Quantization of Super Teichmüller spaces

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based on joint work with Joerg Teschner and Michal Pawelkiewicz arxiv 1510.xxxx

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Motivation

Witten showed how the Quantum Field theory formalism with the Chern-Simons action and compact gauge group can produce important invariants in topology (like the Jones polynomial)

[Witten '89]

- TQFT and Chern Simons with non compact gauge group:
  For CS on a 3-dim manifold $M = \mathbb{R} \times S$, where S is a compact surface, the classical phase space is the space of flat connections on S. One can show that this moduli space contains a component which can be identified with Teichmüller space of S. Quantum CS is obtained by the quantizing the phase space, so quantum Teichmüller can be useful tools for studying quantization of $SL(2, \mathbb{R})$ Chern Simon.

Motivation

- Verlinde conjecture: Space of conformal blocks in quantum Liouville can be identified with Hilbert space of Teichmüller space.

[Velinde '90, Teschner '05]

- N=2 supersymmetric gauge theories via AGT correspondence.
- Integrability: Light cone evolution operator of the discrete Liouville model can be interpreted in pure geometrical term within quantum Teichmüller theory.

[Faddeev and Kashaev'02]

Outline

1. Teichmüller space
   - Definition
   - Geometry
   - Algebra

2. Super Teichmüller space
   - Algebra
   - Geometry

Teichmüller space

- A Riemann surface $\mathcal{C}_{g,n}$ is a real 2d manifold with a conformal structure, with genus $g \geq 0$ and $n \geq 1$ boundary components.
- Our case: Hyperbolic structure i.e. constant negative curvature.
- Our case: Boundary components being punctures, i.e. holes of zero length.

Teichmüller space $\mathcal{T}_{g,n}$

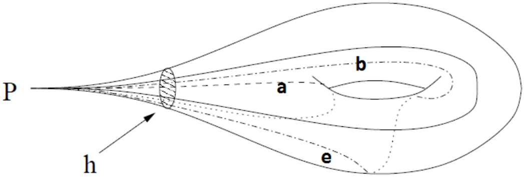
The space of deformations of the metrics of constant negative curvature

$$\mathcal{T}_{g,n} = \{\psi : \pi_1(\mathcal{C}_{g,n}) \to PSL(2,\mathbb{R})\}/PSL(2,\mathbb{R}),$$

- where ψ is an uniformisation map for our Riemann surface, that it is a discrete and faithful representation of fundamental group $\pi_1(\mathcal{C}_{g,n})$ into $PSL(2,\mathbb{R})$.

Useful systems of coordinates

Triangulations of Riemann surfaces:
An ideal triangulation of $\mathcal{C}_{g,n}$ is the isotopy class of a collection of disjointly embedded arcs running between the punctures such that $\mathcal{C}_{g,n}$ decomposes into triangles.



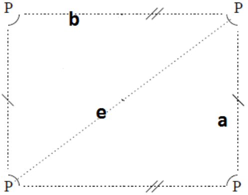


Figure: Triangulation of the one-punctured torus

Corresponding dual graphs (fat graphs)

A trivalent graph embedded in the surface with fixed cyclic order of the edges incident to each vertex

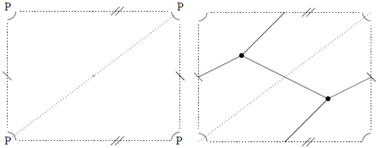


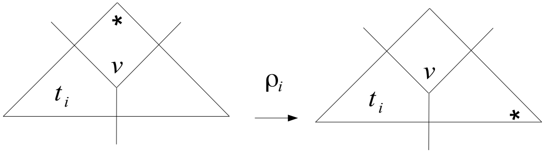
Figure: Representation of the triangulation and the dual fat graph

Penner coordinates

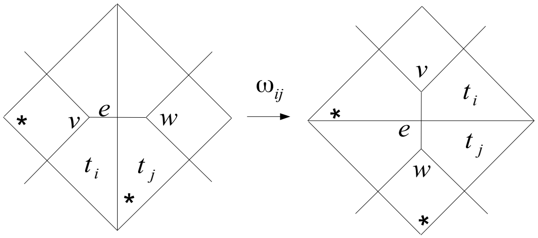
- Given an ideal triangulation Penner assigns a coordinate to each of the edges.
- Each edge e may be straightened to a unique geodesic for the hyperbolic metric.
- The coordinate $l_e(P)$ is defined as the hyperbolic length of e .

Changes of triangulations

Decorated vertex, 120 deg rotation of star



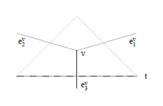
Flip map



Quantization of the Teichmüller space

Kashaev space

Given a fat graph with set of vertices, for each vertex v one may introduce a pair of variables (q_v, p_v) such that $(q_v, p_v) = (l_3 - l_2, l_1 - l_2)$

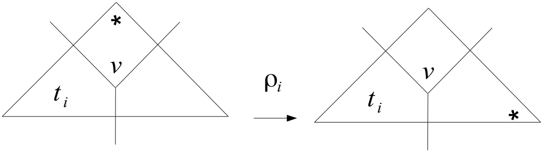


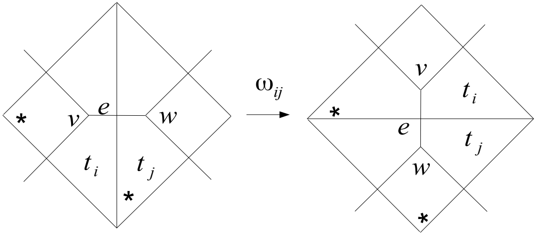
we will consider the vector space V obtained by regarding the variables (q_v, p_v) as the components $(q_v(a), p_v(a))$ of vectors $a \in V$. The space of linear coordinate functions on V will be called the Kashaev space.

- Poisson bracket
- canonical quantization

Change of fat graph

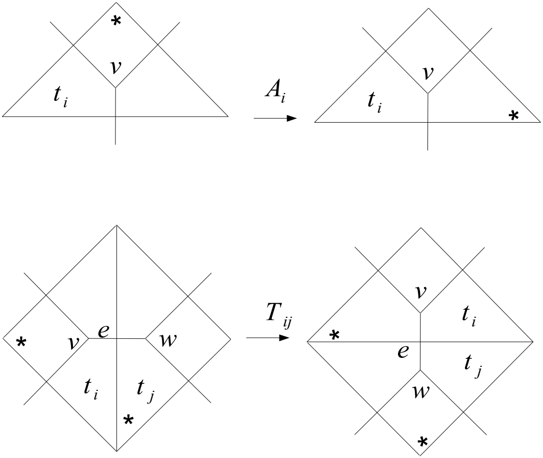
The classical maps





Change of fat graph

Assign a Hilbert space to each triangle



Change of fat graph

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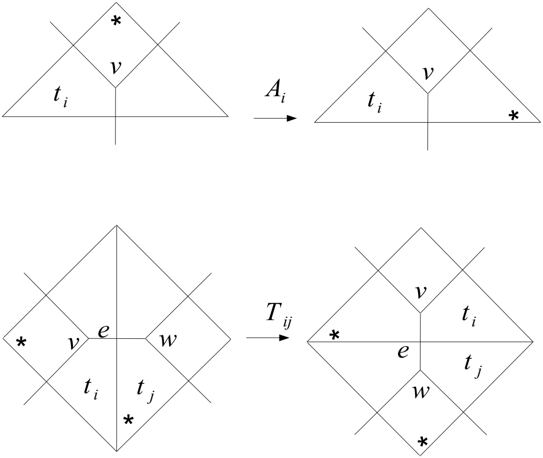


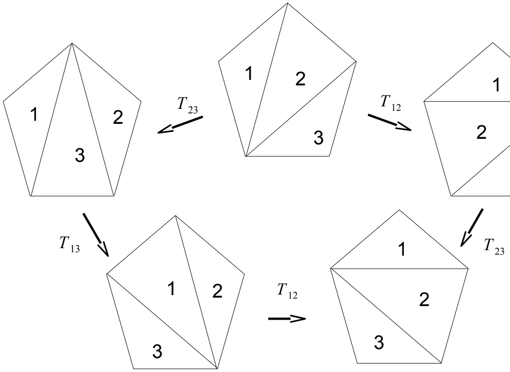
$$A^3 = id$$

$$T_{12} T_{13} T_{23} = T_{23} T_{12}$$

$$A_1 T_{12} A_2 = A_2 T_{21} A_1$$

$$T_{12} A_1 T_{21} = A_1 A_2 P_{(12)}$$

Pentagon as an example



$$T_{12}T_{13}T_{23} = T_{23}T_{12}$$

Algebra/Quantum group

The q-deformed Hopf algebra $U_q(SL(2))$ of the Lie algebra $SL(2)$ is generated by the elements H, H^{-1}, E, F, where, $q = e^{i\pi b^2}$

$$
\begin{aligned}
[H, \hat{H}] &= \frac{1}{2\pi i}, \\
[H, E^{\pm}] &= \mp ibE^{\pm}, \\
[\hat{H}, E^{+}] &= 0, \\
[\hat{H}, E^{-}] &= +ibE^{-}, \\
[E^{+}, E^{-}] &= (q - q^{-1})e^{2\pi bH},
\end{aligned}
$$

$$
\begin{aligned}
\Delta(H) &= 1 \otimes H + H \otimes 1, \\
\Delta(\hat{H}) &= 1 \otimes \hat{H} + \hat{H} \otimes 1, \\
\Delta(E^{+}) &= E^{+} \otimes e^{2\pi bH} + 1 \otimes E^{+}, \\
\Delta(E^{-}) &= E^{-} \otimes e^{-2\pi b\hat{H}} + 1 \otimes E^{-}.
\end{aligned}
$$

Connection with algebra

The observation shows that the quantized coordinate can be associate to the elements of quantum group

$$T^{-1}(1 \otimes x)T = \triangle(x) \quad x \in \text{Borel half of } U_q(SL(2))$$

The Idea

- Replace the quantum group $U_q(SL(2))$ by a suitable quantum super-group.

The goal

- Demonstrate that the resulting quantum theory is the quantum theory of the Teichmüller spaces of super-Riemann surfaces

Super Teichmüller space

$U_q(osp(1 \mid 2))$

Generated by the bosonic operators H, H^{-1} and two fermionic ones $v^{\pm}$, where $q = e^{i\pi b^2}$

$$[H, \hat{H}] = \frac{1}{2\pi i},$$
$$[H, v^+] = -ibv^+,$$
$$[H, v^-] = ibv^-,$$
$$[\hat{H}, v^+] = 0,$$
$$[\hat{H}, v^-] = +ibv^-,$$
$$\{v^+, v^-\} = (q + q^{-1})e^{2\pi bH},$$

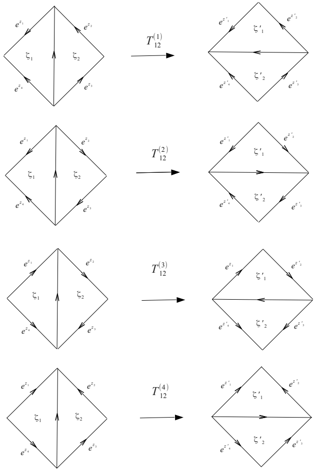
$$\Delta(H) = 1 \otimes H + H \otimes 1,$$
$$\Delta(\hat{H}) = 1 \otimes \hat{H} + \hat{H} \otimes 1,$$
$$\Delta(v^+) = v^+ \otimes e^{2\pi bH} + 1 \otimes v^+,$$
$$\Delta(v^-) = v^- \otimes e^{-2\pi b\hat{H}} + 1 \otimes v^-.$$

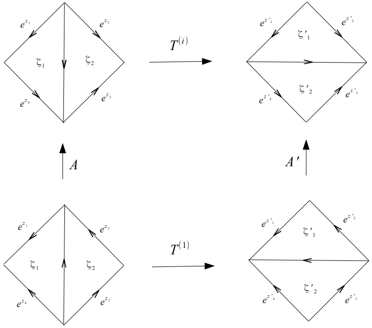
Final Result

$$T = e^{2\pi i p_1 q_2} g_{b_*}^{-1}((-1)^{(-1/2)} e^{2\pi b(q_1+p_2-q_2)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}))$$

Supersymmetric changes of variables

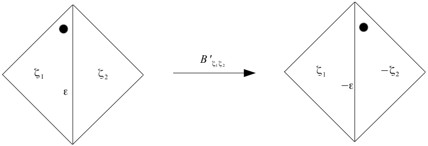
Additional structure called spin structure which is encoded combinatorally with arrows.



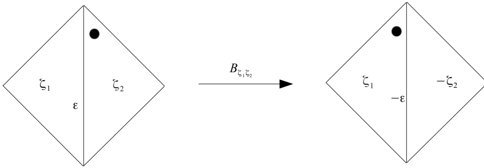


Super Ptolemy groupoid

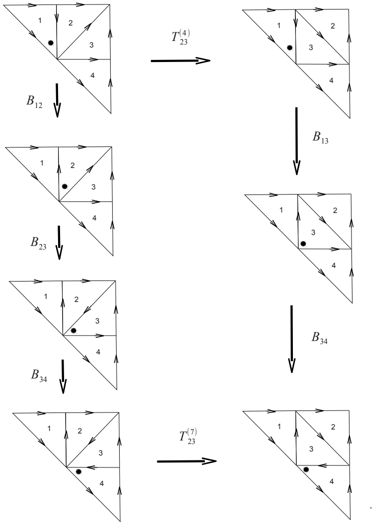
- Push out operator B_{ij}
- Relation with T_{ij}

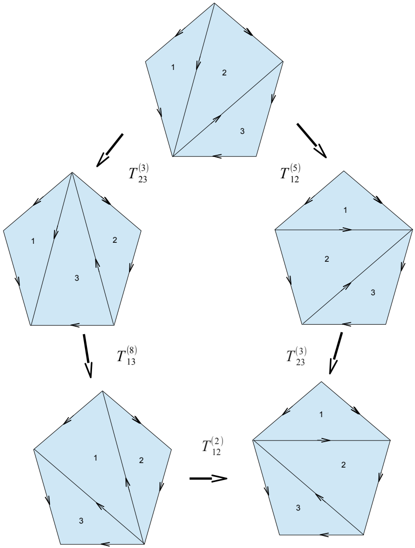


$$T_{12}^{(i)} T_{13}^{(j)} T_{23}^{(k)} = T_{23}^{(l)} T_{12}^{(m)}$$
$$T_{34}^{(i)} B_{24} T_{12}^{(j)} B_{24}^{-1} = B_{23} T_{12}^{(j)} B_{23}^{-1} T_{34}^{(i)}$$
$$B_{13} B_{34} T^{(i)} = T^{(i)} B_{12} B_{23} B_{34}$$



Super Ptolemy groupoid examples





Thanks for your attention