Large Spin Expansion in Conformal Field Theories

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based on the work with Luis F. Alday

Conformal Field Theories

Conformal field theories are ubiquitous in theoretical physics. They describe:

- End-points of RG flows
- Second order phase transitions (3D Ising)
- Quantum gravity and string theory (AdS/CFT)

Conformal Field Theory

There are many different approaches to studying CFTs. These include standard perturbative methods, holography, supersymmetry, integrability, localization and conformal bootstrap.

In this talk I will elaborate on the conformal bootstrap and its connection to some other methods. Its power stems from the fact that it is non-perturbative and completely general. Its weakness that it is hard to ask a specific question.

Conformal Field Theory

CFTs are defined (for this talk) non-perturbatively by specifying the operator algebra

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k c_{ijk}(x)\mathcal{O}_k(0)$$

Each operator is specified by its scaling dimension and its spin $\mathcal{O}_{\Delta,s}$

The list of operators (Δ_i, s_i) together with the OPE coefficients c_{ijk} defines a CFT on the plane.

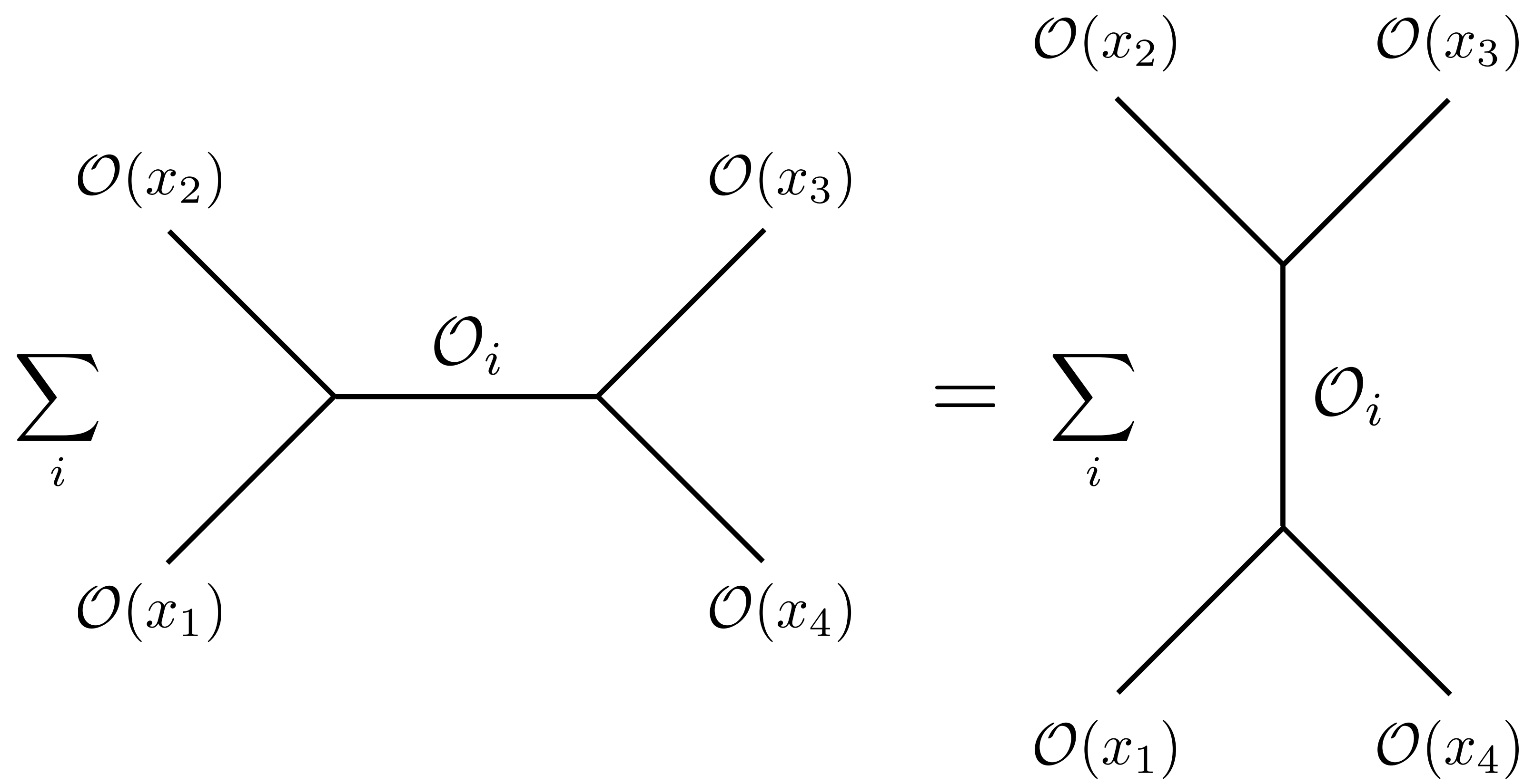
Conformal Bootstrap

The dynamical principle is associativity of the operator algebra which is also known as *crossing*.

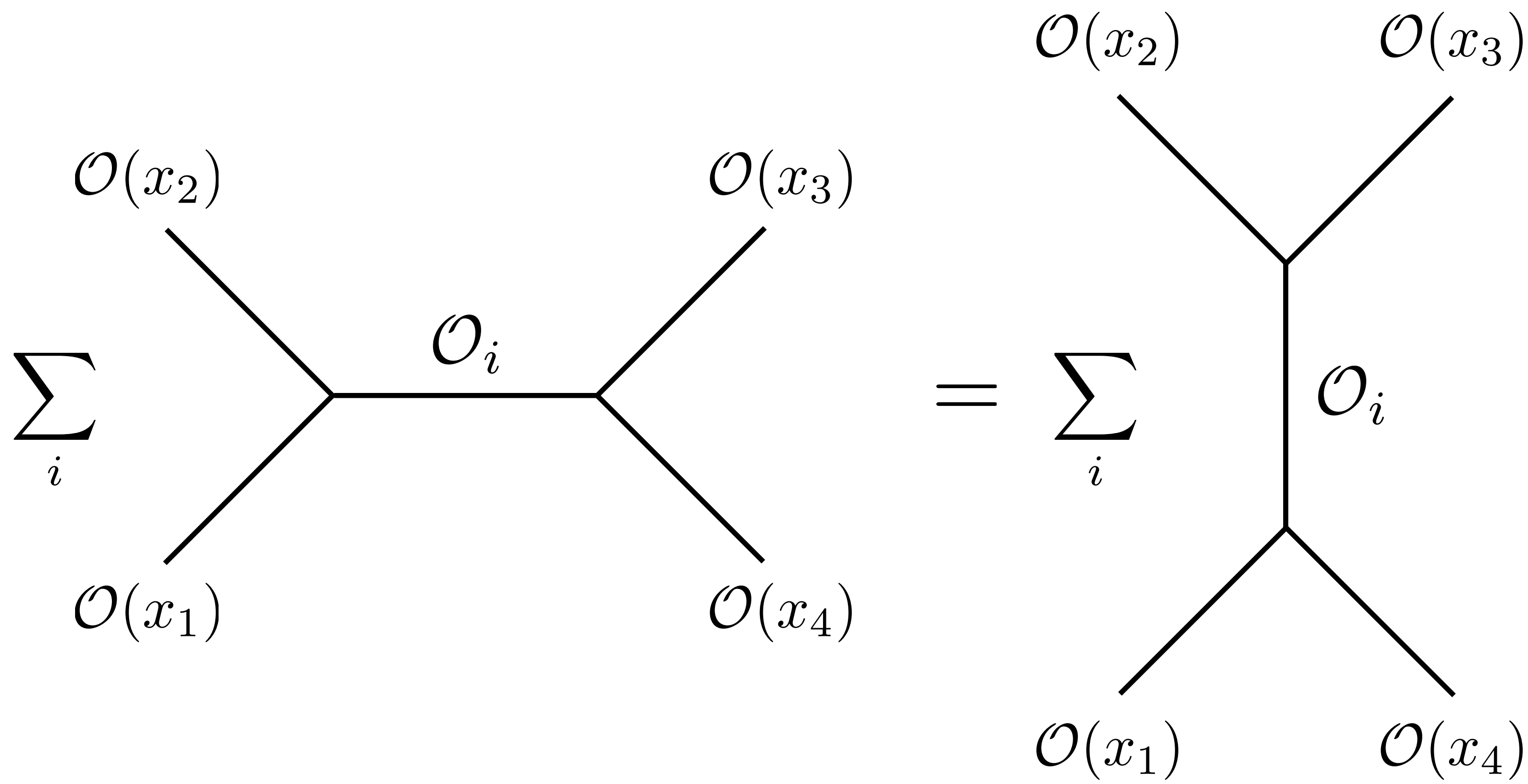
$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle = \frac{G(u,v)}{(x_{12}^2 x_{34}^2)^{\Delta}}$$

[Ferrara, Gatto, Grillo '74] [Polyakov '74]

[Rattazzi, Rychkov, Tonni, Vichi '08]



Conformal Bootstrap



$$v^{\Delta} G(u,v) = v^{\Delta} \sum_i c_i^2 g_{\Delta_i, s_i}(u,v) = u^{\Delta} \sum_i c_i^2 g_{\Delta_i, s_i}(v,u) = u^{\Delta} G(v,u)$$

- Number of primary operators is necessarily infinite
- No simple mapping / physical picture

Conformal Bootstrap

There are two basic approaches to crossing:

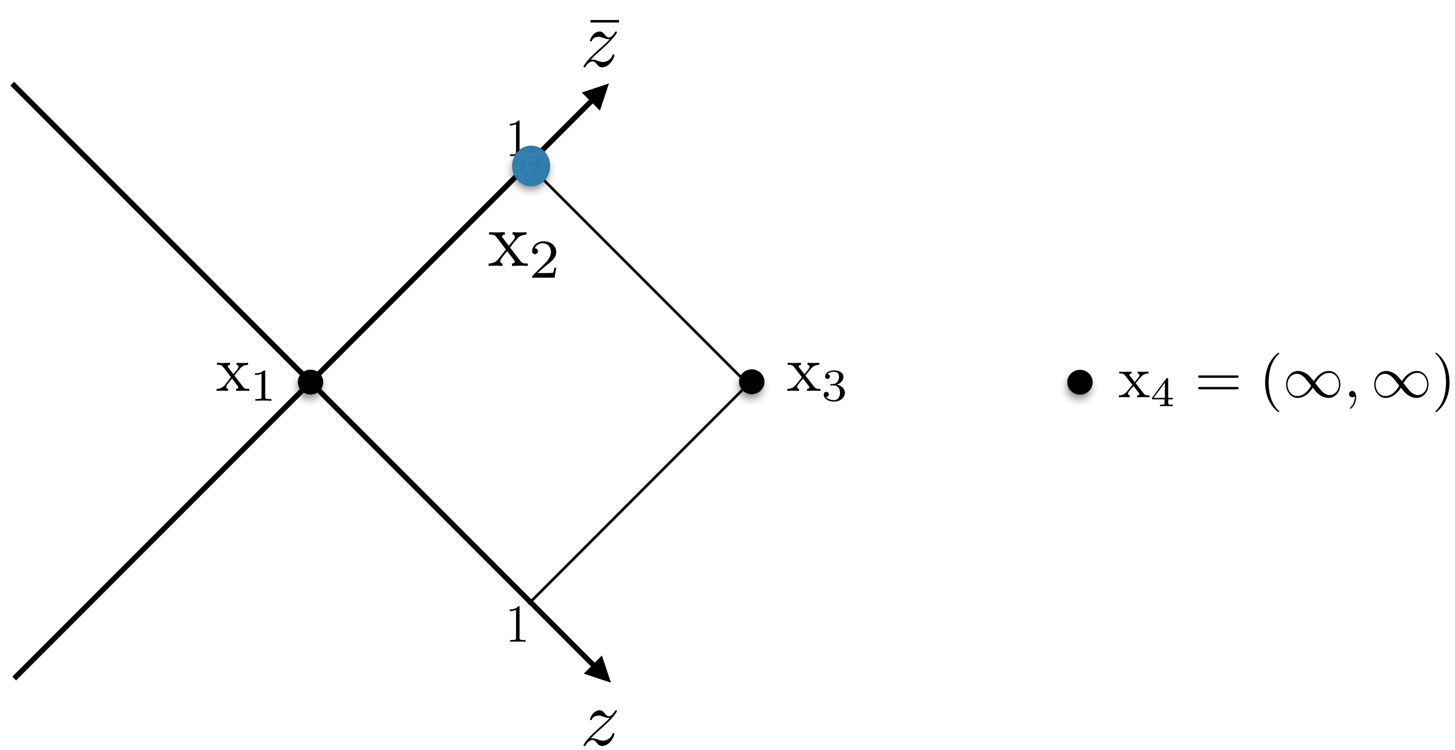
- Focus on the cross ratios (u,v) for which both channels are dominated by *the same light operators* and constrain their properties (numerical bootstrap).
It probes correlators in the Euclidean region.

- *Assume the spectrum of light operators* and derive consequences for the other channel (analytic bootstrap).
It probes correlators in the Lorentzian region.

I will focus on the analytic bootstrap.

Analytic Bootstrap (d>2)

The idea is to consider the limit $v \ll u \ll 1$



$$u = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

$$v = (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

v-channel: low twist low spin operators (assume)

u-channel: twist close to 2Δ large spin operators (derive)

Analytic Bootstrap (d>2)

[Fitzpatrick, Kaplan, Poland, Simmons-Duffin '12]

[Komargodski, AZ '12]

1. The crossing equation:

$$G(u,v) = \left(\frac{u}{v}\right)^{\Delta} G(v,u)$$

2. Consider the limit $v \ll u \ll 1$. Use the v-channel OPE

unit operator

$$G(u,v) = \left(\frac{u}{v}\right)^{\Delta} \left(1 + c_{\tau_{min},s_{min}} v^{\frac{\tau_{min}}{2}} f_{\tau_{min},s_{min}}(u) + \ldots\right)$$

minimal twist operator (stress tensor)

3. Reproduce it using the u-channel OPE

$$\sum_{\tau,s} c_{\tau,s} u^{\frac{\tau}{2}} f_{\tau,s}(v) = \left(\frac{u}{v}\right)^{\Delta} \left(1 + c_{\tau_{min},s_{min}} v^{\frac{\tau_{min}}{2}} f_{\tau_{min},s_{min}}(u) + \ldots\right)$$

$$\tau_i = \Delta_i - s_i$$

Analytic Bootstrap (d>2)

[Fitzpatrick, Kaplan, Poland, Simmons-Duffin '12]

[Komargodski, AZ '12]

$$\sum_{\tau,s} c_{\tau,s} u^{\frac{\tau}{2}} f_{\tau,s}(v) = \left(\frac{u}{v}\right)^{\Delta} \left(1 + c_{\tau_{min},s_{min}} v^{\frac{\tau_{min}}{2}} f_{\tau_{min},s_{min}}(u) + \ldots\right)$$

Collinear blocks are given by

$$f_{\tau,s}(v) = \ {}_2F_1\left(\frac{\tau}{2} + s, \frac{\tau}{2} + s, \tau + 2s, 1 - v\right) \sim \log v$$

They admit the following scaling limit

$$\lim_{v \to 0, s\sqrt{v} - \text{fixed}} f_{\tau,s}(v) \sim e^{-s\sqrt{v}}$$

The mechanism is the following

$$\sum s^{\alpha - 1} e^{-s\sqrt{v}} \sim \frac{1}{v^{\alpha}}$$

$$s \sim \frac{1}{\sqrt{v}}$$

Clustering at large separations

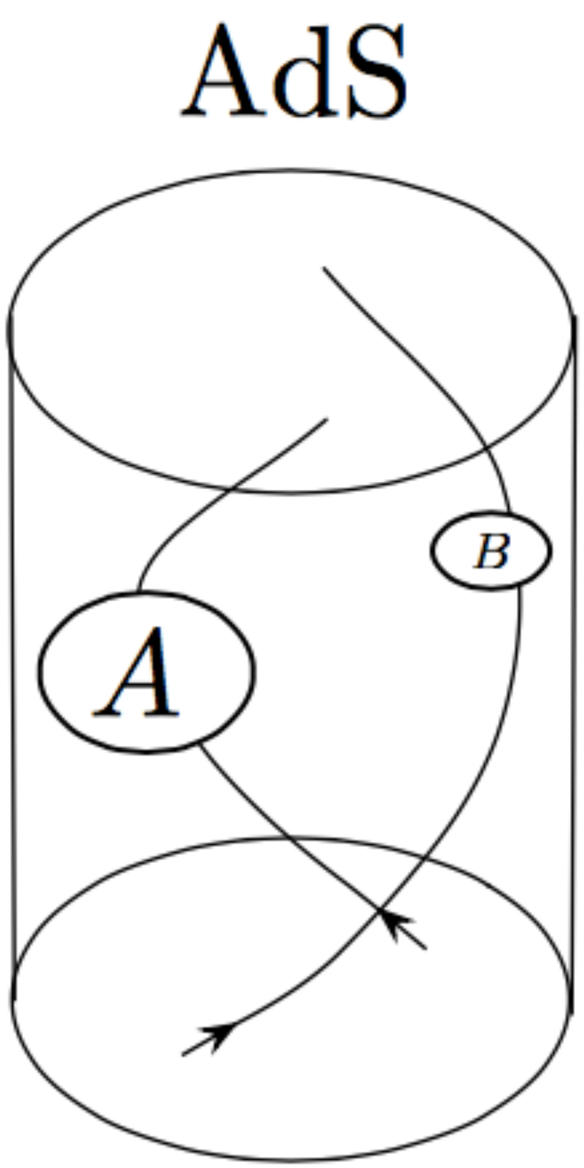
By considering *the unit operator* in the v-channel
we conclude that

The twist spectrum of any CFT has the Fock space-like structure:
Given the operator with twist τ_1 and operator with twist τ_2 there is an infinite set of operators with large spin and twist approaching $\tau_1 + \tau_2$

$$\tau_s = \tau_1 + \tau_2 + \gamma_s$$

$$\lim_{s\to\infty} \gamma_s = 0$$

Clustering at large separations



In AdS it corresponds to considering two objects rotating very fast around each other. When separation becomes large the interaction strength goes to zero.

[Fitzpatrick, Kaplan, Walters '14]

Binding energy

One can consider the correction in the v-channel due to some small twist operator

$$\sum_{\tau,s} c_{\tau,s} u^{\frac{\tau}{2}} f_{\tau,s}(v) = \left(\frac{u}{v}\right)^{\Delta} (1 + c_{\tau_{min},s_{min}} v^{\frac{\tau_{min}}{2}} f_{\tau_{min},s_{min}}(u) + ...)$$

matching to the u-channel introduce some ``binding energy'' for 2-particle operators

$$\tau_s = 2\Delta - \alpha_d \frac{c_{\tau_{min}}}{s^{\tau_{min}}}$$

What are the corrections to this formula?

Descendants. Summary

The correction due to descendants $\partial^k \mathcal{O}$ organize themselves in the following series

$$\gamma_s = -\frac{c_0}{J^{\tau_{min}}}\left(1 + \frac{c_1}{J^2} + \frac{c_2}{J^4} + \cdots\right)$$

$$J^2 = (\Delta + s)(\Delta + s - 1)$$

[Alday, Bissi '13]

For generic quantum numbers we have not found a closed expression, but develop an efficient technique to compute the corrections. Also we can observed that the series is *alternating, asymptotic and Borel-summable,* with

$$\lim_{k\to\infty} |\frac{c_{k+1}}{c_k}| = \frac{k^2}{\pi^2} + \cdots$$

Descendants. Toy Model

To illustrate how it works we can consider the following toy model

$$\int_0^\infty dj K_{toy}(j) h\left(\frac{v}{j^2}\right) = \sum_{n=0} c_n v^n \qquad K_{toy}(j) = \frac{4j^{2\Delta-1}}{\Gamma^2(\Delta)} K_0(2j)$$

On the RHS we have a convergent series coming from the OPE. The solution on the left is

$$h(z) = \sum_{n=0} \frac{\Gamma^2(\Delta)}{\Gamma^2(\Delta-n)} c_n z^n$$

which is generically divergent.

Descendants. Example

As an example consider the first correction. It takes the form

$$c_1 = -\frac{(-2\Delta + \tau_{min} + 2)^2 \left(2\ell^2(d + \tau_{min} - 2) + 2\ell(\tau_{min} - 1)(d + \tau_{min} - 2) + (d-4)\tau_{min}^2\right)}{8(d + 2\ell - 4)(d - 2(\ell + \tau_{min} + 1))}$$
$$- \frac{1}{12}\tau_{min}(\tau_{min}(-3\Delta + \tau_{min} + 3) + 2)$$

For some particular values on the other hand the corrections are very simple

$$\Delta_{min} = 1, d = 3, \Delta = 1/2 \ \rightarrow \qquad c_k = (-1)^{k+1}\frac{(\frac{1}{2})_{k-1}}{2^{2k+1}\Gamma(k+1)}$$

$$\Delta_{min} = 1, d = 3, \Delta = 1 \ \rightarrow \qquad c_k = (-1)^{k}\frac{\Gamma(k+\frac{1}{2})}{4^k\sqrt{\pi}\Gamma(k+1)}$$

Summary of Corrections

We expect the same story to hold for a generic CFT and corrections due to heavier primaries as well.

The series is asymptotic due to the nature of the kernel coming from the crossing equation. Due to convergence of the OPE the series is ``Borel''-summable.

We can now apply this technology to some examples.

Large N in the critical O(N) model

We consider the four-point function of spin fields

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle = \frac{G_{ijkl}(u,v)}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}},$$

$$f_{ijkl}(u,v) = v^{\Delta_\phi} G_{ijkl}(u,v), \quad f_{ijkl}(u,v) = f_{kjil}(v,u)$$

The two-particle operators are higher spin conserved currents which come in three different representations of the O(N) symmetry j_s^S, j_s^T, j_s^A

We can apply the technology of the previous section to any of these families and compare with
the explicit perturbative results.

Large N in the critical O(N) model

As an example consider the two-loop result for the anomalous dimensions of the higher spin currents in the symmetric representation of the O(N)

[Derkachov, Manashov '97]

$$
\begin{aligned}
u^2_{(n)} \quad = \quad & -\eta_1^2 \frac{\mu(\mu-1)}{(\mu-1+n)(\mu-2+n)} \Big\{ \frac{(2\mu^2-3\mu+2)}{\mu-2} R - \frac{2(\mu-1)(2\mu-1)}{\mu-2} S(n,\mu) \\
& + \frac{1}{2}\mu(\mu-1) R(n,\mu) + \frac{1}{\mu-2}\left(\frac{n}{\mu-2+n} + \frac{\mu(\mu^2-5\mu+5)}{(\mu-1)(\mu-2)} \right) \\
& - \frac{\mu(\mu-1)}{2(\mu-1+n)(\mu-2+n)} \left(1 - \frac{1}{(\mu-1+n)} - \frac{1}{(\mu-2+n)} \right) \Big\}, \qquad (5.25)
\end{aligned}
$$

where

$$\eta_1 = 4(2-\mu)\Gamma(2\mu-2)/\Gamma^2(\mu-1)\Gamma(2-\mu)\Gamma(\mu+1), \qquad (5.26)$$

$$R = \psi(1) + \psi(\mu-1) - \psi(2-\mu) - \psi(2\mu-2), \qquad (5.27)$$

$$S(n,\mu) = \psi(\mu-1+n) - \psi(\mu-1), \qquad (5.28)$$

$$R(n,\mu) = \int_0^1 \int_0^1 \mathrm{d}\alpha \mathrm{d}\beta \alpha^{\mu-3} \beta^{\mu-3} (1-\alpha-\beta)^n. \qquad (5.29)$$

Large N in the critical O(N) model

The large spin expansion of this expression can be bootstrapped using the one-loop data

$$\begin{aligned}\gamma_s^S &= \tilde{c}_\alpha \frac{g^2_{\phi\phi\alpha}}{J^{2\Delta_\alpha}}\left(1+\frac{c_{\alpha,1}}{J^2}+...\right)\\ &+\sum_s \tilde{c}_s^{HS}\frac{g^2_{\phi\phi HS}}{J^{\tau_s^{HS}}}\left(1+\frac{c_{HS,1}}{J^2}+...\right)\\ &+\sum_s \tilde{c}_s^{\alpha\partial^s\alpha}\frac{g^2_{\phi\phi\alpha\partial^s\alpha}}{J^{\tau_s^{\alpha\partial^s\alpha}}}\left(1+\frac{c_{\alpha\partial^s\alpha,1}}{J^2}+...\right)\end{aligned}$$

The formulas above, however, are valid to all-loops.

It is also interesting to apply this technique when there is no perturbative expansion.

3D Ising Model

- Conformal invariance and Z_2 invariance
- Contains in the spectrum a scalar operator σ

$$Z_2 : \sigma \to -\sigma$$

$$\Delta_\sigma = \frac{1}{2} + \gamma_\sigma \qquad \gamma_\sigma \simeq 0.018$$

From this it follows that the theory contains an infinite set of light higher spin currents

$$\Delta_s = 1 + s + \gamma_s \qquad s = 2, 4, 6, \ldots$$

$$0 \leq \gamma_s < 2\gamma_\sigma \ll 1$$

[Nachtmann '73]
[Callan, Gross '73]

3D Ising Model

$$\gamma_s \simeq 2\gamma_\sigma - \frac{2\Gamma(\Delta_\varepsilon)}{\Gamma(\frac{\Delta_\varepsilon}{2})^2} \frac{\Gamma(\Delta_\sigma)^2}{\Gamma(\Delta_\sigma - \frac{\Delta_\varepsilon}{2})^2} \frac{f^2_{\sigma\sigma\varepsilon}}{s^{\Delta_\varepsilon}}$$

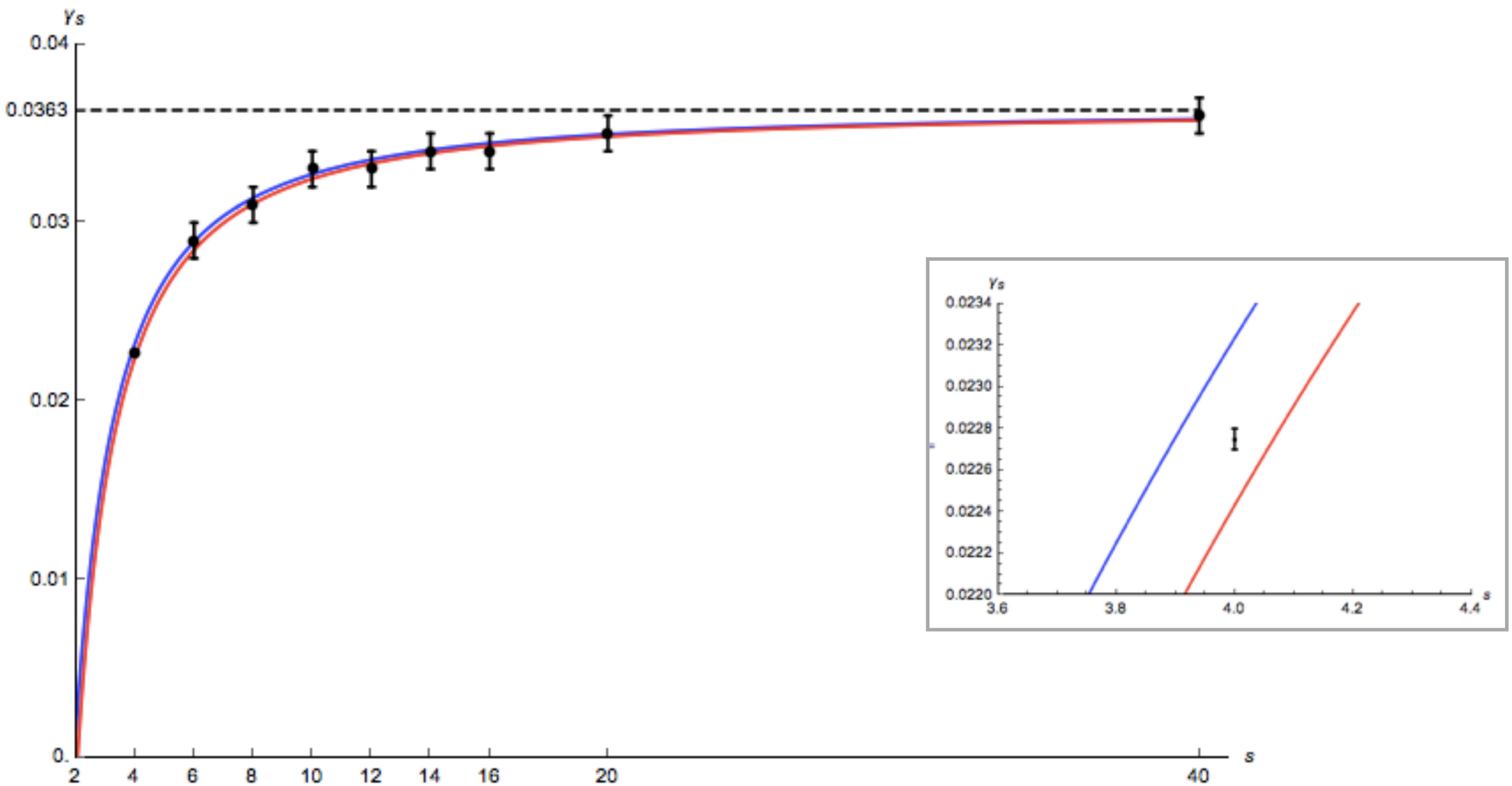
Corrections:

- From heavier operators $\frac{1}{s^\tau}$
- From higher spin currents $\frac{1}{s}$
- From the descendants of Δ_ε $\frac{1}{s^{\Delta_\varepsilon + n}}$

$$\gamma_s \simeq 0.0363 - \frac{0.0926}{s^{1.4126}} + \frac{0.0012}{s^{2.4126}} - \frac{0.0220}{s^{3.4126}} - \frac{0.0026}{s}$$

3D Ising Model

[Numerical bootstrap predictions, unpublished]
(3d Ising collaboration: S. El-Showk, M. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, A. Vichi)



Conclusions

- In the Lorentzian limit the crossing equation can be analyzed analytically (non-perturbatively)

- One can systematically compute the sub-leading corrections to the large spin anomalous dimensions due to descendants and heavier primaries

- Using this technology one can reproduce the known perturbative results and make further all-loop predictions

- Equally one can successfully apply it to strongly coupled theories (3D Ising, O(2), O(3)) where s=4 is already large!

Thank you!

Operators With High Twist

Consider operators made of n fields. We can ask what is the number of primary operators of this type exist. There is sharp transition

$$N(n,s) \sim \frac{s^{n-2}}{\Gamma(n-1)\Gamma(n+1)}$$

- Low twist operators live on finite number of Regge trajectories

- The number of high twist operators grows with spin