm cl})|_{Y=0} = \int d^4x \frac{\delta\Gamma_{\rm cl}}{Y_{\varphi_i}}\frac{\delta\Gamma_{\rm rest}}{\varphi_i} + B\frac{\delta\Gamma_{\rm rest}}{\bar{c}}|_{Y=0} = \tilde{s}\Gamma_{\rm rest}$$

- BRS invariance of rest

General classical solution — step 2b Γ_{rest}

	physical		$N_{gh} > 0$	aux.	$N_{gh} < 0$			
	A^μ	ψ	c	B	$\bar{c}$	Y_{A^μ}	Y_ψ	Y_c
N_{gh}	0	0	1	0	-1	-1	-1	-2
dimension	1	3/2	0	2	2	3	5/2	4

- Theorem: due to $\tilde{s}^2 = 0$, the most general solution of

$$\tilde{s}\Gamma_{\text{rest}}(\psi, A^\mu, c, B, \bar{c}) = 0$$

is

$$\Gamma_{\text{rest}}(\psi, A^\mu, c, B, \bar{c}) = \Gamma_{\text{g.inv.}}(\psi, A^\mu) + \int d^4x \tilde{s} X(\psi, A^\mu, c, B, \bar{c})$$

General classical solution — step 2b Γ_{rest}

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- gauge invariant part and gauge fixing+Faddeev-Popov

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- gauge invariant part and gauge fixing+Faddeev-Popov

4 consequences

General classical solution — step 2b Γ_{rest}

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- gauge invariant part and gauge fixing+Faddeev-Popov

STI is beautiful starting point

General classical solution — step 2b Γ_{rest}

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multiplicative renormalization

General classical solution — step 2b Γ_{rest}

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- gauge invariant part and gauge fixing+Faddeev-Popov

origin of renormalizability

General classical solution — step 2b Γ_{rest}

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- gauge invariant part and gauge fixing+Faddeev-Popov

but first, gauge fixing

Gauge fixing

One possibility: linear gauge fixing

$$\tilde{s}X = \tilde{s}[\bar{c}_a((\partial_\mu A_a^\mu) + \frac{\xi}{2}B_a)]$$

Gauge fixing

One possibility: linear gauge fixing

$$\tilde{s}X = \tilde{s}[\bar{c}_a((\partial_\mu A_a^\mu) + \frac{\xi}{2}B_a)]$$
$$= B_a\partial_\mu A_a^\mu + \frac{\xi}{2}B_a^2 - \bar{c}_a\partial_\mu\tilde{s}A_a^\mu$$

Gauge fixing

One possibility: linear gauge fixing

$$\tilde{s}X = \tilde{s}[\bar{c}_a((\partial_\mu A_a^\mu) + \frac{\xi}{2}B_a)]$$
$$= B_a\partial_\mu A_a^\mu + \frac{\xi}{2}B_a^2 - \bar{c}_a\partial_\mu\tilde{s}A_a^\mu$$

B_a appears nowhere else — appears only quadratically, no vertices

Gauge fixing

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$$\frac{\delta\Gamma}{\delta B_a} = \xi B_a + \partial_\mu A_a^\mu$$

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B_a appears nowhere else — appears only quadratically, no vertices

$$\frac{\delta\Gamma}{\delta B_a} = \xi B_a + \partial_\mu A_a^\mu$$

holds at lowest order *and* exactly, gauge fixing does not renormalize

QCD — Renormalization

Multiplicative renormalization transformation of parameters and fields

generates most general classical solution (with this gauge fixing $\frac{\delta\Gamma}{\delta B_a}$)

$$\begin{aligned} g &\to g^{\text{bare}} = g + \delta g = Z_g g \\ \psi &\to \sqrt{Z_\psi}\psi \\ \{A^\mu, B, \bar{c}, \xi\} &\to \left\{\sqrt{Z_A}A^\mu, \sqrt{Z_A}^{-1}B, \sqrt{Z_A}^{-1}\bar{c}, Z_A\xi\right\} \\ c &\to \sqrt{Z_c}c \end{aligned}$$

Bare Lagrangian

$$\mathcal{L}_{\text{cl}}(g; \psi, A_a^\mu, \ldots) \quad \to \quad \mathcal{L}_{\text{bare}}(g^{\text{bare}}; \psi^{\text{bare}}, A_a^{\mu\,\text{bare}}, \ldots)$$

Proof of renormalizability

- by induction

Assumption:

- $\Gamma^{(n-1)}$ finite up to $(n-1)$-loop level
- all defining equations valid at $(n-1)$-loop level
- and on the regularized level at n-loop level (e.g. dim. reg.)

Claim:

- all n-loop divergences can be absorbed by multiplicative renormalization
- (only free physical parameter: g)

Proof:

$$\Gamma_{\text{reg}}^{(\leq n)} = \Gamma_{\text{fin}}^{(\leq n)} + \Gamma_{\text{div}}^{(n)}$$
$$\Gamma_{\text{div}}^{(n)} = \text{local, equivalent to Lagrangian terms}$$

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Hence, rearrange $0 = S(\Gamma_{\text{reg}}^{(\leq n)})$ as

$$0 = \int \frac{\delta(\Gamma_{\text{fin}}^{(\leq n)} + \Gamma_{\text{div}}^{(n)})}{\delta Y_{\varphi_i}} \frac{\delta(\Gamma_{\text{fin}}^{(\leq n)} + \Gamma_{\text{div}}^{(n)})}{\delta \varphi_i}$$

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$$= \int \frac{\delta(\Gamma_{\text{fin}}^{(\leq n)})}{\delta Y_{\varphi_i}} \frac{\delta(\Gamma_{\text{div}}^{(n)})}{\delta \varphi_i} + (\text{fin} \leftrightarrow \text{div}) + \text{fin.} + \mathcal{O}(2n\text{-loop})$$

Proof:

$$\Gamma^{(\leq n)}_{\text{reg}} = \Gamma^{(\leq n)}_{\text{fin}} + \Gamma^{(n)}_{\text{div}}$$
$$\Gamma^{(n)}_{\text{div}} = \text{local, equivalent to Lagrangian terms}$$

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$$= \int \frac{\delta(\Gamma_{\text{cl}})}{\delta Y_{\varphi_i}} \frac{\delta(\Gamma^{(n)}_{\text{div}})}{\delta \varphi_i} + (\text{cl} \leftrightarrow \text{div}) + \text{fin.} + \mathcal{O}((n+1)\text{-loop})$$

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$$= S(\Gamma_{\text{cl}} + \Gamma_{\text{div}}^{(n)}) + \text{fin.} + \mathcal{O}((n+1)\text{-loop})$$

Hence, the divergences are constrained by the STI,

$$S(\Gamma_{\text{cl}} + \Gamma_{\text{div}}^{(n)}) = 0$$

- Can be absorbed by counterterms generated by the most general classical solution
- thus by multiplicative renormalization ($\Rightarrow$ claim)
- the bare action is thus changed as

$$\Gamma_{\text{bare}}^{(n)} = \Gamma_{\text{bare}}^{(n-1)} + \Gamma_{\text{ct}}^{(n)}$$

- this change does not invalidate the defining equations (STI) ($\Rightarrow$ assumption at order n)

proof complete

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generates most general classical solution (with this gauge fixing $\frac{\delta\Gamma}{\delta B_a}$)

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\begin{aligned}
g &\to g^{\text{bare}} = g + \delta g \\
\psi &\to \sqrt{Z_\psi}\psi \\
\{A^\mu, B, \bar{c}, \xi\} &\to \left\{\sqrt{Z_A}A^\mu, \sqrt{Z_A}^{-1}B, \sqrt{Z_A}^{-1}\bar{c}, Z_A\xi\right\} \\
c &\to \sqrt{Z_c}c \\
Y_{\varphi_i} &\to \sqrt{Z_{\varphi_i}}^{-1}Y_{\varphi_i}
\end{aligned}
$$

Bare Lagrangian generates counterterms

$$
\begin{aligned}
\mathcal{L}_{\text{cl}}(g; \psi, A_a^\mu, \ldots) &\to \mathcal{L}_{\text{bare}}(g^{\text{bare}}; \psi^{\text{bare}}, A_a^{\mu\,\text{bare}}, \ldots) \\
&= \mathcal{L}_{\text{cl}}(g; \psi, A_a^\mu, \ldots) + \mathcal{L}_{\text{ct}}(g; \psi, A_a^\mu; \delta g, \delta Z_{\psi,A,c}, \ldots)
\end{aligned}
$$

Outline

3. Renormalizability of gauge theories — QCD
   - Reminder and overview
   - Definition and proof of renormalizability
   - Outlook: algebraic renormalization
   - Two small but important applications

Systematic analysis: algebraic renormalization [Piguet et al]

QFT at higher orders: Loops + counterterms

$$\Gamma^{\text{ren}} = \Gamma^{\text{reg}} + \Gamma^{\text{ct}}$$

$$\begin{aligned} \Gamma^{\text{ren}} &: \text{ physical content} \\ \Gamma^{\text{reg}}, \Gamma^{\text{ct}} &: \text{ unphysical} \end{aligned}$$

Precise, regularization-independent definition of theory by symmetries, e.g. Slavnov-Taylor identities:

$$S(\Gamma^{\text{ren}}) = 0$$

Theory defined by symmetries: $S(\Gamma^{\mathrm{reg}} + \Gamma^{\mathrm{ct}}) = 0$

Case 1:	$S(\Gamma^{\mathrm{reg}}) = 0$	
Case 2a:	$S(\Gamma^{\mathrm{reg}}) = \Delta,$	$S(\Gamma^{\mathrm{reg}} + \Gamma^{\mathrm{ct}}) = 0$
Case 2b:	$S(\Gamma^{\mathrm{reg}}) = \Delta,$	$S(\Gamma^{\mathrm{reg}} + \Gamma^{\mathrm{ct}}) \neq 0$

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"Textbook case": regularization preserves symmetries

- multiplicative renormalization (cts symmetric)

$$g \to g + \delta g, \quad m \to m + \delta m$$

- most common situation, often assumed without proof

Theory defined by symmetries: $S(\Gamma^{\rm reg} + \Gamma^{\rm ct}) = 0$

Case 1:	$S(\Gamma^{\rm reg}) = 0$	
Case 2a:	$S(\Gamma^{\rm reg}) = \Delta,$	$S(\Gamma^{\rm reg} + \Gamma^{\rm ct}) = 0$
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Nice but not necessary!

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Case 2b:	$S(\Gamma^{\rm reg}) = \Delta,$	$S(\Gamma^{\rm reg} + \Gamma^{\rm ct}) \neq 0$

Nice but not necessary!

- Case 1 $\Leftrightarrow$ Case 2a $\Leftrightarrow$ theory renormalizable
- Renormalizability proof has two steps:
  1. Find Slavnov-Taylor id. $S(\Gamma^{\rm ren}) = 0$
  2. Prove that STI can be satisfied

For SUSY:

[Piguet, Sibold '84], [White '92]

[Piguet et al '96], [Hollik, Kraus, DS '99]. . .

[Hollik,Kraus,Roth,Rupp,Sibold, DS '02]

Theory defined by symmetries: $S(\Gamma^{\rm reg} + \Gamma^{\rm ct}) = 0$

Case 1:	$S(\Gamma^{\rm reg}) = 0$	
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In principle, we don't have to bother whether a regularization preserves symmetries

Theory defined by symmetries: $S(\Gamma^{\mathrm{reg}} + \Gamma^{\mathrm{ct}}) = 0$

Case 1:	$S(\Gamma^{\mathrm{reg}}) = 0$	
Case 2a:	$S(\Gamma^{\mathrm{reg}}) = \Delta,$	$S(\Gamma^{\mathrm{reg}} + \Gamma^{\mathrm{ct}}) = 0$
Case 2b:	$S(\Gamma^{\mathrm{reg}}) = \Delta,$	$S(\Gamma^{\mathrm{reg}} + \Gamma^{\mathrm{ct}}) \neq 0$

In practice, life is easier with a symmetry-preserving regularization!

Outline

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   - Reminder and overview
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   - Outlook: algebraic renormalization
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Useful side result: application to QCD β function

$\beta(g)$ from Z_g

- Possibility 1: from

$$\text{Quark-Quark-Gluon } \delta Z_g + \frac{1}{2}\delta Z_A + \delta Z_\psi,$$
$$\text{Quark s.e. } \delta Z_\psi$$
$$\text{Gluon s.e. } \delta Z_A$$

- Possibility 2: from

$$Y_c cc\text{-interaction } \delta Z_g + \frac{1}{2}\delta Z_c,$$
$$c\text{-s.e. } \frac{1}{2}\delta Z_c - \frac{1}{2}\delta Z_A$$
$$\text{Gluon s.e. } \delta Z_A$$

Second possibility much simpler!

How to obtain Ward/Slavnov-Taylor identities for amplitudes?

Amplitudes

- on-shell, physical polarization vectors
- obtained from full Green functions by LSZ reduction ($\rightarrow$pole part!)

$$iT_{ABC\ldots} = \left.\frac{\langle 0|T\Phi_A\Phi_B\Phi_C\ldots|0\rangle}{\langle 0|T\Phi_A\Phi_A^\dagger|0\rangle\,\langle 0|T\Phi_B\Phi_B^\dagger|0\rangle\ldots}\right|_{\text{on-shell}} \times \text{norm. wave fct.}$$

$$\text{norm. wave fct.} = \langle 0|\Phi_A|A\rangle, \ldots$$

Hence, first consider identities for full Green functions, then LSZ reduction

Identities for full Green functions

For 1PI:

$$S(\Gamma) = \int d^4x \frac{\delta\Gamma}{\delta Y_{\varphi_i}(x)} \frac{\delta\Gamma}{\delta\varphi_i(x)}$$

Identities for full Green functions

Y_{φ_i} is source of loop-corrected BRS transformation:

$$S(\Gamma) = \int d^4x \langle s\varphi_i(x) \rangle \frac{\delta\Gamma}{\delta\varphi_i(x)}$$

Identities for full Green functions

Legendre transformation to full Green functions:

$$S(Z) = \int d^4x \; J_i(x) \, \frac{\delta Z}{\delta Y_{\varphi_i(x)}}$$

Identities for full Green functions

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Taking derivatives of $0 = S(Z)$ leads to identities like

$$0 = \langle (s\Phi_A)\Phi_B \ldots \rangle \pm \langle \Phi_A (s\Phi_B) \ldots \rangle \pm \ldots$$

where $(s\Phi)$ is a renormalized composite operator

On-shell vs. off-shell

$$\langle (s\Phi_A)\Phi_B \ldots \rangle|_{\text{pole-part}}$$

Distinguish two cases

- $(s\Phi_A)$ linear in fields
  - above is just linear combination of ordinary Green functions which have poles for on-shell external momenta
- $(s\Phi_A) \propto c\Phi_A$ or similar $\rightarrow$ non-linear
  - cannot produce a pole in external momentum (in finite order)

Linear BRS transformations in QCD or in QED:

$$sA^\mu = \partial^\mu c + \ldots \qquad s\bar{c} = B$$

$$sc_a = \frac{1}{2} g f_{abc} c_b c_c (= 0(\text{QED})) \qquad s\psi \propto c\psi$$

Amplitudes with many gluons/photons, and quarks/electrons (all on-shell)

$$\epsilon^{\mu_1}\epsilon^{\mu_2}\ldots\mathcal{M}_{\mu_1\mu_2\ldots}(k_1,k_2,\ldots) \leftrightarrow \langle A_{\mu_1}A_{\mu_2}\ldots\rangle|_{\text{on-shell,pole-part}}$$

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Obtain STI:

$$0=\langle(s\bar{c})A_{\mu_2}\ldots\rangle+\langle\bar{c}(sA_{\mu_2})\ldots\rangle+\ldots$$

other terms in sA_μ or $s\psi$ do not contribute

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Obtain STI: note: $s\bar{c}=B=-\frac{1}{\xi}\partial^\mu A_\mu$; take on-shell,pole-part

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Obtain STI:

$$0=\langle(s\bar{c})A_{\mu_2}\dots\rangle+\langle\bar{c}(sA_{\mu_2})\dots\rangle+\dots$$
$$0=-\frac{1}{\xi}\langle\partial^{\mu_1}A_{\mu_1}A_{\mu_2}\dots\rangle+\langle\bar{c}(\partial_{\mu_2}c)\dots\rangle+\dots$$

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Hence, in obvious notation

$$\frac{1}{\xi}k_1^{\mu_1}\mathcal{M}_{\mu_1\mu_2\ldots}(k_1,k_2,\ldots)=k_{2\,\mu_2}\mathcal{M}_{\bar{c}c\ldots}(k_1,k_2,\ldots)+\ldots$$

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In QED, ghosts are free, r.h.s. cannot contribute if $k_1+k_2\neq 0$:

$$\text{QED}:\quad k_1^{\mu_1}\mathcal{M}_{\mu_1\mu_2\ldots}(k_1,k_2,\ldots)=0$$

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In QCD, the r.h.s. vanishes after contraction with physical ϵs:

$$\text{QCD}:\quad k_1^{\mu_1}\epsilon^{\mu_2}\ldots\mathcal{M}_{\mu_1\mu_2\ldots}(k_1,k_2,\ldots)=0$$

[discussion and more general QCD result: Leader/Predazzi 2011]

Outline

- $gg \to H$ is very important process
- operator renormalization necessary
- nice application of BRS/ST identities and quantum action principle

  [Joglekar, Lee; Kluberg-Stern, Zuber; Spiridonov]
- changes in FDH/DRED

Integrate out top-loop $\rightarrow$ effective operator

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}\lambda\, H\, F_a^{\mu\nu} F_{a,\mu\nu}$$

gauge invariant dimension-5 operator, λ=effective coupling

Integrate out top-loop $\rightarrow$ effective operator

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}\lambda\, H\, F_a^{\mu\nu} F_{a,\mu\nu}$$

gauge invariant dimension-5 operator, λ=effective coupling

- Interested only in QCD corrections
- Higgs appears only as external field, no propagator

Integrate out top-loop → effective operator

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}\lambda\, H\, F_a^{\mu\nu} F_{a,\mu\nu}$$

gauge invariant dimension-5 operator, λ=effective coupling

$\Rightarrow$ Treat

$$\lambda\, H(x) \equiv Y_1(x)$$

as external field (source in generating functional)

Starting point

$$\mathcal{L}_{\text{eff}} = Y_1(x)\; O_1(x)$$
$$O_1 = -\frac{1}{4} F_a^{\mu\nu} F_{a,\mu\nu}$$

Task:

- compute renormalized Green functions with one external Y_1
  $\Leftrightarrow$ with one insertion of operator O_1

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Difficulty:

- $\mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{eff}}$ not multiplicatively renormalizable!
- need many more terms in $\mathcal{L}_{\text{eff}}$! (e.g. $H \to q\bar{q}$, $H \to c\bar{c}$ etc)

Correct procedure

- repeat proof of renormalizability of QCD, but one change:
- additional external field $Y_1(x)$, bosonic, dim=0, $N_{gh} = 0$

Steps:

- write down Slavnov-Taylor identity — literally unchanged
- most general classical solution — changed, can depend on $Y_1(x)$!
- most general structure of divergences — same change
- theory QCD$\oplus Y_1(x)$ is multiplicatively renormalizable
  if we start with most general classical solution of STI

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  $\Leftrightarrow$ general BRS transformations $\tilde{s}\varphi_i$

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Find this most general classical solution of QCD$\oplus Y_1(x)$!

- Step 1: Dependence on sources Y_{φ_i} of BRS transformations
  $\Leftrightarrow$ general BRS transformations $\tilde{s}\varphi_i$
  as before, arise from standard form by multiplicative renormalization **but**

$$Z_{A,\psi,c,g} = Z_{A,\psi,c,g}(Y_1(x))$$

(power series not only in coupling but also in external field $Y_1(x)$)

What needs to be done explicitly?

Find this most general classical solution of QCD$\oplus Y_1(x)$!

- Step 2: Lagrangian without BRS sources, $Y_{\varphi_i} = 0$:
  as before,

$$\Gamma_{\text{rest}}(\psi, A^\mu, c, B, \bar{c}) = \Gamma_{\text{g.inv.}}(\psi, A^\mu) + \int d^4x \tilde{s} X(\psi, A^\mu, c, B, \bar{c})$$

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$$\tilde{s}X = \tilde{s}[\bar{c}_a((\partial_\mu A_a^\mu) + \frac{\xi}{2}B_a)]$$

- **but:** BRS and gauge transformations as usual, but for

$$g_{\text{bare}}(x)A^\mu_{\text{bare}}(x) = \sqrt{Z_A(Y_1(x))}Z_g(Y_1(x))gA^\mu(x), \quad \text{etc}$$

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$$g_{\text{bare}}(x)A^\mu_{\text{bare}}(x) = \sqrt{Z_A(Y_1(x))}Z_g(Y_1(x))gA^\mu(x), \quad \text{etc}$$

- **here:** x-dependent prefactors don't change the possible terms!
  different for scalars: $Z_\phi(\partial^\mu\phi)(\partial_\mu\phi) \longrightarrow Z_\phi(Y_1)(\partial^\mu\phi)(\partial_\mu\phi) + (\Box z_{\text{new}}(Y_1))\phi\phi$ [Gnendiger,Signer,DS]

What needs to be done explicitly?

Find this most general classical solution of QCD$\oplus Y_1(x)$!

- Step 2: Lagrangian without BRS sources, $Y_{\varphi_i} = 0$:
  as before,

$$\Gamma_{\text{rest}}(\psi, A^\mu, c, B, \bar{c}) = \Gamma_{\text{g.inv.}}(\psi, A^\mu) + \int d^4x \tilde{s}X(\psi, A^\mu, c, B, \bar{c})$$

$$\tilde{s}X = \tilde{s}[\bar{c}_a((\partial_\mu A^\mu_a) + \frac{\xi}{2}B_a)]$$

- **but:** BRS and gauge transformations as usual, but for

$$g_{\text{bare}}(x)A^\mu_{\text{bare}}(x) = \sqrt{Z_A(Y_1(x))}Z_g(Y_1(x))gA^\mu(x), \quad \text{etc}$$

- **here:** x-dependent prefactors don't change the possible terms!
  different for scalars: $Z_\phi(\partial^\mu\phi)(\partial_\mu\phi) \longrightarrow Z_\phi(Y_1)(\partial^\mu\phi)(\partial_\mu\phi) + (\Box z_{\text{new}}(Y_1))\phi\phi$ [Gnendiger,Signer,DS]

Result:

- Most general classical solution as for QCD alone, but Y_1-dependent renormalization constants:
- Obtained from $\mathcal{L}_{\text{QCD}}$ (without Y_1 and O_1!) by applying

$$
\begin{aligned}
g &\to g^{\text{bare}} = g + \delta g = Z_g g \\
\psi &\to \sqrt{Z_\psi}\psi \\
\{A^\mu, B, \bar{c}, \xi\} &\to \left\{\sqrt{Z_A}A^\mu, \ldots (\text{related})\right\} \\
c &\to \sqrt{Z_c}c \\
Y_{\varphi_i} &\to \sqrt{Z_{\varphi_i}}^{-1} Y_{\varphi_i} \\
Z_{A,\psi,c,g} &= Z_{A,\psi,c,g}(Y_1(x))
\end{aligned}
$$

- the arising $\mathcal{L}_{\text{bare}}$ is sufficient to cancel all divergences

$F^{\mu\nu}(A) \longrightarrow \frac{1}{g_{\text{bare}}} F^{\mu\nu}(g_{\text{bare}} A_{\text{bare}})$

Application to operators and $gg \to H$

only one external Higgs/one external $Y_1(x)$/one insertion of $O_1(x)$

- Such Green functions will be finite (since everything is finite)
- but they require only

$$\mathcal{L}_{\text{bare}}\Big|_{\text{up to one power of } Y_1} =: \mathcal{L}_{\text{bare,QCD}} + Y_1 \sum_j z_j O_j^{\text{bare}}$$

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- Now compute the most general structure of the Y_1-terms
- have to apply the renormalization transformation to $\mathcal{L}_{\text{QCD}}$ and only take Y_1-terms!

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- Now compute the most general structure of the Y_1-terms
- have to apply the renormalization transformation to $\mathcal{L}_{\text{QCD}}$ and only take Y_1-terms!
- i.e. expand the four ren. constants only as

$$Z_i(Y_1) = 1 + z_i Y_1$$

- then the desired result is a linear combination of the z_i-terms

Now compute the most general structure of the Y_1-terms

$$\mathcal{L}_{\text{QCD}} = \bar{\psi} i\gamma^\mu D_\mu \psi - \frac{1}{4} F_a^{\mu\nu} F_{a\mu\nu} + s[\bar{c}_a((\partial_\mu A_a^\mu) + \frac{\xi}{2} B_a)]$$

- four ren. constants — expand each as $Z_i(Y_1) = 1 + z_i Y_1$
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Start with $\sqrt{Z_\psi}$, $\psi \to (1 + \frac{1}{2} z_\psi Y_1 + \ldots)\psi$:

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$$\mathcal{L}_{\text{QCD}} \longrightarrow \ldots + z_\psi \frac{1}{2} \left((Y_1 \bar{\psi}) i \gamma^\mu D_\mu \psi + \bar{\psi} i \gamma^\mu D_\mu (Y_1 \psi) \right)$$

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Now compute the most general structure of the Y_1-terms

$$\mathcal{L}_{\rm QCD} = \bar\psi i\gamma^\mu D_\mu \psi - \frac{1}{4} F_a^{\mu\nu} F_{a\mu\nu} + s[\bar c_a((\partial_\mu A_a^\mu) + \frac{\xi}{2} B_a)]$$

Finally, $c \to (1 + \frac{1}{2} z_c Y_1)c$

on top of all of this, we can apply ordinary renormalization transformation!! ($O_i \to O_i^{\rm bare}$)

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$$\mathcal{L}_{\text{QCD}} \longrightarrow \ldots + z_c \frac{1}{2} Y_1 \, (-O_5)$$
$$O_5 = (D^\mu \partial_\mu \bar{c})_a c_a.$$

Result: most general terms linear in $Y_1(x)$

[Kluberg-Stern, Zuber '74; Joglekar, Lee '75]

$$\mathcal{L}_{\text{bare}} = \mathcal{L}_{\text{QCD,bare}} + Y_1(x) \sum_{j=1}^{5} z_j O_j^{\text{bare}}(x) + \mathcal{O}(Y_1^2)$$

$$O_1 = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu,a},$$

$$O_2 = 0,$$

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basis of operators required to cancel divergences
$O_{4,5}$ not gauge invariant, O_3 vanishes by eq. of motion

Study the renormalization of these operators in more detail — notation

$$\sum_j Y_1 z_j O_j^{\text{bare}}$$

- z_j= bare quantities, contain $\frac{1}{\epsilon}$, depend on couplings
- New parameters: z_j^{tree}

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Notation: operator renormalization

$$\sum_j z_j O_j^{\text{bare}} = \sum_{ij} z_i^{\text{tree}}(\delta_{ij} + \delta Z_{ij}) O_j^{\text{bare}} \equiv \sum_i z_i^{\text{tree}} O_{i,\text{ren}}$$

Next question: what is the result of this operator renormalization? Answered in DREG by trick

[Kluberg-Stern, Zuber '74][Spiridonov '84]

Desired:

- result of $O_{i,\text{ren}} = (\delta_{ij} + \delta Z_{ij}) O_j^{\text{bare}}$???

Idea: could be possible to obtain from QCD, since all operators already exist in $\mathcal{L}_{\text{QCD}}$

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Observation (valid here, not in general):

- If we know $\int Y_1 \sum z_j O_j^{\text{bare}}$ for Y_1 =const, we know it in general! (no total derivative appears)
- All operators can be expressed by differential operators

Example: $O_1 = -\frac{1}{4} F_a^{\mu\nu} F_{a\mu\nu}$

B-field is eliminated now

$$\int O_1 = \left(\frac{1}{2}\int A\frac{\delta}{\delta A} + D_1\right)\int \mathcal{L}_{\text{QCD}}$$

$$D_1 = -\frac{1}{2} g\partial_g + \xi\partial_\xi$$

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Spiridonov: Now use regularized quantum action principle in DREG

$$\text{finite} = D_1 Z(J) = \int \mathcal{D}\phi\, D_1\; e^{i\int \mathcal{L}_{\text{bare}} + J\phi}$$

$$= \int \mathcal{D}\phi\, i \quad \underbrace{\left(D_1 \int \mathcal{L}_{\text{bare}}\right)} \quad e^{i\int \mathcal{L}_{\text{bare}} + J\phi}$$

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Operator renormalization: results

Can represent

$$\int O_{1,\text{ren}} = D_1 \int \mathcal{L}_{\text{bare}} = (D_1 g^{-2} Z_g^{-2}) \int g^2 Z_g^2 O_1^{\text{bare}} + \text{rest}$$

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$$\int O_{1,\text{ren}} = D_1 \int \mathcal{L}_{\text{bare}} = \sum_j (D_1 \log \mathbf{Z}_j) \int O_j^{\text{bare}}$$

$$\mathbf{Z}_1 = g^{-2} Z_g^{-2}$$

$$\mathbf{Z}_3 = Z_\psi$$

$$\mathbf{Z}_4 = Z_g \sqrt{Z_A}$$

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In particular, often used result:

$$Z_{11} = 1 + D_1 \log Z_g^{-2} = 1 + \alpha_s \partial_{\alpha_s} \log Z_{\alpha_s}$$

Outline

Outline

5 Additional topics

- Algebraic renormalization of SUSY
- More information on operator renormalization
- Custodial symmetry
- renormalization of vevs

on non invariant regularizations

- QFT at higher orders: Loops + counterterms $\Gamma^{\text{ren}} = \Gamma^{\text{reg}} + \Gamma^{\text{ct}}$
- Theory defined by symmetries: $S(\Gamma^{\text{reg}} + \Gamma^{\text{ct}}) = 0$

Case 1:	$S(\Gamma^{\text{reg}}) = 0$	
Case 2a:	$S(\Gamma^{\text{reg}}) = \Delta,$	$S(\Gamma^{\text{reg}} + \Gamma^{\text{ct}}) = 0$
Case 2b:	$S(\Gamma^{\text{reg}}) = \Delta,$	$S(\Gamma^{\text{reg}} + \Gamma^{\text{ct}}) \neq 0$

- Case 1: "Textbook case", case 2a: equally good.
  Decide algebraically whether possible
  ("Algebraic renormalization" [BRS, Piguet, ...])
  symmetry-restoring c.t.s uniquely fixed; rest: multiplicative renorm.
- Case 2b: anomaly — theory inconsistent

Questions/Tasks

- as for QCD: finiteness, physical meaning (gauge invariance, SUSY)?
- minimal or full field renormalization?

Tasks:

- Find suitable STI for gauge invariance + SUSY
- Prove that STI can be satisfied (even if regularization breaks it)
- Use STI to obtain answers
- Can draw further interesting conclusions

Difficult to find STI

Gauge fixing required

- SUSY gauge in superfield formalism $\rightarrow$ solved [Piguet, Sibold '84]
- Wess-Zumino gauge: fewer unphysical d.o.f.
  Breaks SUSY — Renorm., sym. identities difficult

  [Breitenlohner, Maison '85][White '92, Maggiore, Piguet, Wolf '96]

Peculiarities of the WZ gauge

- Algebra modified

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot\alpha}\} &= 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \quad + \sim \delta_{\text{gauge}} \\ [Q_\alpha, \delta_{\text{gauge}}] &= 0, \quad [P_\mu, \delta_{\text{gauge}}] \neq 0 \end{aligned}$$

- Gauge fixing $-\frac{1}{2\xi}(\partial^\mu A_\mu)^2$ breaks SUSY

Must treat gauge invariance, SUSY together

Construction of STI

Generalize BRST, Batalin/Vilkovisky formalism to gauge invariance+SUSY, then STI follows as usual [White '92, Maggiore, Piguet, Wolf '96]

- Ghosts for all generators in symmetry algebra $\longrightarrow$ BRS:

$$s\varphi = (c_a \delta_{\text{gauge},a} + \epsilon^\alpha Q_\alpha + \bar{Q}_{\dot\alpha}\overline{\epsilon}^{\dot\alpha} - \omega^\mu P_\mu)\varphi$$

- BRS transformations of ghosts $\leftrightarrow$ $s^2 = 0$:

$$\begin{aligned} sc_a &= \frac{1}{2} g f_{abc} c_b c_c + 2i\epsilon\sigma^\mu\overline{\epsilon} A_{a\mu} - i\omega^\mu \partial_\mu c_a \\ s\omega^\mu &= 2\epsilon\sigma^\mu\overline{\epsilon} \\ \Leftrightarrow \{Q_\alpha, \bar{Q}_{\dot\alpha}\} &= 2\sigma^\mu_{\alpha\dot\alpha}(P_\mu - iA_{a\mu}\delta_{\text{gauge},a}) \end{aligned}$$

MSSM specifics

- abelian subgroup (simpler but less constraining) [Hollik, Kraus, DS '99]
- soft SUSY breaking, e.g.

$$\begin{array}{rcl} H_1 QD + H_2 QU & \rightarrow & \text{allowed} \\ H_2^\dagger QD + H_1^\dagger QU & \rightarrow & \text{forbidden} \end{array}$$

  use coupling to spurions [Maggiore,Piguet, Wolf '96][Hollik, Kraus, DS '01][Golterman, Shamir '10]
- Gauge fixing vs mixing $A^0/G^0/Z^\mu_{\rm long}$

Resulting STI describes softly broken SUSY, and gauge invariance, WZ gauge fixing [Hollik, Kraus, Roth, Rupp, Sibold, DS '02]

Results very satisfactory

Renormalizability/answers to questions: [Hollik, Kraus, Roth, Rupp, Sibold, DS '02]

- STI can be satisfied: no SUSY or gauge anomalies
- If regularization symmetric: mult., minimal renormalization sufficient
- If not: symmetry-restoring counterterms uniquely determined
- Full field renormalization possible

- complete on-shell unmixing possible (also for unphysical d.o.f.)
- no infrared off-shell div.s

MSSM renormalizable, all above renormalization tranformations ok

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- Full field renormalization possible

  $\Leftrightarrow$ after renormalization: $\begin{pmatrix} \tilde{f}_L \\ \tilde{f}_R \end{pmatrix} \to \mathcal{R} \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix}$ also in STI
- complete on-shell unmixing possible (also for unphysical d.o.f.)
- no infrared off-shell div.s

MSSM renormalizable, all above renormalization tranformations ok

Results very satisfactory

Physical meaning:

- Gauge invariant, SUSY, finite theory, renormalized gauge/SUSY transformations defined
- can define S-matrix, phys. Hilbert space, SUSY operator Q_α^{in}:

$$[Q_\alpha^{\text{in}}, S] = [Q^{\text{BRS}}, \ldots]$$

$\Rightarrow$ Q_α^{in} conserved on phys. Hilbert space [Rupp, Scharf, Sibold '01]

Further results possible on:

- gauge dependence
- non-renormalization theorems

Outline

5. Additional topics
   - Algebraic renormalization of SUSY
   - More information on operator renormalization
   - Custodial symmetry
   - renormalization of vevs

Various statements on operator renormalization

- use equations of motion of lower orders to modify higher-dimension terms

$$S_{\text{eff}} \to S_{\text{eff}} + \Delta\phi \frac{\delta S_{\text{eff}}}{\delta\phi} + \mathcal{O}((\Delta\phi)^2)$$

  this is a field redefinition and thus does not change physical quantities but only Green functions
- non-gauge invariant operators (which have to be total BRS-variations) do not contribute to observables (but to Green functions)

Outline

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   - Algebraic renormalization of SUSY
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Properties of general, non-SM electroweak theory

$$SU(2)_L \times U(1)_Y \to U(1)_{Q=T^3+Y}$$

- gauge invariance has four generators, four gauge bosons:

$$T^A = (T^a, Y), \qquad A = 1,2,3,4; \qquad a = 1,2,3.$$
$$V_A^\mu = (W_a^\mu, B^\mu)$$
$$D^\mu = \partial^\mu + ig^A T^A V_A^\mu, \qquad g^A = (g, g, g, g')$$

- commutators are defined by $SU(2)_L \times U(1)_Y$
- vacuum invariant under $Q = T^3 + Y$

Option 1: elementary scalar fields ϕ exist and break symmetry at tree-level

Option 2: different

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Option 2: different

Theorem 1: state general mass matrix

$$\mathcal{M}^2_{AB} = \langle\phi\rangle^\dagger \{g^A T^A, g^B T^B\} \langle\phi\rangle = \begin{pmatrix} g^2 v^2 & & & \\ & g^2 v^2 & & \\ & & g^2 u^2 & -g' g u^2 \\ & & -g' g u^2 & g'^2 u^2 \end{pmatrix}$$

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v and u are two unknowns: $\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = \frac{v^2}{u^2}$

- mass matrix has $\mathrm{U}(1)_Q$ invariance $\leftrightarrow$ O(2) invariance:
- $\rho = 1$ would mean $u = v$ — an additional O(3) or SU(2) custodial symmetry!

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Only have to compute $|D^\mu\phi|^2$, set $\phi \to \langle\phi\rangle$

Result: always has the form $\frac{1}{2} V_A^\mu \mathcal{M}^2_{AB} V_{B\mu}$ with

$$\mathcal{M}^2_{AB} = \langle\phi\rangle^\dagger \{g^A T^A, g^B T^B\} \langle\phi\rangle = \begin{pmatrix} g^2 v^2 & & & \\ & g^2 v^2 & & \\ & & g^2 u^2 & -g' g u^2 \\ & & -g' g u^2 & g'^2 u^2 \end{pmatrix}$$

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Proof of this general form of the mass matrix

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Use $U(1)_{Q=T^3+Y}$ invariance of vacuum: $(T^3 + Y)\langle\phi\rangle = 0$

- $\mathcal{M}^2_{A3} = -\frac{g}{g'}\mathcal{M}^2_{A4}$ etc $\Rightarrow$ lower right block
- $0 = \langle\phi\rangle^\dagger [T^3 + Y, g^A T^A g^B T^B] \langle\phi\rangle$ leads to

$$(A = 1, B = 2) : \mathcal{M}^2_{11} = \mathcal{M}^2_{22}$$
$$(A = B = 1, 2) : \mathcal{M}^2_{12} = \mathcal{M}^2_{21} = 0$$
$$(A = 1, B = 3) : \mathcal{M}^2_{13} = \mathcal{M}^2_{14} = 0 \text{ etc}$$

- this proves the block structure

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- this proves the block structure

Theorem 2: Goldstone boson kinetic terms in the corresponding non-gauge theory

Answer: the 3×3-submatrix $\mathcal{M}^2_{ab}$!

$$\mathcal{L}_{\text{kin},G} = \frac{1}{2}\langle\phi\rangle^\dagger\{T^a, T^b\}\langle\phi\rangle(\partial^\mu G^a)(\partial_\mu G^b) = \frac{\mathcal{M}^2_{ab}}{2g^2}(\partial^\mu G^a)(\partial_\mu G^b)$$

Theorem 2: Goldstone boson kinetic terms in the corresponding non-gauge theory

- still work at tree level, with elementary scalar fields
- consider the corresponding non-gauge theory with $g = g' = 0$

$$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_{Q=T^3+Y}$$

- three physical, massless Goldstone bosons G^a, $a = 1, 2, 3$:

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What are their kinetic terms?

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Proof of this general form of the kinetic terms

Define Goldstone boson fields in nonlinear form by splitting off factor from ϕ:

$$\phi(x) = e^{iT^a G^a(x)}\tilde{\phi}(x) \equiv U(x)\tilde{\phi}(x)$$

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Kinetic terms:

$$\mathcal{L}_{\text{Higgs}} = |\partial^\mu \phi|^2 = \langle\tilde{\phi}\rangle^\dagger \partial^\mu U^\dagger \partial_\mu U \langle\tilde{\phi}\rangle + \ldots$$

lead to the above statement!

Consequence: relation to custodial symmetry

What is custodial symmetry?

A symmetry of the non-gauge theory for $g = g' = 0$, under which the Goldstone bosons transform as an SU(2) (or SO(3)) triplet:

$$G^a \to R_{ab} G^b, R \in \mathrm{SO}(3)$$

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If custodial symmetry holds, then the Goldstone kinetic terms are

$$\propto \mathcal{M}^2_{ab} \propto \delta_{ab}$$

and thus $u = v$ in the vector boson mass matrix:

$$\mathcal{M}^2_{AB} = \langle\phi\rangle^\dagger \{g^A T^A, g^B T^B\} \langle\phi\rangle = \begin{pmatrix} g^2 v^2 & & & \\ & g^2 v^2 & & \\ & & g^2 v^2 & -g' g v^2 \\ & & -g' g v^2 & g'^2 v^2 \end{pmatrix}$$

Custodial Symmetry in SM

- rewrite SM Higgs doublet and Higgs potential using

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \longrightarrow \Phi = \begin{pmatrix} \phi^{0*} & \phi^+ \\ -\phi^- & \phi^0 \end{pmatrix}$$

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$$V(\Phi) = \mu^2 \mathrm{Tr}(\Phi^\dagger \Phi) + \lambda \mathrm{Tr}(\Phi^\dagger \Phi)^2$$

- symmetric under $SU(2)_L \times SU(2)_R$, $\Phi \to L \Phi R^\dagger$
- vacuum $\langle \Phi \rangle_{\mathrm{vac}} = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$ invariant under $SU(2)_{L=R}$
- ($U(1)_Y$ and $U(1)_Q$ are subgroups)

Violation of Custodial Symmetry by Higgs Triplet

- Triplet Φ, Y=0, SU(2)⇔O(3)-rotations
- but in vacuum: $\langle\Phi\rangle = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$, no remnant SU(2) or O(3)
- mass term

$$\mathcal{M}^2_{ab} = \langle\phi\rangle^\dagger\{T^a, T^b\}\langle\phi\rangle = \begin{pmatrix} g_2^2 v^2 & & & \\ & g_2^2 v^2 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}$$

- $M_W^2 = g_2^2 v^2$, $M_Z^2 = 0$

…but it can be well motivated to consider such models

Outline

5. Additional topics
   - Algebraic renormalization of SUSY
   - More information on operator renormalization
   - Custodial symmetry
   - renormalization of vevs

Renormalization of VEVs

Higgs/spontaneously broken gauge invariance:

$$\phi \to \phi + v$$

such that $\langle\phi\rangle = 0$, i.e. tadpoles vanish
Need to renormalize:

$$\phi \to \sqrt{Z}\phi, \quad v \to v + \delta v$$

Details and questions

Most generic renormalization transformation:

$$(\phi + v) \to \sqrt{Z}\phi + v + \delta v$$
$$\text{or} \quad (\phi + v) \to \sqrt{Z}(\phi + v + \delta\bar{v})$$

Ultimately δv is important for $\delta \tan\beta$, β functions, etc.

$\delta\bar{v}$ characterizes to what extent v renormalizes differently from ϕ.

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1. When/why $\delta \bar{v} \neq 0$?
2. Properties of $\delta \bar{v}$?
3. β_v and applications.

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Idea:

- $\delta\bar{v} = v\delta\hat{Z}$
- compute $\delta\hat{Z}$ (using STI)

Current status for RGE coefficients

needed by SUSY spectrum generators (Spheno, Softsusy, SuseFlav, FlexibleSUSY, Sarah)

Model	β(phys. parameter)	γ(fields)
$\forall$ gauge theory	Machacek,Vaughn '83, Luo et al '03	
$\forall$ SUSY model	Martin, Vaughn; Jack, Jones; Yamada '93	partially

Note in SUSY: $\gamma(\text{scalar in WZ gauge+Landau or } R_\xi \text{ gauge}) \neq \gamma(\text{superfield}) \stackrel{?}{=} \gamma(\text{light cone gauge})$

Model	$\beta_v^{(1)}$	$\beta_v^{(2)}$
MSSM	Chankowski Nucl.Phys. B423 Athron, DS, Voigt '12	Yamada 94 $O(g^2 Y^2)$
$\forall$ gauge theory	?	
$\forall$ SUSY model	?	

Here: fill the gaps

Meaning of running *v*, alternative treatment

Fix renormalization scale μ, renormalize all divergences in $\overline{MS}$ or $\overline{DR}$

- adjust *v* such that tadpoles $\langle\phi\rangle = 0$
- *v*= minimum of renormalized effective scalar potential at scale μ

Change μ, change all parameters, including *v*, according to β functions

- all Green functions unchanged, including $\langle\phi\rangle = 0$

Minimum of renormalized effective scalar potential is μ-dependent and gauge dependent $\Rightarrow$ not an observable

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Very different treatment of v possible,

e.g. [Jegerlehner, Kalmykov, Kniehl '13][Bednyakov, Pikelner, Velizhanin '13]:

- always define v_{bare} =Minimum of bare eff. scalar potential
- then v_{bare}=abbreviation of combination of bare parameters
- In this scheme, $\delta v, \delta M_W, \delta \tan\beta$=gauge independent, but tadpoles are divergent (physical quantities unchanged)

Influence of global gauge invariance in a nutshell

When does $\delta\bar{v}$ appear?

$$\begin{aligned} \text{global gauge invariance} \quad &\Rightarrow \quad \delta\bar{v} = 0 \\ \text{no global gauge invariance} \quad &\Rightarrow \quad \delta\bar{v} \neq 0 \end{aligned}$$

R_ξ gauge fixing:

$$F = \partial^\mu A_\mu - \xi e v (2\,\mathrm{Im}\phi)$$

R_ξ breaks global gauge invariance for $\xi \neq 0 \Rightarrow \delta\bar{v} \neq 0$.

Sketch of the usual counterterm procedure

Compute loops in one of the possible regularizations.

- all divergences correspond to local terms in the Lagrangian
- can be absorbed by adding counterterms

$$\mathcal{L}_{\text{cl}} + \mathcal{L}_{\text{ct}} = \ldots - e\,\bar{\psi}\gamma^{\mu}\psi A_{\mu} \\ + \ldots - \delta e(\epsilon)\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

- by choosing ϵ-dependence appropriately, all divergences cancel
- arbitrary finite parts $\leftrightarrow$ local terms allowed by unitarity/causality

- results take a form like

$$e^2 + \delta e^2(\epsilon) - e^2 \Pi(q;\epsilon)$$

- concrete ($|q^2| \gg m_e^2$):

$$\Pi(q;\epsilon) = \frac{\alpha}{3\pi}\left(-\frac{1}{\epsilon} - \log\left(\frac{-q^2}{\bar{\mu}^2}\right) + \frac{5}{3} + \mathcal{O}(\epsilon)\right)$$

$$\frac{\delta e^2(\epsilon)}{e^2} = \frac{\alpha}{3\pi}\left(-\frac{1}{\epsilon} + \text{fin.const.}\right)$$

- Now two directions:
  - renormalization schemes
  - first, play around a little; bare quantities

Play around a little with the expression

regroup in two ways

$$e^2 \qquad\qquad +\delta e^2(\epsilon) - e^2\Pi(q;\epsilon)$$

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$$e^2 \qquad +\delta e^2(\epsilon) - e^2\Pi(q;\epsilon)$$
$$= e^2 \qquad - e^2\Pi_{\text{ren}}(q)$$

- manifestly finite

Play around a little with the expression

regroup in two ways

$$e^2 \qquad +\delta e^2(\epsilon) - e^2\Pi(q;\epsilon)$$

$$= e^2_{\text{bare}}(\epsilon) \qquad - e^2(\epsilon)\Pi(q;\epsilon)$$

$$= e^2_{\text{bare}}(\epsilon) \qquad - e^2_{\text{bare}}(\epsilon)\Pi(q;\epsilon) + \ldots$$

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regroup in two ways

$$
\begin{aligned}
& e^2 && +\delta e^2(\epsilon) - e^2\Pi(q;\epsilon) \\
&= e^2_{\text{bare}}(\epsilon) && - e^2(\epsilon)\Pi(q;\epsilon) \\
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\end{aligned}
$$

- manifestly finite
- only e_{bare} matters

$$e_{\text{bare}}(\epsilon) = e + \delta e(\epsilon)$$

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$$\mathcal{L}_{\text{cl}} + \mathcal{L}_{\text{ct}} = \dots - e\,\bar{\psi}\gamma^\mu\psi A_\mu$$

$$+ \dots - \delta e(\epsilon)\bar{\psi}\gamma^\mu\psi A_\mu$$

$$= \mathcal{L}_{\text{bare}} = \dots - e_{\text{bare}}(\epsilon)\bar{\psi}\gamma^\mu\psi A_\mu$$

Outline

- Renormalization schemes, scheme independence
- DRED, quantum action principle, and Higgs mass

$$\Pi(q;\epsilon) = \frac{\alpha}{3\pi}\left(-\frac{1}{\epsilon} - \log\left(\frac{-q^2}{\bar{\mu}^2}\right) + \frac{5}{3} + \mathcal{O}(\epsilon)\right)$$

$$\frac{\delta e(\epsilon)}{e} = \frac{\alpha}{6\pi}\left(-\frac{1}{\epsilon} + \text{fin.const.}\right)$$

$$\Pi_{\text{ren}}(q) = \Pi(q) - 2\frac{\delta e}{e}$$

Renormalization scheme = choice of fin.const.

on-shell	$\Pi_{\text{ren}}(0) = 0$
$\overline{MS}$	$\text{fin.const.} = 0$
“$\Delta\alpha(M_Z)$”	$2\frac{\delta e}{e} = \Pi^{\text{fermion}}(M_Z) + \Pi^{\text{rest}}(0)$

for Π_{ren}, a QED Ward identity was used to eliminate field renormalization

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Renormalization scheme = choice of fin.const.

Possibilities (all equivalent):

on-shell	$\Pi_{\text{ren}}(0) = 0$
$\overline{MS}$	$\text{fin.const.} = 0$
"$\Delta\alpha(M_Z)$"	$2\frac{\delta e}{e} = \Pi^{\text{fermion}}(M_Z) + \Pi^{\text{rest}}(0)$

for Π_{ren}, a QED Ward identity was used to eliminate field renormalization

- Lagrangian contains ϵ-dependent, “bare” quantities:

$$\mathcal{L}_{\text{bare}} = \ldots - e_{\text{bare}}(\epsilon)\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

which can be split into renormalized and counterterm quantities

$$\begin{aligned} e_{\text{bare}}(\epsilon) &= e_{\text{ren}} + \delta e^{1\text{L}} + \delta e^{2\text{L}} + \ldots \\ &= e_{\text{ren}} + a_1(\epsilon) e_{\text{ren}}^3 + \ldots \end{aligned}$$

- Renormalization group
- power counting/multiplicative renormalization

Outline

- Renormalization schemes, scheme independence
- DRED, quantum action principle, and Higgs mass

Another possibility: Quantum Action Principle

Task: consider e.g. SUSY of Green's functions

SUSY Ward/ST identities: $i\,\delta_{\text{SUSY}}\langle T\phi_1\ldots\phi_n\rangle \stackrel{?}{=} 0$

Quantum action principle: $i\,\delta_{\text{SUSY}}\langle T\phi_1\ldots\phi_n\rangle = \langle T\phi_1\ldots\phi_n\Delta\rangle$

- $\Delta \equiv \int \delta_{\text{SUSY}}\mathcal{L}$ in D dimensions
- if $\Delta = 0$ were true, all SUSY Ward and Slavnov-Taylor identities would be satisfied on the regularized level

Very useful theorem, valid at all orders

Proof of Quantum Action Principle

Depends on regularization:

$$i\,\delta_{\text{sym}}\langle T\phi_1\dots\phi_n\rangle = \langle T\phi_1\dots\phi_n\Delta\rangle, \quad \Delta = \int \delta_{\text{sym}}\mathcal{L}$$

Proofs:

BPHZ [Lowenstein et al '71]
DREG [Breitenlohner, Maison '77]
DRED [DS '05]

Application of Quantum action principle

Example 1: QCD and gauge invariance!

$$\Delta \equiv \delta_{\mathrm{BRS}} \mathcal{L}^{\mathrm{DREG}}_{\mathrm{QCD}} = 0 \text{ vanishes!!}$$

Quantum action principle:
$$i\,\delta_{\mathrm{SUSY}}\langle T\phi_1 \ldots \phi_n\rangle = \langle T\phi_1 \ldots \phi_n \Delta\rangle = 0$$

$\Rightarrow$ We know for decades that DREG preserves QCD gauge invariance at all orders! [Breitenlohner, Maison '77]

Application of Quantum Action Principle

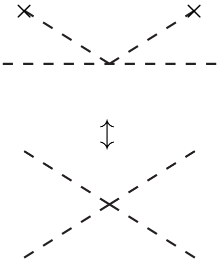
Example 2: SUSY of DRED:

$$\Delta \equiv \delta_{\mathrm{SUSY}} \mathcal{L}^{\mathrm{DRED}} \neq 0$$

gives rise to Feynman rules [DS '05]

- DRED might break some SUSY-identities $\rightarrow$ study each case seperately
- quantum action principle still useful to check which identities are valid

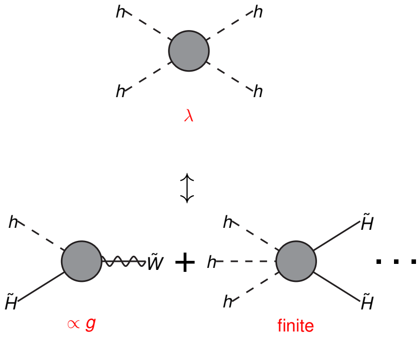
Higgs boson mass and quartic coupling



Higgs mass

- M_h governed by quartic Higgs self coupling λ
- $\lambda \propto g^2$ in SUSY

Quartic coupling and SUSY

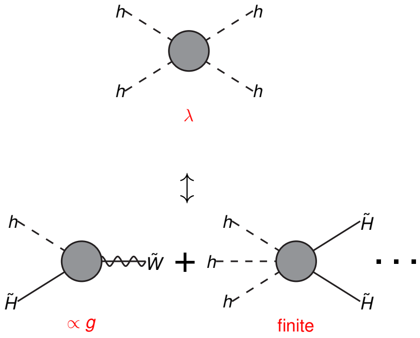


Slavnov-Taylor identity

- expresses $\lambda \propto g^2$
- Needs to be verified

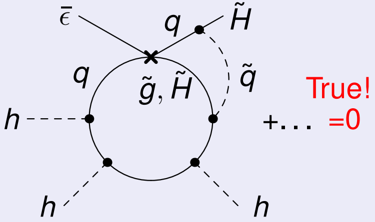
$$0 \stackrel{?}{=} \delta_{\rm SUSY}\langle hhh\tilde{H}\rangle$$

Quartic coupling and SUSY



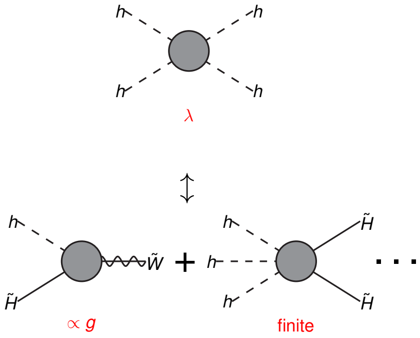
STI valid if

$\langle \Delta hhh\tilde{H} \rangle = 0 \Leftrightarrow$



Explicit computation ⇒ STI valid in DRED at two-loop level [Hollik, DS '05]

Quartic coupling and SUSY



Results:

- Two-loop STI valid in DRED (in Yukawa-approximation, $\mathcal{O}(\alpha_{t,b}^2, \alpha_{t,b}\alpha_s)$)
- for M_h-calculation at this order, multiplicative renormalization correct
- Previous calculations sufficient

Explicit computation ⇒ STI valid in DRED at two-loop level [Hollik, DS '05]

Practical consequences

- DREG preserves QCD gauge invariance but breaks SUSY
- DRED preserves SUSY in many but not all cases
- The Quantum Action Principle holds and is useful

Current status ok but should be improved in view of future more precise SUSY computations