Cosmological aspects of the next-to-minimal supersymmetric standard model

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based on K. Kadota, M. Kawasaki, KS, hep-ph/1503.06998. (accepted in JCAP)

Supersymmetry (SUSY)

Fermion $\longleftrightarrow$ Boson

SM particles $\longleftrightarrow$ superpartners

- Hierarchy problem
- Coupling constant unification
- Dark matter candidate (assuming R-parity conservation)
  - Leading candidate of the physics beyond SM, which will be investigated in the forthcoming experiments
  - Implications for physics in the early universe ?
  - Cosmology: alternative probe of the model

Abstract

- We discuss cosmological aspects of the next-to-minimal supersymmetric standard model (NMSSM):

NMSSM =

Minimal Supersymmetric Standard Model (MSSM)

+

Additional gauge singlet superfield S

- Formation of domain walls in the early universe
- Estimate the gravitational wave signatures from domain walls and their parameter dependence

NMSSM as a solution to the μ-problem

Renormalizable superpotential of the MSSM

$$W_{\rm MSSM} = \mu H_u H_d + \lambda^e_{ij} H_d L_i E^c_j + \lambda^d_{ij} H_d Q_i D^c_j - \lambda^u_{ij} H_u Q_i U^c_j$$

μ-term

$i, j, k = 1, 2, 3$: family indices

- μ-problem: Why $\mu \sim M_{\rm SUSY}$ rather than $\mu \sim M_{\rm GUT}$ or $M_{\rm Pl}$?
- Introduce a gauge singlet S and replace the μ-term

$$\mu H_u H_d \rightarrow \lambda S H_u H_d$$

- Singlet acquires a VEV to induce an effective μ-term

$$\boxed{\mu_{\rm eff} = \lambda \langle S \rangle = \frac{\lambda}{\sqrt{2}} v_s} \qquad \langle S \rangle = \frac{1}{\sqrt{2}} v_s$$

- No dimensionful parameter except for soft SUSY breaking effects $\sim M_{\rm SUSY}$

$\Rightarrow$ naively expected that $\mu_{\rm eff} \sim \mathcal{O}(M_{\rm SUSY})$

- Need to forbid any dimensionful parameters like

$$\mu H_u H_d, \quad \mu'^2 S, \quad \text{and} \quad \mu'' S^2$$

- Impose a Z3 symmetry

$$\boxed{Z_3 : \ \Phi \to e^{2\pi i/3}\Phi}$$

$\Phi = (L, E^c, Q, U^c, D^c, H_u, H_d, S)$: every chiral supermultiplets of the NMSSM

$$W_{\text{NMSSM}} = \boxed{\lambda S H_u H_d + \frac{\kappa}{3} S^3}$$
$$+ \lambda^e_{ij} H_d L_i E^c_j + \lambda^d_{ij} H_d Q_i D^c_j - \lambda^u_{ij} H_u Q_i U^c_j$$

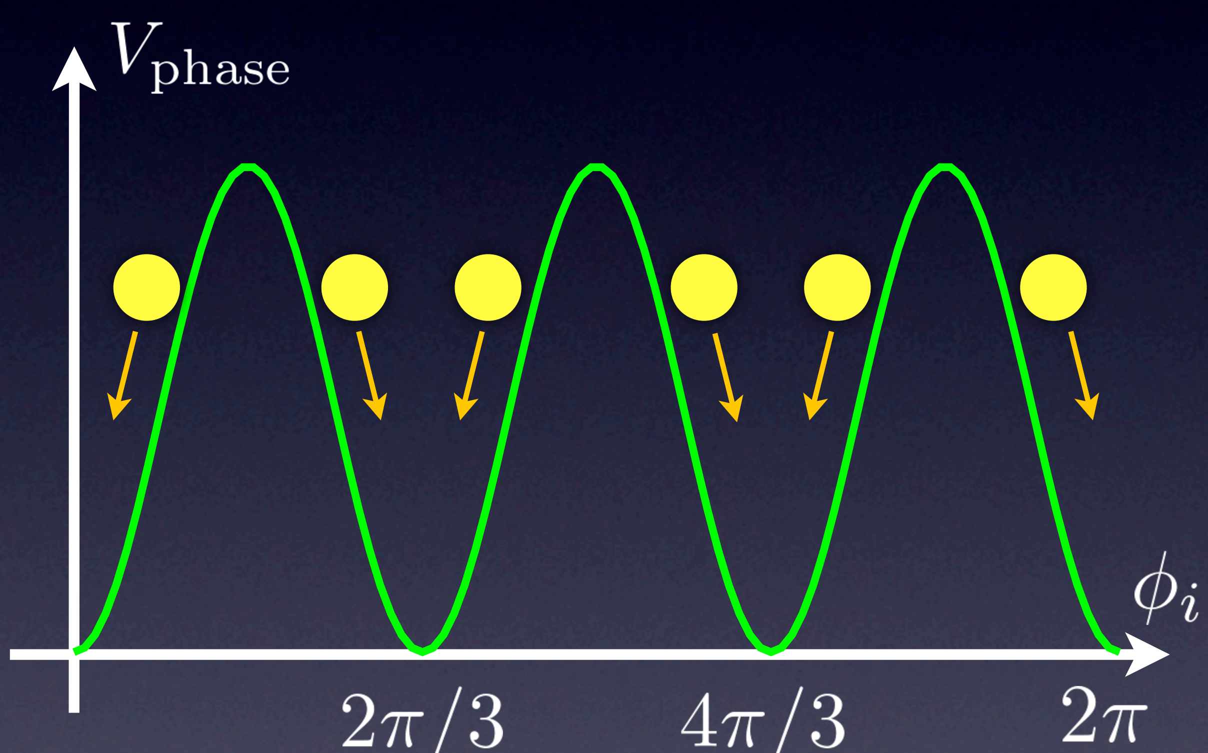
- Z3 is spontaneously broken when S, H_u, H_d acquire VEVs

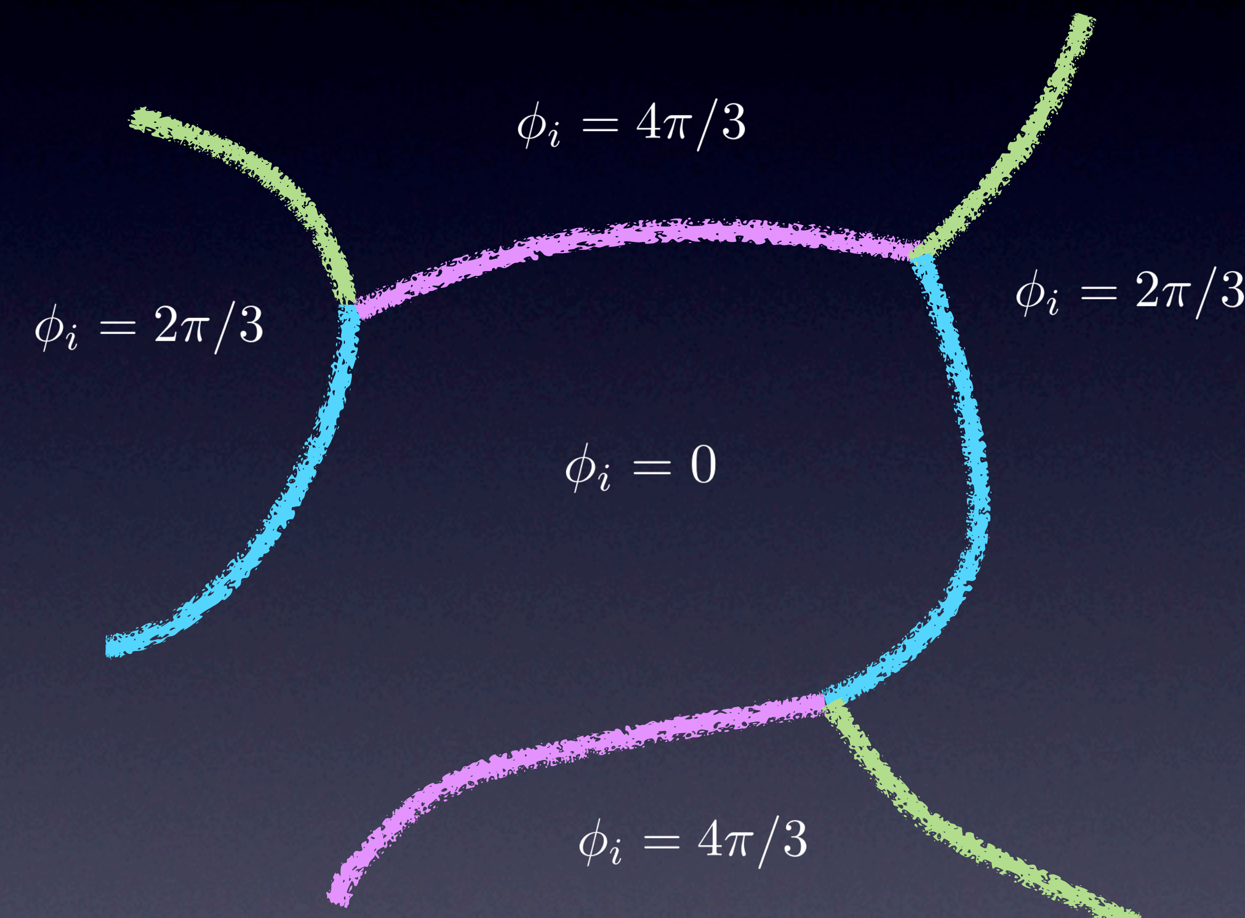
Formation of domain walls,
which can be the source of the gravitational wave background

Phase dependent terms in the scalar potential

$$V_{\rm phase} = \frac{1}{3\sqrt{2}}\kappa A_\kappa v_s^3 \cos(3\phi_s) - \frac{1}{2}\lambda\kappa v_u v_d v_s^2 \cos(\phi_u + \phi_d - 2\phi_s)$$
$$- \frac{1}{\sqrt{2}}\lambda A_\lambda v_u v_d v_s \cos(\phi_u + \phi_d + \phi_s)$$

$$\boxed{\langle H_u\rangle = \frac{1}{\sqrt{2}} v_u e^{i\phi_u},\ \langle H_d\rangle = \frac{1}{\sqrt{2}} v_d e^{i\phi_d},\ \langle S\rangle = \frac{1}{\sqrt{2}} v_s e^{i\phi_s}}$$





- Three degenerate minima related by

$$\phi_i \to \phi_i + \frac{2\pi k}{3}, \qquad k = 0, 1, 2 \qquad\qquad i = u, d, s$$

- Domain walls are formed around their boundaries

Domain wall problem and its solution

- Energy density of domain walls

$$\rho_{\rm wall} \sim \sigma_{\rm wall}/t$$

Press, Ryden, Spergel, ApJ 347, 590 (1989)

$\sigma_{\rm wall}$: surface mass density of domain walls [energy/area]

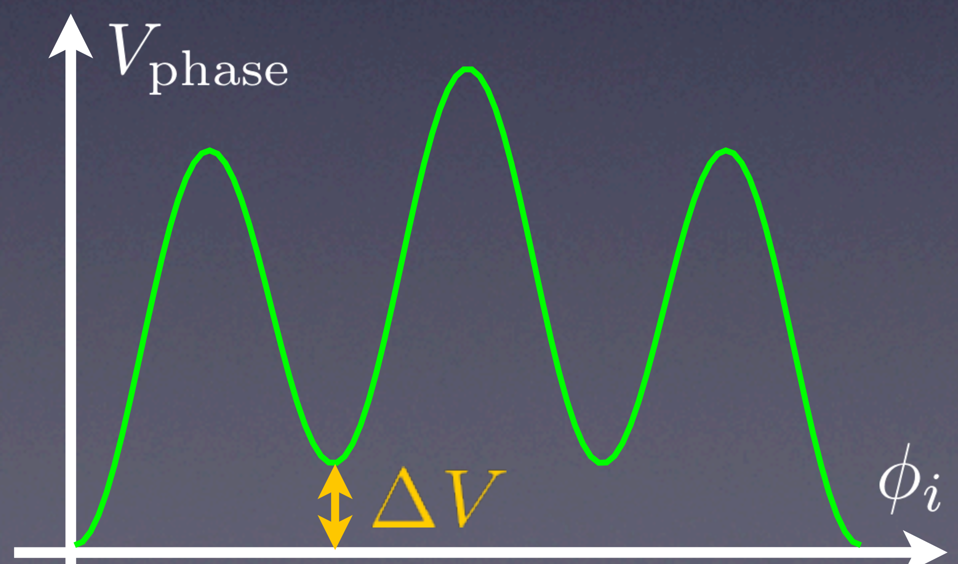
It decays slower than $\rho_{\rm matter} \propto a^{-3}$ and $\rho_{\rm radiation} \propto a^{-4}$

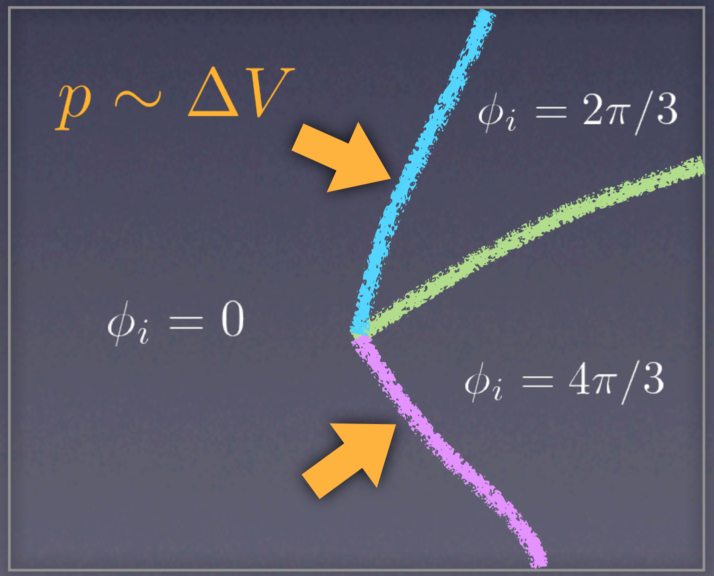
- Walls come to overclose the universe (domain wall problem)

Zel'dovich, Kobzarev, Okun, JETP 40, 1 (1975)

- This problem can be avoided by introducing an explicit symmetry breaking term ΔV Panagiotakopoulos, Tamvakis, Phys. Lett. B446, 224 (1999)

(collapse at a later time $t_{\rm dec} \propto 1/\Delta V$ due to the "pressure" between different vacua)

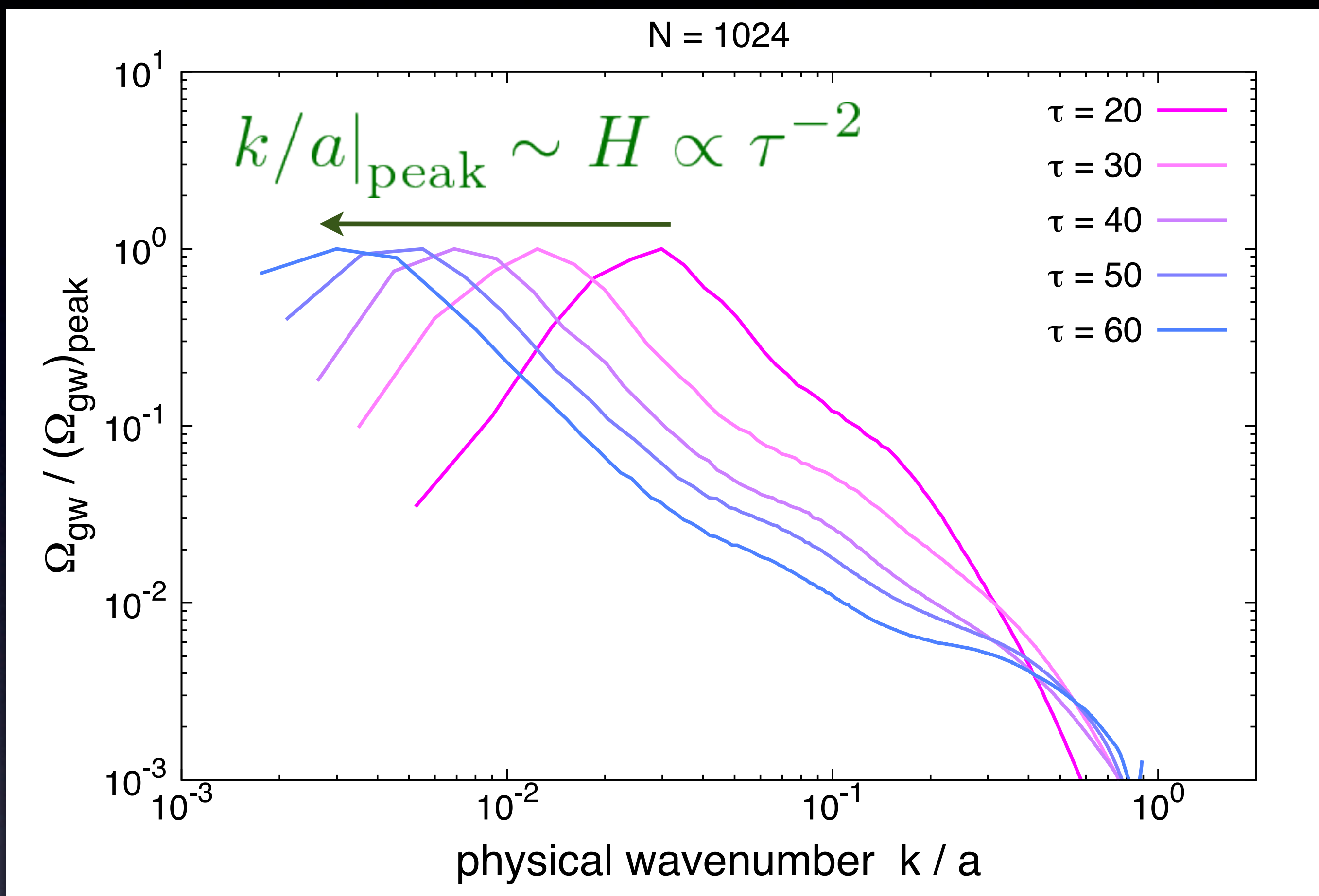


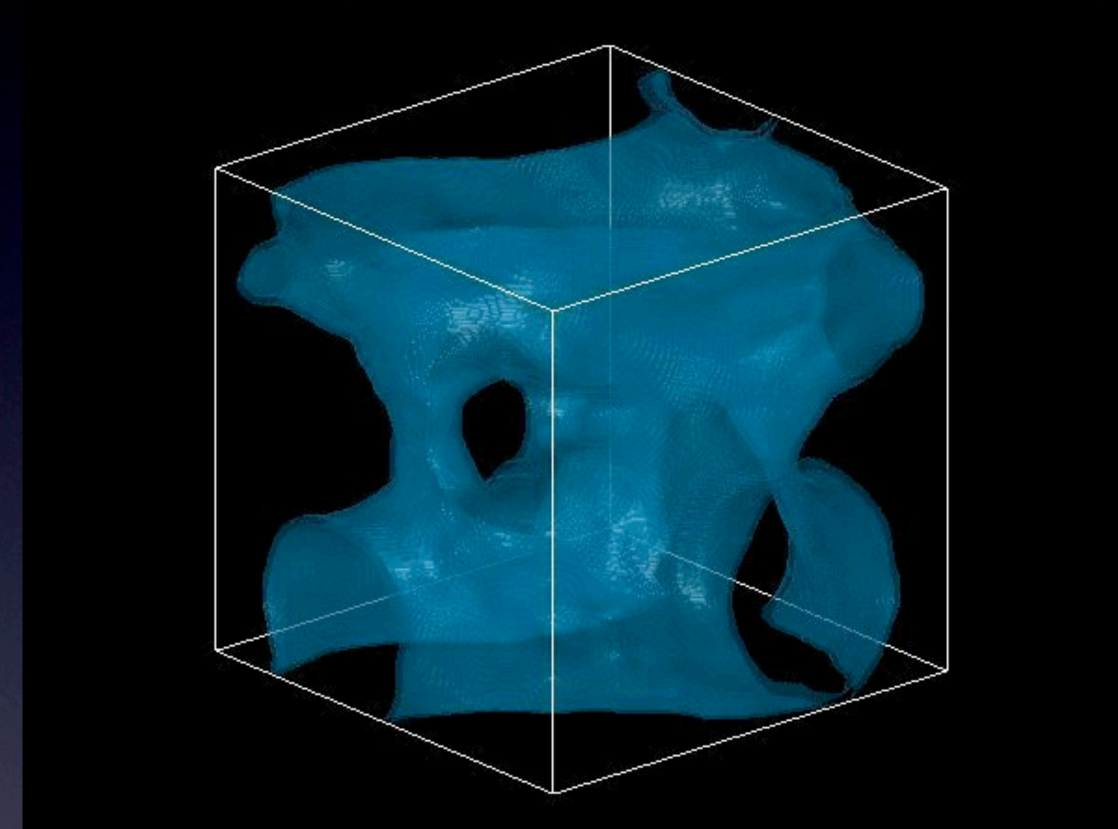


Gravitational waves from domain walls

Hiramatsu, Kawasaki, KS, JCAP02(2014)031

- Simulation of scalar fields in 3D lattice with 512^3 and 1024^3





- Amplitude $\boxed{\Omega_{\rm gw} h^2(t_0) \simeq 1.7 \times 10^{-17} \left(\frac{\sigma_{\rm wall}}{1{\rm TeV}^3}\right)^2 \left(\frac{t_{\rm dec}}{0.01{\rm sec}}\right)^2}$

- Peak frequency $\boxed{f_{\rm peak}(t_0) = \frac{a(t_{\rm dec})}{a(t_0)} H(t_{\rm dec}) \simeq 10^{-9}{\rm Hz} \left(\frac{0.01{\rm sec}}{t_{\rm dec}}\right)^{1/2}}$

- Decay before BBN: $t_{\rm dec} \lesssim 0.01{\rm sec} \rightarrow f \gtrsim 10^{-9}{\rm Hz}$

cf. pulsar timing $\Omega_{\rm gw} h^2 \sim 10^{-8}$ at $f \sim 10^{-9} - 10^{-8}{\rm Hz}$

Decoupling limit

- $v_s \gg v_u, v_d$ is possible if $\lambda \ll 1$

  cf. $\mu = \frac{1}{\sqrt{2}}\lambda v_s \approx \mathcal{O}(M_{\rm SUSY})$ $\quad |\langle S\rangle| = \frac{1}{\sqrt{2}}v_s,\ |\langle H_u\rangle| = \frac{1}{\sqrt{2}}v_u,\ |\langle H_d\rangle| = \frac{1}{\sqrt{2}}v_d$

- In this limit, the potential can be approximated as

$$V \simeq \kappa^2|S|^4 + m_S^2|S|^2 + \left[\frac{\kappa}{3}A_\kappa S^3 + \text{h.c.}\right] \Rightarrow v_s \simeq -\frac{\sqrt{2}A_\kappa}{4\kappa}\left(1+\sqrt{1-\frac{8m_S^2}{A_\kappa^2}}\right)$$

- λ and κ should be of the same order of magnitudes since $\mu \sim \lambda v_s \sim (\lambda/\kappa)A_\kappa \sim M_{\rm SUSY}$

- Decoupling limit is given by

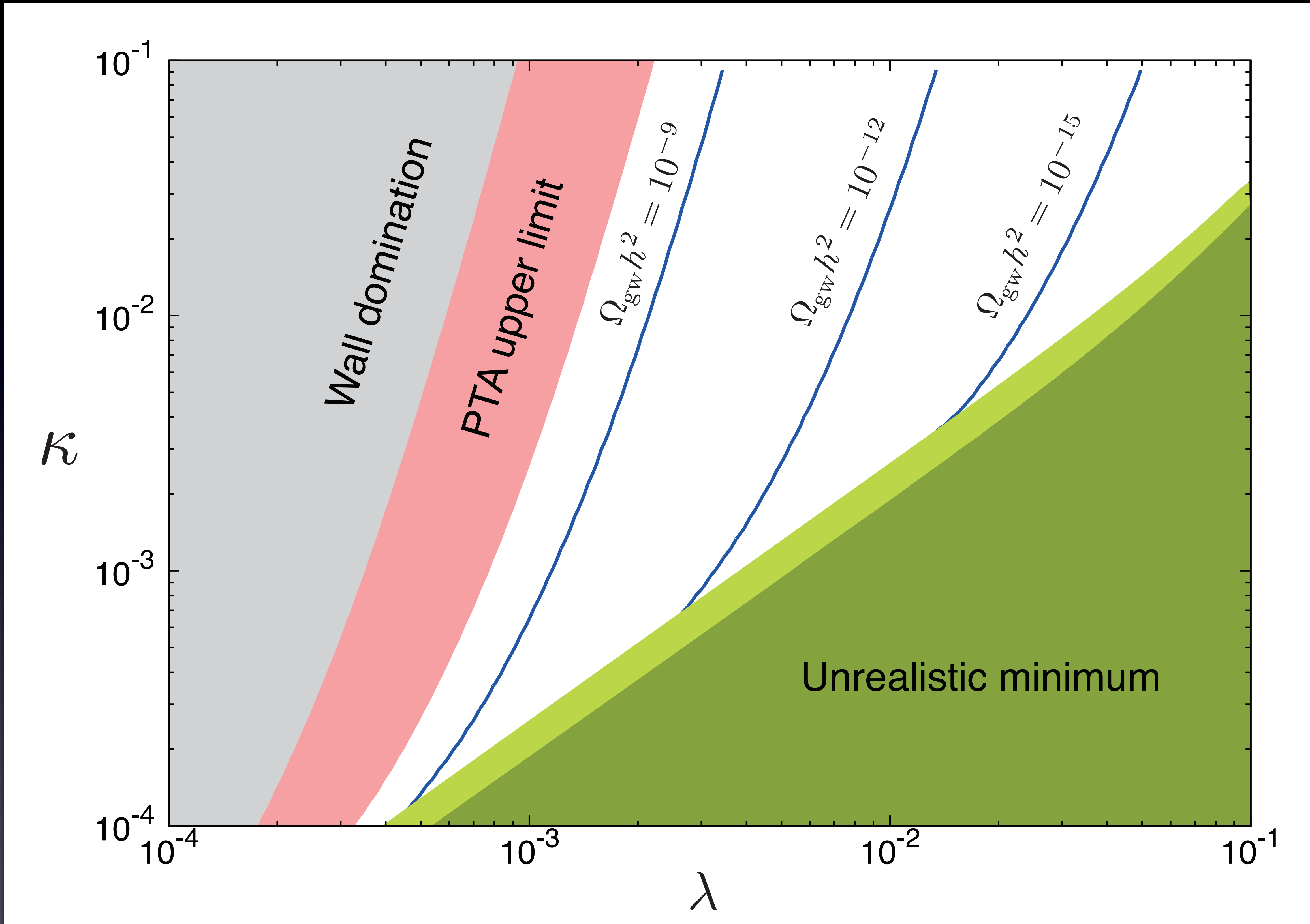
$$v_s \sim |A_\kappa/\kappa| \gg M_{\rm SUSY} \quad \text{for} \quad \lambda \sim \kappa \to 0$$

- Surface mass density of domain walls

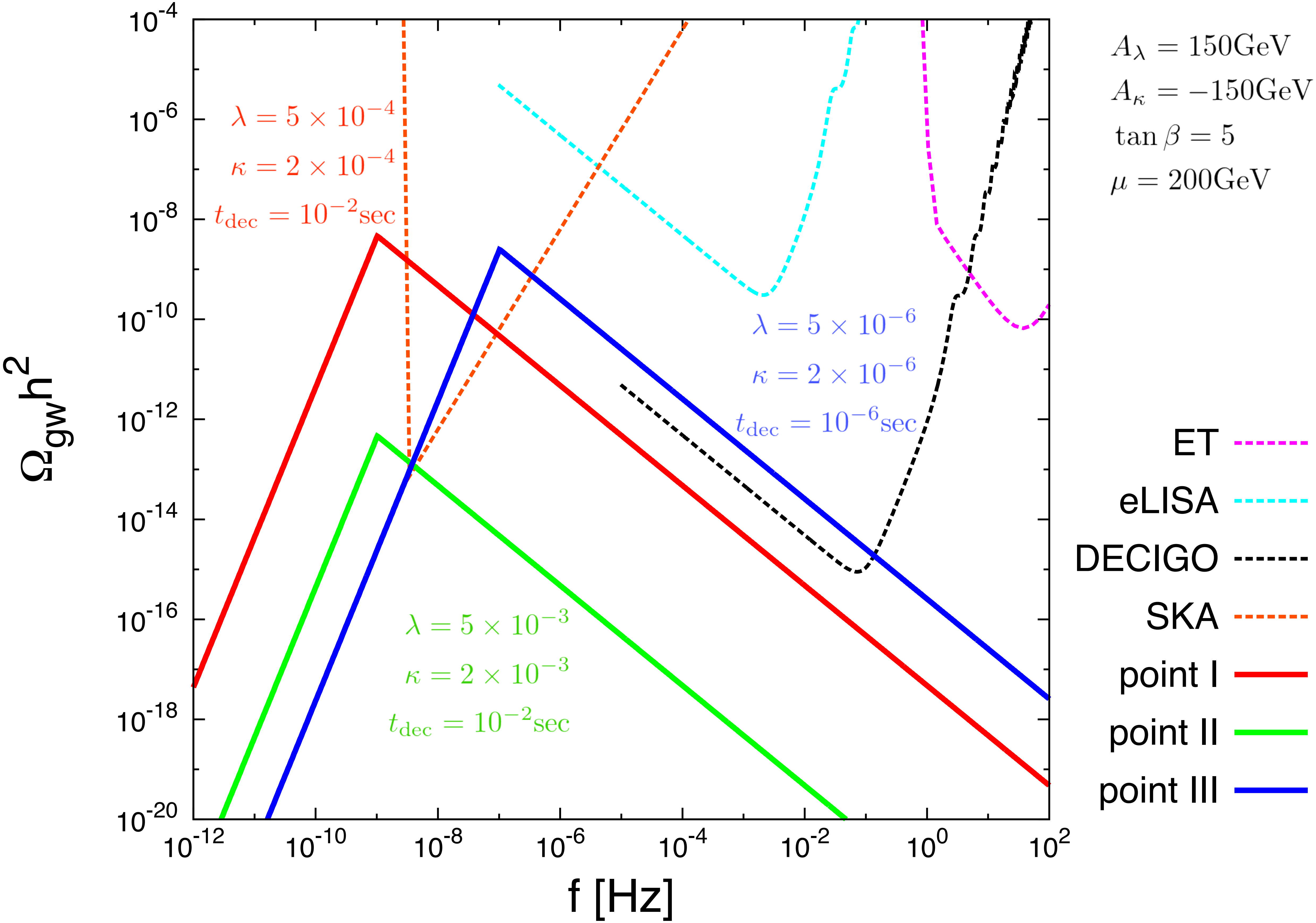
$$\sigma_{\rm wall} \sim \underbrace{M_{\rm SUSY}^{-1}}_{\text{width of domain walls}} \times V \sim \mathcal{O}(\kappa v_s^3) \gg \mathcal{O}(M_{\rm SUSY}^3) \quad \text{for} \quad \kappa, \lambda \ll 1$$

Amplitude of gravitational waves $\Omega_{\rm gw} \propto \sigma_{\rm wall}^2$ is enhanced

$$t_{\mathrm{dec}} = 0.01\mathrm{sec}$$



$$\boxed{\Omega_{\mathrm{gw}}h^2 \propto \kappa^2 v_s^6 t_{\mathrm{dec}}^2 \propto \kappa^2 \lambda^{-6} \mu^6 t_{\mathrm{dec}}^2 \quad \text{for} \quad \lambda, \kappa \ll 1}$$



$A_\lambda = 150\text{GeV}$
$A_\kappa = -150\text{GeV}$
$\tan\beta = 5$
$\mu = 200\text{GeV}$
$\lambda = 5 \times 10^{-4}$
$\kappa = 2 \times 10^{-4}$
$t_{\rm dec} = 10^{-2}\text{sec}$
$\lambda = 5 \times 10^{-6}$
$\kappa = 2 \times 10^{-6}$
$t_{\rm dec} = 10^{-6}\text{sec}$
$\lambda = 5 \times 10^{-3}$
$\kappa = 2 \times 10^{-3}$
$t_{\rm dec} = 10^{-2}\text{sec}$
ET
eLISA
DECIGO
SKA
point I
point II
point III
$\Omega_{gw}h^2$
f [Hz]

Conclusions

- NMSSM predicts the formation of domain walls due to the spontaneous breaking of the discrete Z_3 symmetry
  - Such domain walls can produce gravitational wave background
- Amplitude of gravitational waves is large in the decoupling limit

$$\Omega_{\rm gw}h^2 \propto \kappa^2\lambda^{-6}\mu^6 t_{\rm dec}^2 \qquad \text{for} \qquad \lambda, \kappa \ll 1$$

- Peak frequency is determined by the decay time of domain walls
  - $t_{\rm dec} \lesssim 0.01\text{sec} \;\rightarrow\; f \gtrsim 10^{-9}\text{Hz}$: typically pulsar timing range

Cosmology has a complementary role to probe the model

Backup

μ -problem of the MSSM

Renormalizable superpotential of the MSSM

$$W_{\mathrm{MSSM}} = \underbrace{\mu H_u H_d}_{\mu\text{-term}} + \lambda^e_{ij} H_d L_i E^c_j + \lambda^d_{ij} H_d Q_i D^c_j - \lambda^u_{ij} H_u Q_i U^c_j$$

$i, j, k = 1, 2, 3$: family indices

- μ cannot vanish

$$|\mu| > m_{\text{lightest chargino}} > \mathcal{O}(100\mathrm{GeV})$$

from chargino (mixtures of winos and charged higgsinos) mass

- μ must not be too large

$|\mu| \lesssim M_{\mathrm{SUSY}}$ to induce electroweak symmetry breaking $\langle H^0_u \rangle, \langle H^0_d \rangle \neq 0$

cf. $$V_{\text{neutral,quad}} = (m^2_{H_u} + |\mu|^2)|H^0_u|^2 + (m^2_{H_d} + |\mu|^2)|H^0_d|^2 + B\mu(H^0_u H^0_d + \mathrm{h.c.})$$

$|m_{H_u}|, |m_{H_d}|, B \sim \mathcal{O}(M_{\mathrm{SUSY}})$: soft SUSY breaking mass

- Why $\mu \sim M_{\mathrm{SUSY}}$ rather than $\mu \sim M_{\mathrm{GUT}}$ or M_{Pl} ?

Scalar potential

$$V = \kappa^2|S|^4 + \lambda^2|H_uH_d|^2 + m_{H_u}^2|H_u|^2 + m_{H_d}^2|H_d|^2 + m_S^2|S|^2 + \lambda^2|S|^2\left(|H_u|^2+|H_d|^2\right)$$
$$+ \frac{g_1^2+g_2^2}{8}\left(|H_u|^2-|H_d|^2\right)^2 + \underbrace{\left[\frac{\kappa}{3}A_\kappa S^3 - \lambda\kappa S^{*2}H_uH_d - \lambda A_\lambda H_uH_dS + \mathrm{h.c.}\right]}_{V_{\mathrm{phase}}}$$

g_1, g_2 : $U(1)_Y$ and $SU(2)_L$ gauge couplings

$|m_{H_u}|, |m_{H_d}|, |m_S|, |A_\lambda|, |A_\kappa| \sim \mathcal{O}(M_{\mathrm{SUSY}})$: soft SUSY breaking parameters

Hereafter $H_u \equiv H_u^0,\ H_d \equiv H_d^0$

Parameterize three VEVs

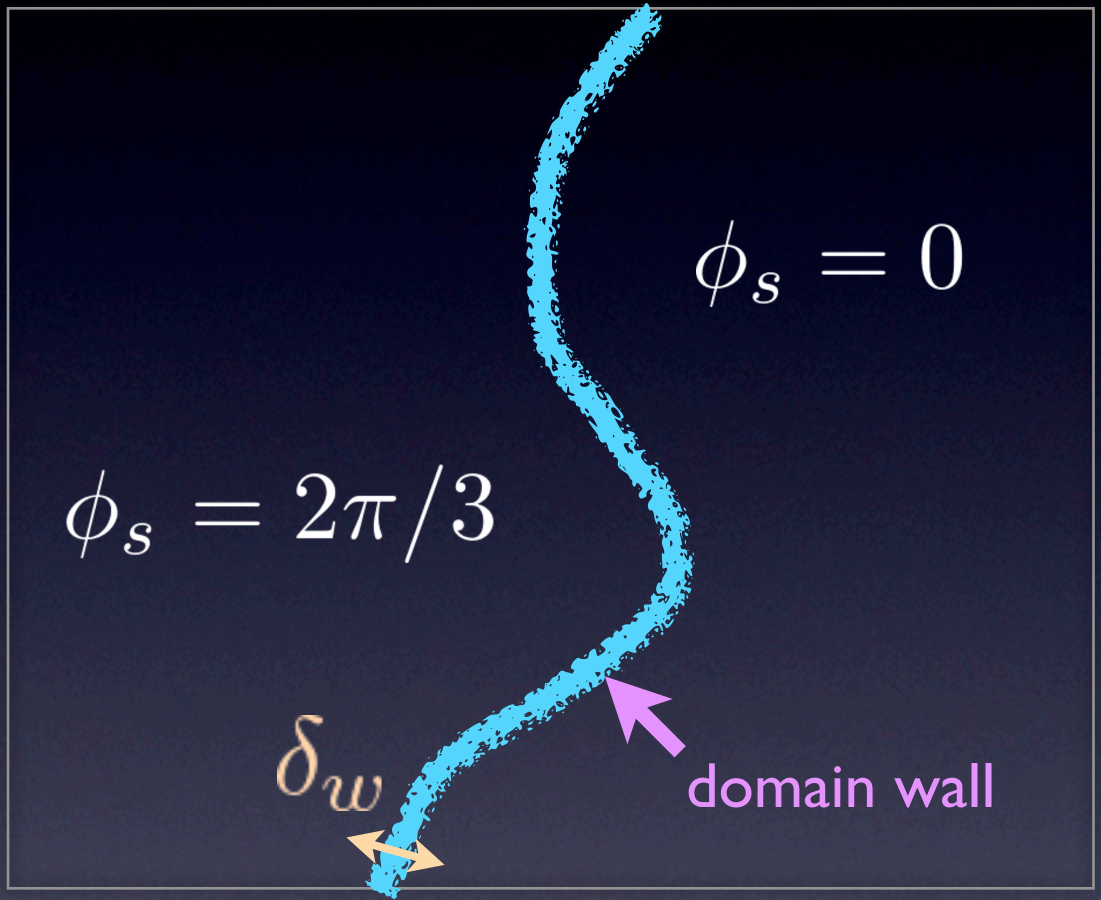
$$\langle H_u\rangle = \frac{1}{\sqrt{2}}v_u e^{i\phi_u},\ \langle H_d\rangle = \frac{1}{\sqrt{2}}v_d e^{i\phi_d},\ \langle S\rangle = \frac{1}{\sqrt{2}}v_s e^{i\phi_s}$$

$$V_{\mathrm{phase}} = \frac{1}{3\sqrt{2}}\kappa A_\kappa v_s^3\cos(3\phi_s) - \frac{1}{2}\lambda\kappa v_u v_d v_s^2\cos(\phi_u+\phi_d-2\phi_s)$$
$$- \frac{1}{\sqrt{2}}\lambda A_\lambda v_u v_d v_s\cos(\phi_u+\phi_d+\phi_s)$$

Properties of domain walls

$$V \simeq \kappa^2|S|^4 + m_S^2|S|^2 + \left[\frac{\kappa}{3}A_\kappa S^3 + \text{h.c.}\right]$$

$$S = \frac{1}{\sqrt{2}}v_s e^{i\phi_s} \quad v_s \sim \left|\frac{A_\kappa}{\kappa}\right|$$



- Width of the wall

$$\delta_w \sim \left|\frac{\partial^2 V}{\partial(v_s\phi_s)^2}\right|^{-1/2}$$

$$\sim |\kappa A_\kappa v_s|^{-1/2} \sim \mathcal{O}(M_{\rm SUSY}^{-1})$$

- Surface mass density

$$\sigma_{\rm wall} = \int dz \rho_{\rm wall}(z)$$

$$\sim \delta_w \times V \sim \mathcal{O}(\kappa v_s^3)$$

$\rho_{\rm wall}(z)$: energy density of the wall

z : coordinate perpendicular to the surface of the wall

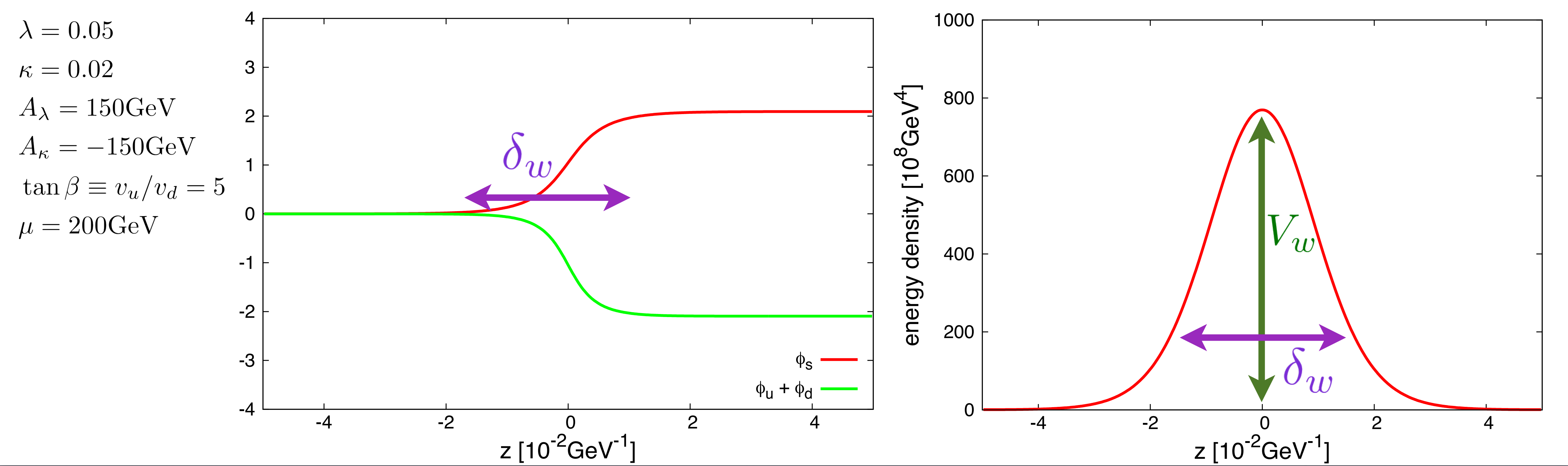
Note that $\boxed{\sigma_{\rm wall} \sim \mathcal{O}(\kappa v_s^3) \gg \mathcal{O}(M_{\rm SUSY}^3)}$ for $\kappa \ll 1$

Numerical estimation

- Solve static field equations with boundary conditions

$$\frac{d^2\Phi_i}{dz^2} = \frac{\partial V}{\partial \Phi_i^*}, \quad \Phi_i(z=\infty) = \frac{v_i}{\sqrt{2}}, \quad \Phi_i(z=-\infty) = \frac{v_i}{\sqrt{2}} e^{\frac{2\pi}{3} i}$$

$$\Phi_i = (S, H_u, H_d) \qquad v_i = (v_s, v_u, v_d)$$



$$V_w \sim |\kappa A_\kappa v_s^3|, \ \delta_w \sim |\kappa A_\kappa v_s|^{-1/2}, \ v_s \sim |A_\kappa/\kappa|$$

- The behavior in the decoupling limit is confirmed

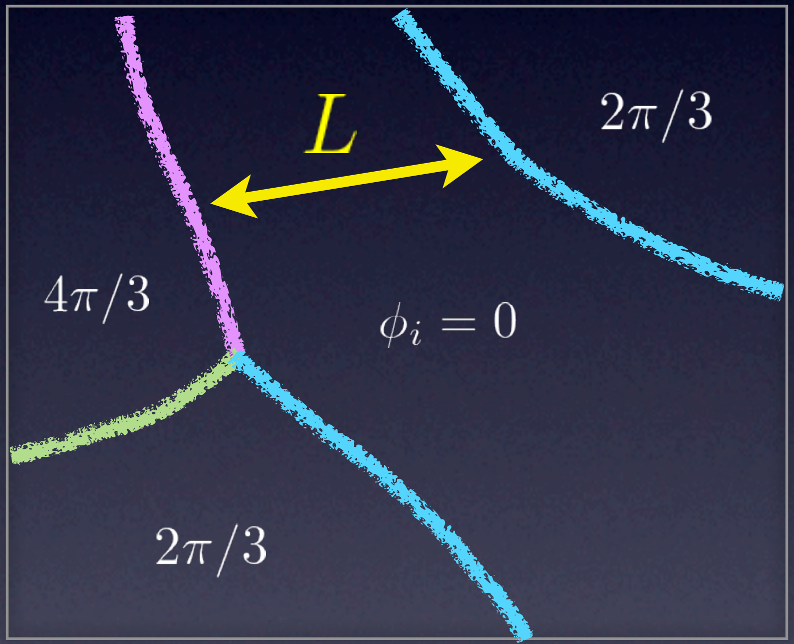
$$\boxed{\sigma_{\text{wall}} \sim \kappa v_s^3 \quad \text{for} \quad \lambda, \kappa \ll 1}$$

Domain wall problem

- Friction force from the thermal plasma becomes irrelevant for $T < \mathcal{O}(0.1-1)\mathrm{GeV}$ Abel, Sarkar, White, Nucl. Phys. B454, 663 (1995)

  ➡ After that tension force dominates
  Walls are stretched up to the Hubble radius

- Scaling solution Press, Ryden, Spergel, ApJ 347, 590 (1989)



  - $\mathcal{O}(1)$ wall(s) per Hubble radius

$$L \sim H^{-1} \sim t$$

L : distance between two neighboring walls

  - Energy density

$$\rho_{\rm wall} \sim \sigma_{\rm wall} L^2/L^3 \sim \sigma_{\rm wall}/t$$

decays slower than $\rho_{\rm matter} \propto a^{-3}$ and $\rho_{\rm radiation} \propto a^{-4}$

- Walls come to overclose the universe (domain wall problem)

Zel'dovich, Kobzarev, Okun, JETP 40, 1 (1975)

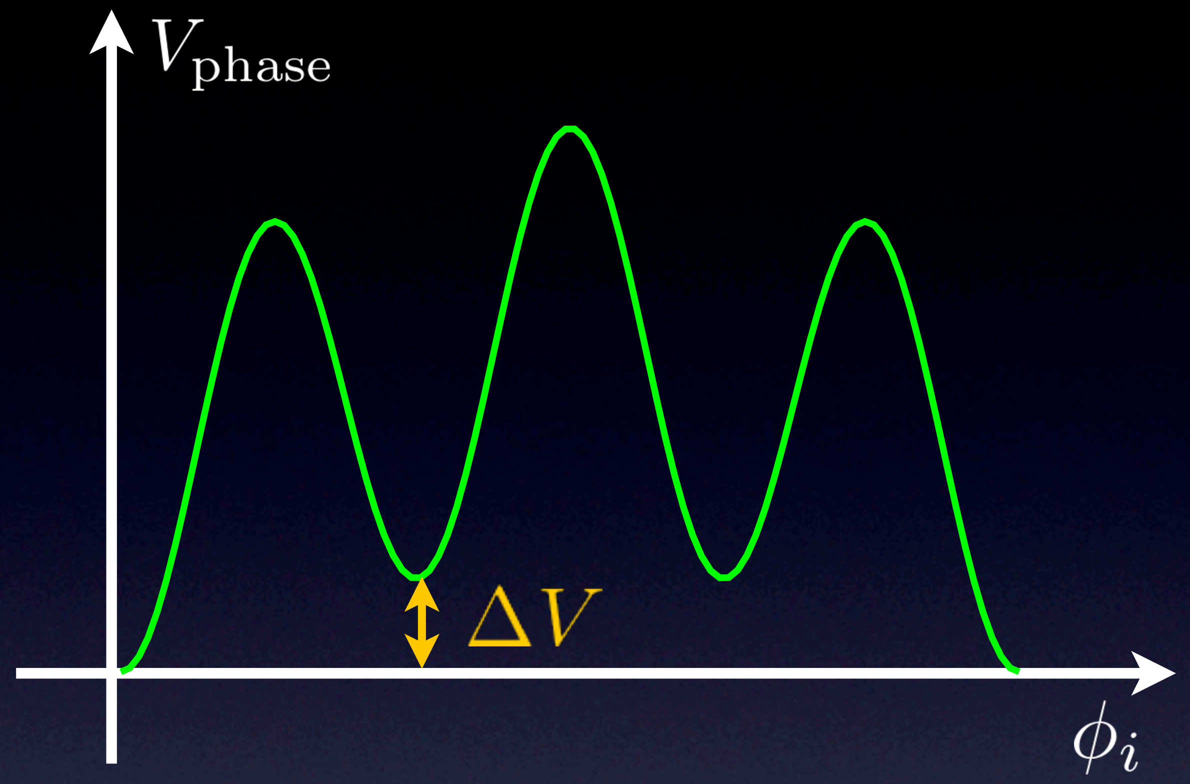
Collapse of domain walls

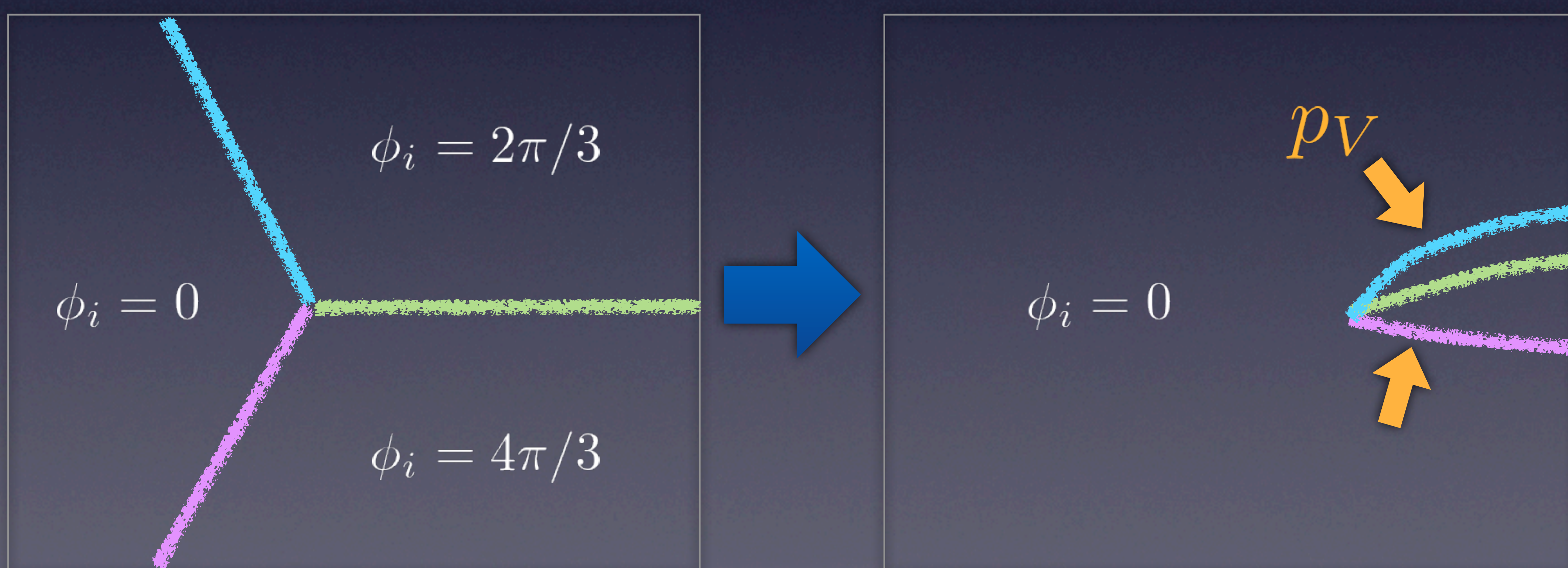
Approximate discrete symmetry (bias)

Vilenkin, PRD23, 852 (1981)

$$\Delta V \sim \Lambda^4 \qquad \Lambda \ll v_s$$

- Degenerate vacua are lifted
- Domain walls are annihilated due to the pressure $p_V \sim \Delta V$





Annihilation occurs when

$$p_V \sim p_T$$

$p_T \sim \sigma_{\rm wall}/R$: tension

R : curvature radius of walls

Decay time

$$t_{\rm dec} \sim R|_{p_V = p_T} \sim \frac{\sigma_{\rm wall}}{\Lambda^4}$$

$$\sim 7{\rm sec}\left(\frac{\sigma_{\rm wall}}{1{\rm TeV}^3}\right)\left(\frac{0.1{\rm MeV}}{\Lambda}\right)^4$$

- Decay products of domain walls may dissociate light element created during Big Bang Nucleosynthesis (BBN)

Require that $\boxed{t_{\rm dec} \lesssim 0.01{\rm sec}}$

- Domain walls exist after the electroweak phase transition and decay before the epoch of BBN

Gravitational waves are expected to be produced

Gravitational waves from domain walls

- Continuously produced during the scaling regime
  → It terminates at $t \sim t_{\rm dec}$
- Magnitude of gravitational waves
  - Quadrupole formula

$$P \sim G \dddot{Q}_{ij} \dddot{Q}_{ij} \sim M_{\rm wall}^2 / t^2$$: Power [energy / time]

$$Q_{ij} \sim M_{\rm wall} t^2$$

$$M_{\rm wall} \sim \sigma_{\rm wall} \mathcal{A} t^2$$

During the scaling regime

$$L \sim t$$

$$\rho_{\rm wall} \sim \sigma_{\rm wall} / t$$

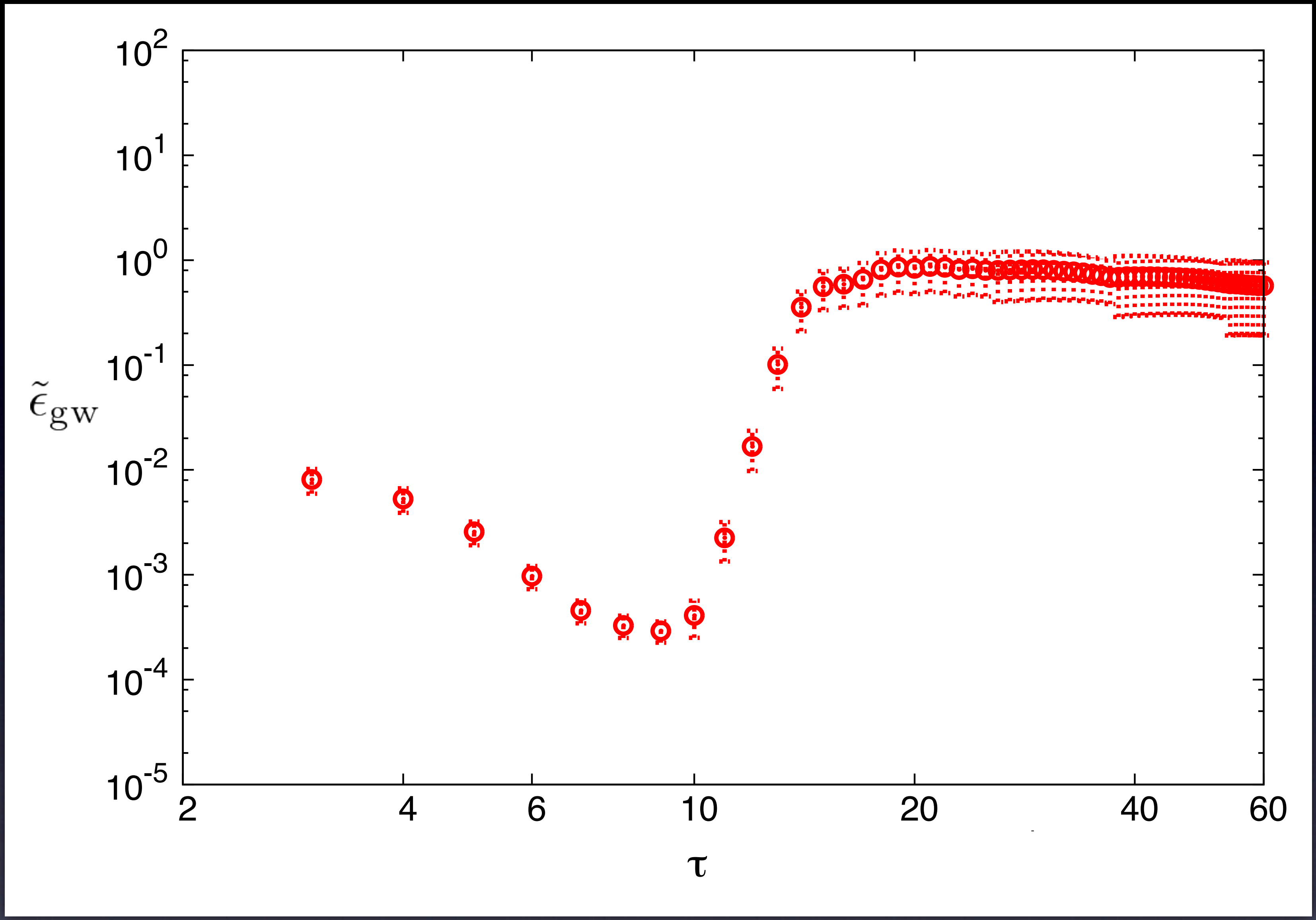
$$\mathcal{A} \equiv \frac{\rho_{\rm wall}}{\sigma_{\rm wall}} t \simeq \text{const. of } \mathcal{O}(1)$$

  - Energy density

$$\rho_{\rm gw} \sim \frac{Pt}{t^3} \sim G \mathcal{A}^2 \sigma_{\rm wall}^2$$

Magnitude of gravitational waves

Hiramatsu, Kawasaki, KS, JCAP02(2014)031



$$\tilde{\epsilon}_{\rm gw} \equiv \frac{1}{G\mathcal{A}^2\sigma_{\rm wall}^2}\left(\frac{d\rho_{\rm gw}}{d\ln k}\right)_{\rm peak}$$

c.f.

$$\rho_{\rm gw} \sim G\mathcal{A}^2\sigma_{\rm wall}^2 \sim {\rm const.}$$

during the scaling regime

$$\tilde{\epsilon}_{\rm gw} \simeq 0.7$$

Estimation of the present density

- Assume that the production of gravitational waves terminated at $t = t_{\rm dec}$

- Peak amplitude

$$\Omega_{\rm gw}(t_{\rm dec}) = \frac{1}{\rho(t_{\rm dec})}\left(\frac{d\rho_{\rm gw}}{d\ln k}\right)_{\rm peak} = \frac{8\pi\tilde{\epsilon}_{\rm gw}G^2\mathcal{A}^2\sigma_{\rm wall}^2}{3H^2(t_{\rm dec})} \qquad H(t_{\rm dec}) = \frac{1}{2t_{\rm dec}}$$

$$\boxed{\Omega_{\rm gw}h^2(t_0) \simeq 1.20\times10^{-17}\times\tilde{\epsilon}_{\rm gw}\mathcal{A}^4\left(\frac{10.75}{g_*}\right)^{1/3}\left(\frac{\sigma_{\rm wall}}{1{\rm TeV}^3}\right)^2\left(\frac{t_{\rm dec}}{0.01{\rm sec}}\right)^2}$$

g_* : relativistic degrees of freedom at $t_{\rm dec}$

$\tilde{\epsilon}_{\rm gw} \simeq 0.7$ $\mathcal{A} \simeq 1.2$ from simulations

- Peak frequency

$$\boxed{f_{\rm peak}(t_0) = \frac{a(t_{\rm dec})}{a(t_0)}H(t_{\rm dec}) \simeq 1.02\times10^{-9}{\rm Hz}\left(\frac{0.01{\rm sec}}{t_{\rm dec}}\right)^{1/2}}$$

- Decay before BBN: $t_{\rm dec} \lesssim 0.01{\rm sec} \;\rightarrow\; f \gtrsim 10^{-9}{\rm Hz}$

cf. pulsar timing $\Omega_{\rm gw}h^2 \sim 10^{-8}$ at $f \sim 10^{-9} - 10^{-8}{\rm Hz}$

Cosmological constraints

- Gravitational waves

$$\boxed{\Omega_{\rm gw} h^2 < \mathcal{O}(10^{-8})}$$ from pulsar timing observations

$$\sigma_{\rm wall} \sim \kappa v_s^3 \qquad \mu = \lambda v_s/\sqrt{2} \approx \mathcal{O}(100\text{GeV})$$

$$\Rightarrow \Omega_{\rm gw} h^2 \propto \sigma_{\rm wall}^2 t_{\rm dec}^2 \propto \kappa^2 v_s^6 t_{\rm dec}^2 \propto \kappa^2 \lambda^{-6} \mu^6 t_{\rm dec}^2$$

- Avoiding unrealistic minima of the potential
  - There might be some unrealistic minima on which

$$\langle H_u \rangle, \langle H_d \rangle \neq \left(\begin{array}{c} \text{correct} \\ \text{electroweak} \\ \text{value} \end{array} \right) \quad \text{and} \quad \langle S \rangle \neq \left(\begin{array}{c} \text{correct value} \\ \text{for } \mu\text{-term} \end{array} \right)$$

  - For the height of the potential $V_{\rm min}$, we should confirm that

$$\boxed{V_{\rm min,true} < V_{\rm min,unrealistic}}$$

Origin of the bias term

- Non-renormalizable operators such as $\Delta V \sim S^5/M_{\rm Pl}$ cannot be the bias since it radiatively induces a large tadpole operator $\Delta V \sim M_{\rm SUSY}^2 M_{\rm Pl}(S+S^*)$ which destabilizes S Abel, Sarkar, White, Nucl. Phys. B454, 663 (1995)

- The action (including non-renormalizable terms) can be arranged to induce a small bias term

$$\Delta V \sim (16\pi^2)^{-n} M_{\rm SUSY}^3 (S+S^*)$$

Panagiotakopoulos, Tamvakis, Phys. Lett. B446, 224 (1999)

rather than the dangerous term of $\sim M_{\rm SUSY}^2 M_{\rm Pl}(S+S^*)$ by imposing a discrete subgroup $Z_n^{(R)}$ of R-symmetry with n even
(consider Z_3 symmetry as an accidental symmetry)

Table 1. Three bench mark points and estimated surface mass density of domain walls, peak amplitude of gravitational waves, and its frequency. Here we used $g_* = 10.75$ for $t_{\rm dec} = 10^{-2}$sec and $g_* = 68.8$ for $t_{\rm dec} = 10^{-6}$sec [30] to estimate $\Omega_{\rm gw} h^2(t_0)_{\rm peak}$.

	I	II	III
λ	5×10^{-4}	5×10^{-3}	5×10^{-6}
κ	2×10^{-4}	2×10^{-3}	2×10^{-6}
A_λ	150GeV	150GeV	150GeV
A_κ	-150GeV	-150GeV	-150GeV
$\tan\beta$	5	5	5
μ	200GeV	200GeV	200GeV
$t_{\rm dec}$	10^{-2}sec	10^{-2}sec	10^{-6}sec
$\sigma_{\rm wall}$	1.96×10^4 TeV^3	1.96×10^2 TeV^3	1.96×10^8 TeV^3
$\Omega_{\rm gw} h^2(t_0)_{\rm peak}$	4.66×10^{-9}	4.66×10^{-13}	2.51×10^{-9}
$f(t_0)_{\rm peak}$	1.02×10^{-9}Hz	1.02×10^{-9}Hz	1.02×10^{-7}Hz

Calculation of GW spectrum

- Linearized theory

$$ds^2 = -dt^2 + a^2(\delta_{ij} + h_{ij})dx^i dx^j$$

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{16\pi G}{a^2}T^{TT}_{ij} \qquad \rho_{\rm gw}(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}(t,\mathbf{x})\dot{h}_{ij}(t,\mathbf{x})\rangle_V$$

- "Green function" method J. Dufaux, et al., PRD76, 123517 (2007)

$$\Omega_{\rm gw}(k,\tau) \equiv \frac{d\rho_{\rm gw}/d\ln k}{\rho_c(\tau)} = \frac{4}{3\pi V}\frac{G^2}{a^4 H^2}S_k(\tau)$$

$$S_k(\tau) \equiv k\int d\Omega_k \sum_{ij}\left(\left|C^{(1)}_{ij}\right|^2 + \left|C^{(2)}_{ij}\right|^2\right) \qquad \rho_c(\tau) = \frac{3H^2}{8\pi G}$$

$$C^{(1)}_{ij} = -\int_{\tau_i}^{\tau} kd\tau' \sin(k\tau')a(\tau')T^{TT}_{ij}(\mathbf{k},\tau')$$

$$C^{(2)}_{ij} = \int_{\tau_i}^{\tau} kd\tau' \cos(k\tau')a(\tau')T^{TT}_{ij}(\mathbf{k},\tau')$$

$$T^{\rm TT}_{ij}(\mathbf{k},\tau) = \Lambda_{ij,kl}\{\partial_k\phi\partial_l\phi\}(\mathbf{k},\tau)$$

$$\Lambda_{ij,kl}(\hat{k}) = P_{ik}(\hat{k})P_{jl}(\hat{k}) - \frac{1}{2}P_{ij}(\hat{k})P_{kl}(\hat{k})$$

$$P_{ij}(\hat{k}) = \delta_{ij} - \hat{k}_i\hat{k}_j \qquad \hat{k} = \mathbf{k}/|\mathbf{k}|$$

Spectrum in small k

- Correlation function of the anisotropic stress tensor

$$a^{-2}(\tau)T_{ij}^{\mathrm{TT}}(\tau,\mathbf{k}) \equiv (\rho_c + p_c)\Pi_{ij}(\tau,k)$$

$$\sum_{ij}\langle\Pi_{ij}(\tau_1,\mathbf{k})\Pi_{ij}^{*}(\tau_2,\mathbf{k}')\rangle \equiv (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\Pi(k,\tau_1,\tau_2)$$

$\rho_c,\ p_c$: energy density & pressure of the background fluids

- Requirement from causality

$$\Pi(|\mathbf{x}-\mathbf{x}'|,\tau_1,\tau_2) = 0 \quad \text{for} \quad |\mathbf{x}-\mathbf{x}'| > l_c$$

$l_c \gtrsim H^{-1}$: correlation length

$$\Pi(k,\tau_1,\tau_2) = \int d^3z e^{i\mathbf{k}\cdot\mathbf{z}}\Pi(z,\tau_1,\tau_2) = \int_0^{l_c} dz\frac{4\pi z}{k}\sin(kz)\Pi(z,\tau_1,\tau_2)$$

$$\xrightarrow{kl_c\to 0} (\text{indep. of } k)$$

- $\boxed{\Omega_{\mathrm{gw}} \propto k^3}$ at small k

$$\Omega_{\mathrm{gw}}(k,t) = \frac{1}{\rho_c(t)}\frac{d\rho_{\mathrm{gw}}(t)}{d\ln k} = \frac{4}{3\pi^2}k^3\int_{\tau_p}^{\tau}\frac{d\tau_1}{\tau_1}\int_{\tau_p}^{\tau}\frac{d\tau_2}{\tau_2}\cos(k(\tau_1-\tau_2))\Pi(k,\tau_1,\tau_2)$$

$$\propto k^3 \qquad \text{for} \qquad kl_c \ll 1$$