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Functional equations for solid-on-solid models

with domain walls and a reflecting end

out soon arXiv:1510.00xxx [math-ph]

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Boundary conditions in statistical physics

- Ultimate **goal** for statistical-physical model: compute

$$\mathcal{Z} = \sum_{\text{configs}} \underbrace{\text{weight(config)}}_{\text{e.g.}\ \ \mathrm{e}^{-E(\text{config})/k_{\mathrm{B}}T}}$$

  i. Study model for arbitrary finite size L

  ii. Take thermodynamic limit $L \to \infty$

- **Suprising fact**: thermodynamics can be sensitive to choice of boundary conditions!

[Korepin Zinn-Justin '00]

Outline

- **Case study**: solid-on-solid model
  with domain walls and a reflecting end
- **Our goal**: compute $\mathcal{Z}$, for $L < \infty$

  Prior status: determinant formula [Filali '11]
- **Results**
  - Functional equation for the partition function

$$\sum_{\nu=0}^{L} M_\nu(\lambda_0;\lambda_1,\cdots,\lambda_L)\,\mathcal{Z}(\lambda_0,\cdots,\widehat{\lambda_\nu},\cdots,\lambda_L) = 0$$

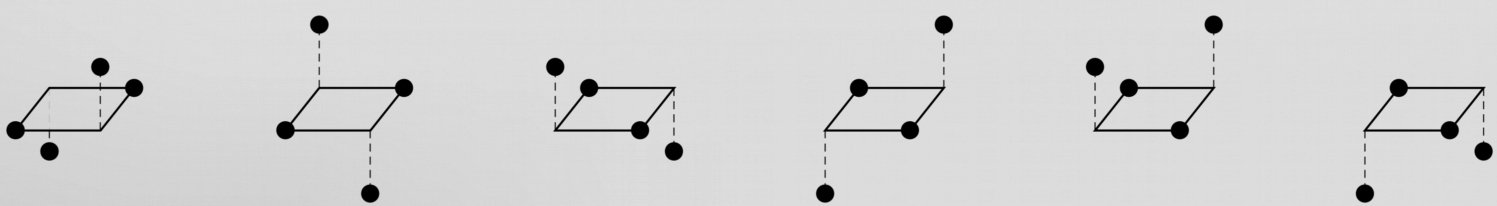
  - Solution, which is unique up to normalization, gives new (multiple-integral) expression for $\mathcal{Z}$

Solid-on-solid models

- Growth of (*bcc*) crystals
- Square 2d lattice with height variables at sites

$$\mathcal{Z} = \sum_{\substack{\text{height}\\ \text{configs}}} \text{weight}(\text{config}) = \sum_{\substack{\text{height}\\ \text{configs}}} \prod_{\text{faces}} \text{weight}\left(\begin{smallmatrix}\text{face}\\ \text{config}\end{smallmatrix}\right)$$

- Height difference between neighbours = 1



[Baxter '72]

Specific model

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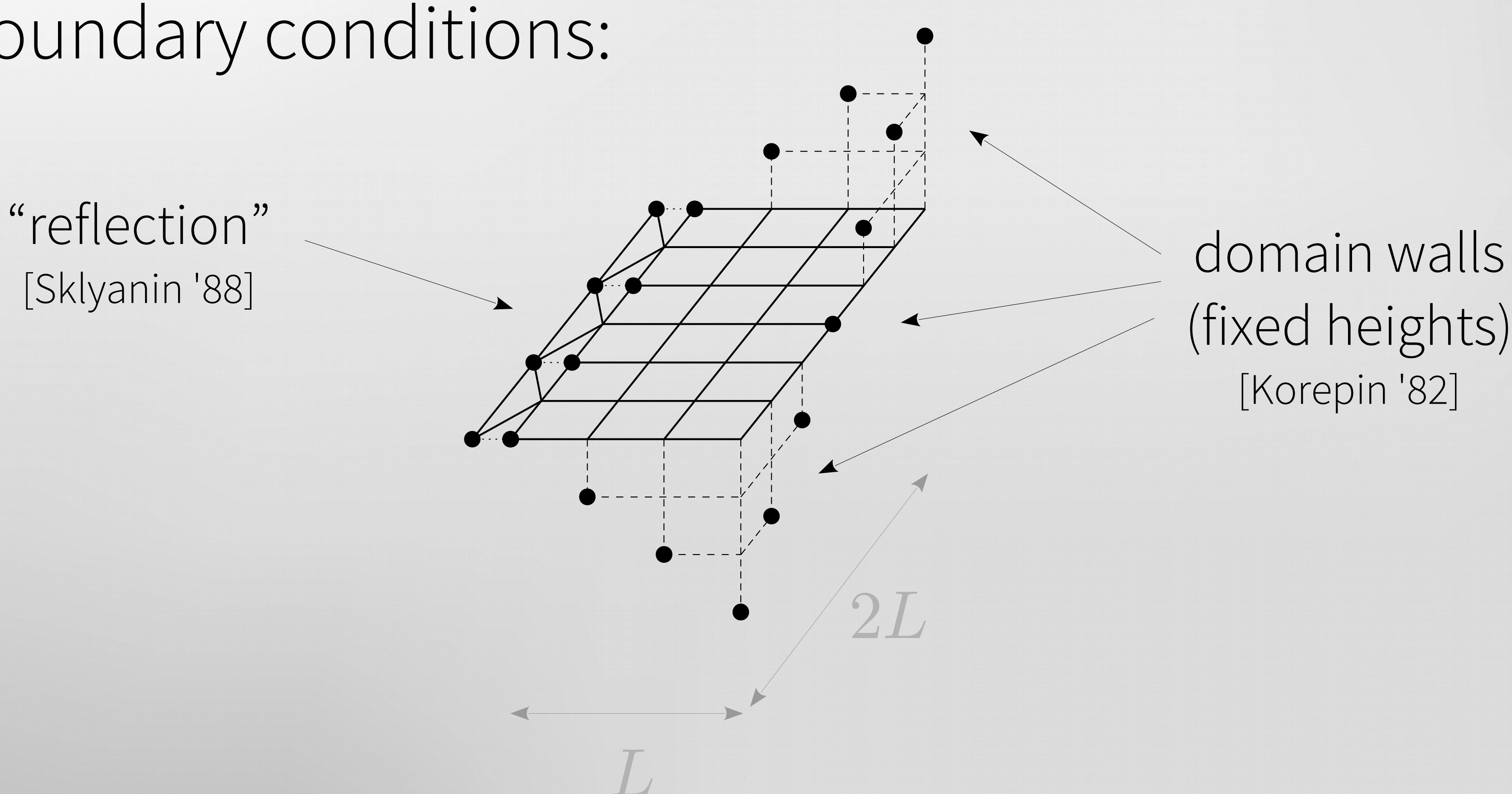
solid-on-solid with domain walls and a reflecting end

- Solid-on-solid model on $L \times 2L$ lattice
- Boundary conditions:



Algebraic reformulation

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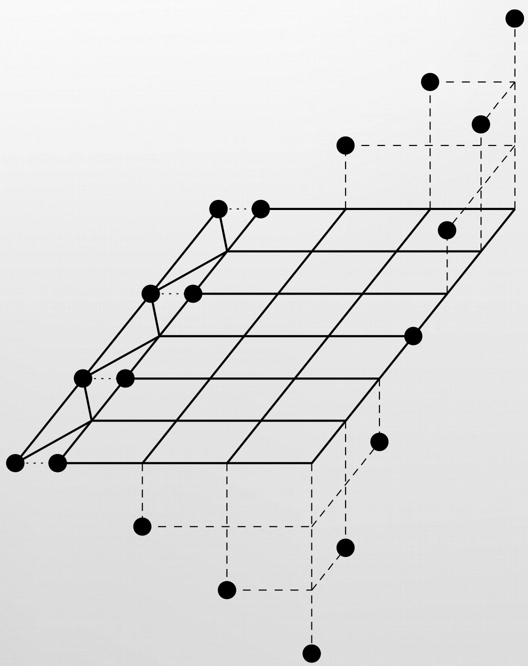
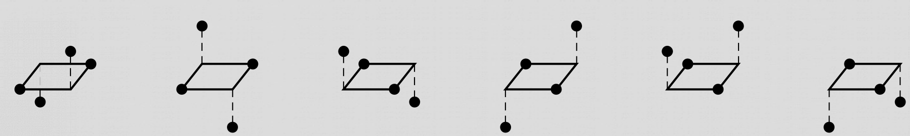
diagrammatics for partition function



$$= \sum_{\substack{\text{height} \\ \text{configs} \\ \text{in interior}}} \prod_{\text{faces}} \text{weight}\left(\substack{\text{face} \\ \text{config}}\right) = \mathcal{Z}$$



Algebraic reformulation

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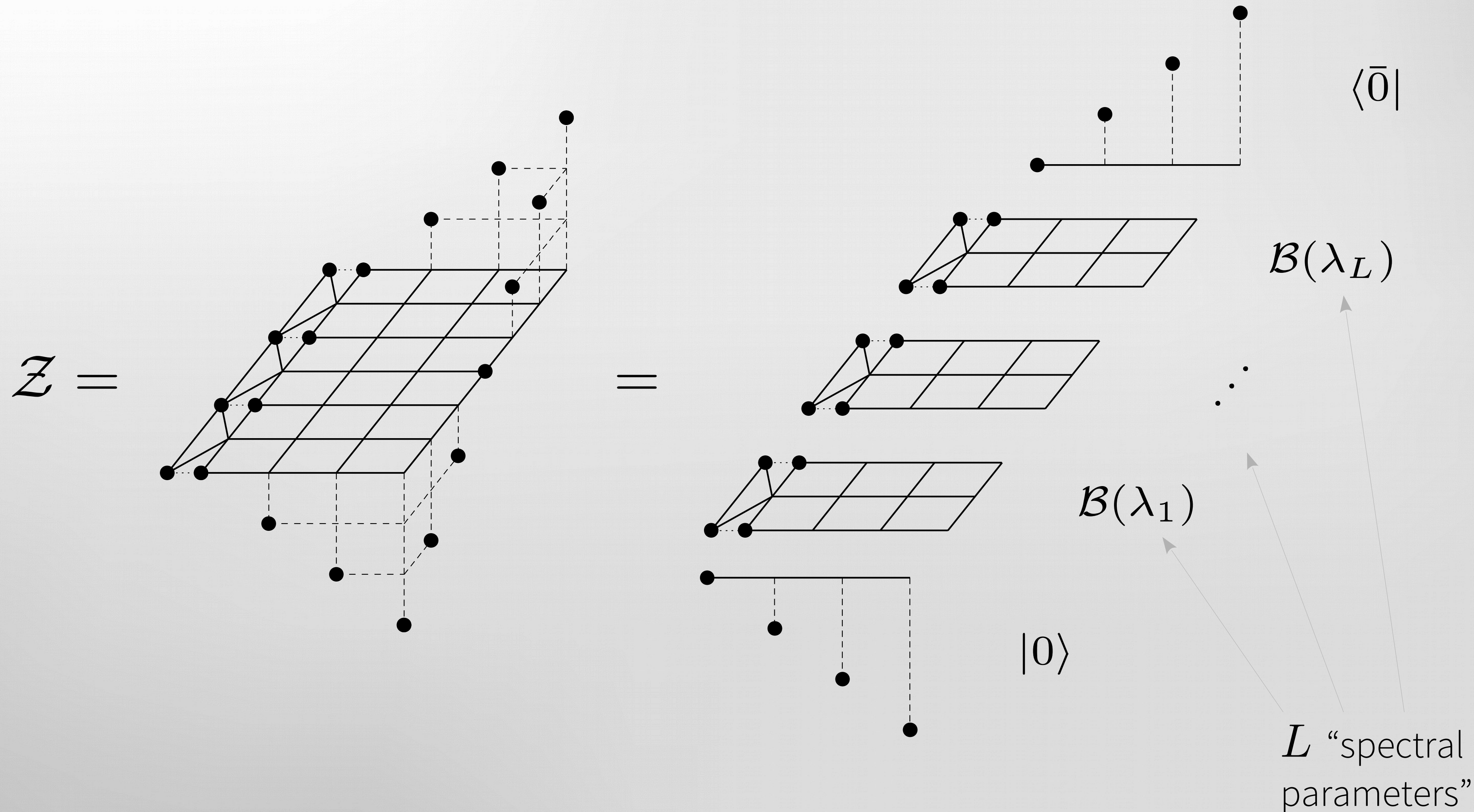
partition function as an L-point correlator



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Algebraic reformulation

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dynamical reflection algebra

- $\mathcal{B}$ generates,
  together with operators $\mathcal{A}$, $\mathcal{C}$, $\tilde{\mathcal{D}}$, H,
  the **dynamical reflection algebra**
  - Various relations
    e.g.

[Gervais Neveu '84]
[Sklyanin '88]
[Felder '95]

commute | commute & exchange λ's | commute & turn into $\tilde{\mathcal{D}}$

$$\mathcal{A}(\lambda_0)\,\mathcal{B}(\lambda_1) = \ldots\,\mathcal{B}(\lambda_1)\,\mathcal{A}(\lambda_0) + \ldots\,\mathcal{B}(\lambda_0)\,\mathcal{A}(\lambda_1) + \ldots\,\mathcal{B}(\lambda_1)\,\tilde{\mathcal{D}}(\lambda_0)$$

ratios of higher-order (Jacobi) theta functions in λ's

Functional equation

- Algebraic-functional method [Galleas '08 '10]
  - Start from $\mathcal{Z}(\lambda_1, \cdots, \lambda_L) = \langle\bar{0}|\mathcal{B}(\lambda_1)\cdots\mathcal{B}(\lambda_L)|0\rangle$
  - Insert $\mathcal{A}(\lambda_0)$ via $\langle\bar{0}| \propto \langle\bar{0}|\mathcal{A}(\lambda_0)$
  - Move $\mathcal{A}$ past all $\mathcal{B}$'s using e.g.

$$\mathcal{A}(\lambda_0)\,\mathcal{B}(\lambda_1) = \ldots\,\mathcal{B}(\lambda_1)\,\mathcal{A}(\lambda_0) + \ldots\,\mathcal{B}(\lambda_0)\,\mathcal{A}(\lambda_1) + \ldots\,\mathcal{B}(\lambda_1)\,\tilde{\mathcal{D}}(\lambda_0)$$

- **Result**: functional equation for $\mathcal{Z}$

$$\sum_{\nu=0}^{L} M_\nu(\lambda_0; \lambda_1, \cdots, \lambda_L)\,\mathcal{Z}(\lambda_0, \cdots, \widehat{\lambda_\nu}, \cdots, \lambda_L) = 0$$

[JL '15]

Results

[JL '15]

1) Functional equation for partition function

2) Characterizes $\mathcal{Z}$ up to normalization factor

3) Solution: multiple-integral formula

$$\mathcal{Z}(\lambda_1, \cdots, \lambda_L) = \Omega_L \, f(\gamma)^L f'(0)^L \oint \frac{\mathrm{d}^L \boldsymbol{z}}{(2\pi\mathrm{i})^L} \frac{\prod_{i \neq j}^L f(z_i - z_j)}{\prod_{i,j=1}^L f(z_i - \lambda_j)}$$

$$\times \prod_{i<j}^L f(z_i - \mu_j) f(z_i + \mu_j + \gamma) \prod_{l=1}^L \frac{f(2z_l)}{f(2z_l + \gamma)} \, m_l(z_1, \cdots, z_l)$$

f : odd Jacobi theta function

polynomial in f

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Summary and beyond

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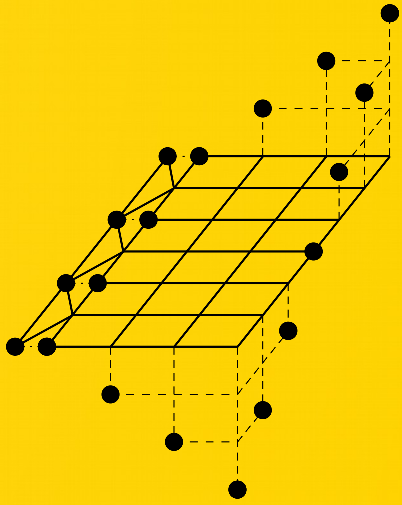

- Solid-on-solid model
  with domain walls and a reflecting end
- **Results**
  - Functional equation

$$\sum_{\nu=0}^{L} M_\nu(\lambda_0;\lambda_1,\cdots,\lambda_L)\,\mathcal{Z}(\lambda_0,\cdots,\widehat{\lambda_\nu},\cdots,\lambda_L)=0$$

  - Multiple-integral formula

$$\mathcal{Z}(\lambda_1,\cdots,\lambda_L)=\Omega_L\,f(\gamma)^L f'(0)^L\oint\frac{\mathrm{d}^L\boldsymbol{z}}{(2\pi\mathrm{i})^L}\frac{\prod_{i\neq j}^L f(z_i-z_j)}{\prod_{i,j=1}^L f(z_i-\lambda_j)}$$

$$\times\prod_{i<j}^{L} f(z_i-\mu_j)f(z_i+\mu_j+\gamma)\prod_{l=1}^{L}\frac{f(2z_l)}{f(2z_l+\gamma)}\,m_l(z_1,\cdots,z_l)$$

- **Outlook**
  - Thermodynamic limit
  - Comparison with other boundary conditions