Calculation of 3-loop operator matrix elements with two masses

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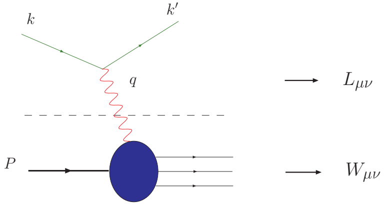
Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q}$$ Bjorken–x

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P, s \rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x, Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x, Q^2) \; .$$

Structure Functions: $F_{2,L}$
contain light and heavy quark contributions.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).

$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz\, \delta(x - yz) f(y) g(z) \ .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx\, x^{N-1} f(x) \ .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)\ .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j \mid O_i \mid j\rangle\ .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO [Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

The inclusive DIS structure functions can be represented in the FFNS in terms of massless and heavy quark contrubutions:

$$F_i(x,Q^2) = F_i^{\mathrm{massless}}(x,Q^2) + F_i^{\mathrm{heavy}}(x,Q^2)$$

$$\begin{aligned}\frac{1}{x}F_i^{\mathrm{massless}}(x,Q^2) \quad &= \quad \sum_q e_q^2 \Bigg\{ \frac{1}{N_F}\left[\Sigma(x,\mu^2)\otimes C_{i,Q}^{\mathrm{S}}\left(x,\frac{Q^2}{\mu^2}\right) + G\left(x,\mu^2\right)\otimes C_{i,g}\left(x,\frac{Q^2}{\mu^2}\right)\right] \\ &+\Delta_q(x,\mu^2)\otimes C_{i,q}^{\mathrm{NS}}\left(x,\frac{Q^2}{\mu^2}\right)\Bigg\}\,, \quad i=2,L\end{aligned}$$

where Σ and Δ_k are the flavor singlet and non-singlet distributions given by

$$\begin{aligned}\Sigma \quad &= \quad \sum_{k=1}^{N_F}(f_k + f_{\overline{k}}) \\ \Delta_k \quad &= \quad f_k + f_{\overline{k}} - \frac{1}{N_F}\Sigma,\end{aligned}$$

and G denotes the gluon density.

Starting at 3-loop order, we have to consider the simultaneous contributions of quarks of different mass, since m_b is not much larger than m_c

$$\frac{m_c}{m_b} \sim 0.1$$

The heavy flavor contribution with two masses is given by

$$\begin{aligned}
&\frac{1}{x} F_{(2,L)}{}^{\text{heavy}}(x, N_F+2, Q^2, m_1^2, m_2^2) = \\
&\quad \sum_{k=1}^{N_F} e_k^2 \Bigg\{ L_{q,(2,L)}^{\mathrm{NS}} \left(x, N_F+2, \frac{Q^2}{\mu^2}, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \otimes \left[f_k(x,\mu^2,N_F) + f_{\overline{k}}(x,\mu^2,N_F) \right] \\
&\qquad + \frac{1}{N_F} L_{q,(2,L)}^{\mathrm{PS}} \left(x, N_F+2, \frac{Q^2}{\mu^2}, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,N_F) \\
&\qquad + \frac{1}{N_F} L_{g,(2,L)}^{\mathrm{S}} \left(x, N_F+2, \frac{Q^2}{\mu^2}, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \otimes G(x,\mu^2,N_F) \Bigg\} \\
&\quad + \tilde{\tilde{H}}_{q,(2,L)}^{\mathrm{PS}} \left(x, N_F+2, \frac{Q^2}{\mu^2}, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,N_F) \\
&\quad + \tilde{\tilde{H}}_{g,(2,L)}^{\mathrm{S}} \left(x, N_F+2, \frac{Q^2}{\mu^2}, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \otimes G(x,\mu^2,N_F)
\end{aligned}$$

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2 + A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big]\\
L_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= a_s^2 A_{gg,Q}^{(1)}(N_F+2) N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+2)\ \delta_2\\
&\quad + A_{gg,Q}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+2) + A_{gg,Q}^{(2)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\\
&\quad + A_{Qg}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ ,\\
L_{q,(2,L)}^{\mathsf{NS}}(N_F+2) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2)\ \delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big]\\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+2)\ \delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2) C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big]\\
\tilde{\tilde{H}}_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2 a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ \delta_2 +\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big] + a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2\\
&\quad + \sum_{i=1}^{2} e_{Q_i}^2\Big[\tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+2) + A_{gq,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\\
&\quad + A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2)\Big]\Big]\ ,\\
\tilde{\tilde{H}}_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2\Big[a_s\Big[\ A_{Qg}^{(1)}(N_F+2)\ \delta_2 +\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\Big] + a_s^2\Big[\ A_{Qg}^{(2)}(N_F+2)\ \delta_2\\
&\quad + A_{Qg}^{(1)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) +\ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\\
&\quad + \tilde{C}_{g,(2,L)}^{(2)}(N_F+2)\Big]\Big]\\
&\quad + a_s^3\Big[\ A_{Qg}^{(3)}(N_F+2)\ \delta_2 +\ \sum_{i=1}^{2} e_{Q_i}^2\Big[A_{Qg}^{(2)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2)\\
&\quad + A_{gg,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) +\ A_{Qg}^{(1)}(N_F+2)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+2)\\
&\quad + \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big\} +\ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+2) +\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+2)\Big]\Big]
\end{aligned}
$$

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2 + A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big] \\
L_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= a_s^2 A_{gg,Q}^{(1)}(N_F+2) N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+2)\ \delta_2 \\
&\quad + A_{gg,Q}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+2) + A_{gg,Q}^{(2)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\quad + A_{Qg}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ , \\
L_{q,(2,L)}^{\mathsf{NS}}(N_F+2) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2)\ \delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+2)\ \delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2) C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big] \\
\tilde{\tilde{H}}_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2 a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ \delta_2 +\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big] + a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2 \\
&\quad + \sum_{i=1}^{2} e_{Q_i}^2\Big[\tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+2) + A_{gq,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\quad + A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2)\Big]\Big]\ , \\
\tilde{\tilde{H}}_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2\Big[a_s\Big[\ A_{Qg}^{(1)}(N_F+2)\ \delta_2 +\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\Big] + a_s^2\Big[\ A_{Qg}^{(2)}(N_F+2)\ \delta_2 \\
&\quad + A_{Qg}^{(1)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) +\ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\quad + \tilde{C}_{g,(2,L)}^{(2)}(N_F+2)\Big]\Big] \\
&\quad + a_s^3\Big[\ A_{Qg}^{(3)}(N_F+2)\ \delta_2 +\ \sum_{i=1}^{2} e_{Q_i}^2\Big[A_{Qg}^{(2)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) \\
&\quad + A_{gg,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) +\ A_{Qg}^{(1)}(N_F+2)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+2) \\
&\quad + \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big\} +\ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+2) +\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+2)\Big]\Big]
\end{aligned}
$$

The Wilson Coefficients at large Q^2

$$L_{q,(2,L)}^{\mathsf{PS}}(N_F+2) = a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2 + A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big]$$

$$\begin{aligned} L_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= a_s^2 A_{gg,Q}^{(1)}(N_F+2) N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+2)\ \delta_2 \\ &+ A_{gg,Q}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+2) + A_{gg,Q}^{(2)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\ &+ A_{Qg}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ , \end{aligned}$$

$$\begin{aligned} L_{q,(2,L)}^{\mathsf{NS}}(N_F+2) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2)\ \delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\ &+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+2)\ \delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2) C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big] \end{aligned}$$

$$\begin{aligned} \tilde{\tilde{H}}_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2 a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ \delta_2 +\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big] + a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2 \\ &+\ \sum_{i=1}^{2} e_{Q_i}^2\Big[\tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+2) + A_{gq,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\ &+ A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2)\Big]\Big]\ , \end{aligned}$$

$$\begin{aligned} \tilde{\tilde{H}}_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2\Big[a_s\Big[\ A_{Qg}^{(1)}(N_F+2)\ \delta_2 +\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\Big] + a_s^2\Big[\ A_{Qg}^{(2)}(N_F+2)\ \delta_2 \\ &+\ A_{Qg}^{(1)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) +\ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\ &+\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+2)\Big]\Big] \\ &+\ a_s^3\Big[\ A_{Qg}^{(3)}(N_F+2)\ \delta_2 +\ \sum_{i=1}^{2} e_{Q_i}^2\Big[A_{Qg}^{(2)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) \\ &+\ A_{gg,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) +\ A_{Qg}^{(1)}(N_F+2)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+2) \\ &+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big\} +\ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+2) +\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+2)\Big]\Big] \end{aligned}$$

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2 + A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big] \\
L_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= a_s^2 A_{gg,Q}^{(1)}(N_F+2)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+2)\ \delta_2 \\
&+ A_{gg,Q}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+2) + A_{gg,Q}^{(2)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&+ A_{Qg}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ , \\
L_{q,(2,L)}^{\mathsf{NS}}(N_F+2) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2)\ \delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+2)\ \delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big] \\
\tilde{\tilde{H}}_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2 a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ \delta_2 +\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big] + a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2 \\
&+\ \sum_{i=1}^{2} e_{Q_i}^2\Big[\tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+2) + A_{gq,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&+ A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2)\Big]\Big]\ , \\
\tilde{\tilde{H}}_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2\Big[a_s\Big[\ A_{Qg}^{(1)}(N_F+2)\ \delta_2 +\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\Big] + a_s^2\Big[\ A_{Qg}^{(2)}(N_F+2)\ \delta_2 \\
&+\ A_{Qg}^{(1)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) +\ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&+\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+2)\Big]\Big] \\
&+\ a_s^3\Big[\ A_{Qg}^{(3)}(N_F+2)\ \delta_2 +\ \sum_{i=1}^{2} e_{Q_i}^2\Big[A_{Qg}^{(2)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) \\
&+\ A_{gg,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) +\ A_{Qg}^{(1)}(N_F+2)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+2) \\
&+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big\} +\ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+2) +\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+2)\Big]\Big]
\end{aligned}
$$

Variable Flavor Number Scheme

$$f_k(n_f+2,\mu^2)+f_{\overline{k}}(n_f+2,\mu^2) = A_{qq,Q}^{\mathrm{NS}}\left(n_f+2\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right)\otimes\left[f_k(n_f,\mu^2)+f_{\overline{k}}(n_f,\mu^2)\right]$$

$$+\frac{1}{n_f}A_{qq,Q}^{\mathrm{PS}}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right)\otimes\Sigma(n_f,\mu^2)$$

$$+\frac{1}{n_f}A_{qg,Q}^{\mathrm{S}}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right)\otimes G(n_f,\mu^2)$$

$$f_{Q+\bar{Q}}(n_f+2,\mu^2) = A_{Qq}^{\mathrm{PS}}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right)\otimes\Sigma(n_f,\mu^2)+A_{Qg}^{\mathrm{S}}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right)\otimes G(n_f,\mu^2)\,.$$

$$G(n_f+2,\mu^2) = A_{gq,Q}^{\mathrm{S}}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right)\otimes\Sigma(n_f,\mu^2)+A_{gg,Q}^{S}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right)\otimes G(n_f,\mu^2)\,.$$

$$\Sigma(n_f+2,\mu^2) = \sum_{k=1}^{n_f+2}\left[f_k(n_f+2,\mu^2)+f_{\overline{k}}(n_f+2,\mu^2)\right]$$

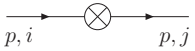
$$= \left[A_{qq,Q}^{\mathrm{NS}}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right)+A_{qq,Q}^{\mathrm{PS}}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right)+A_{Qq}^{\mathrm{PS}}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right)\right]$$

$$\otimes\Sigma(n_f,\mu^2)$$

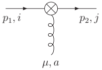
$$+\left[A_{qg,Q}^{\mathrm{S}}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right)+A_{Qg}^{\mathrm{S}}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right)\right]\otimes G(n_f,\mu^2)$$

Calculation of the 3-loop operator matrix elements

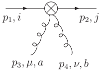
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



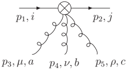
$$\delta^{ij} \not\Delta \gamma_\pm (\Delta \cdot p)^{N-1}\ , \quad N \geq 1$$



$$g t^a_{ji} \Delta^\mu \not\Delta \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}\ , \quad N \geq 2$$



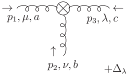
$$g^2 \Delta^\mu \Delta^\nu \not\Delta \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2}$$
$$\Big[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1}\Big]\ ,$$
$$N \geq 3$$



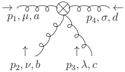
$$g^3 \Delta_\mu \Delta_\nu \Delta_\rho \not\Delta \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta.p_2)^j (\Delta.p_1)^{N-m-2}$$
$$\Big[\ (t^a t^b t^c)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1}$$
$$+ (t^a t^c t^b)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1}$$
$$+ (t^b t^a t^c)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1}$$
$$+ (t^b t^c t^a)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1}$$
$$+ (t^c t^a t^b)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1}$$
$$+ (t^c t^b t^a)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1}\Big]\ ,$$
$$N \geq 4$$

$$\gamma_+ = 1\ , \quad \gamma_- = \gamma_5\ .$$

$$\frac{1+(-1)^N}{2} \delta^{ab} (\Delta \cdot p)^{N-2}$$
$$\Big[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu\Big]\ , \quad N \geq 2$$

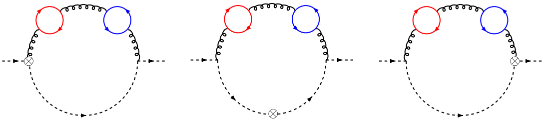


$$-ig \frac{1+(-1)^N}{2} f^{abc} \Bigg($$
$$\Big[(\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu)\Big] (\Delta \cdot p_1)^{N-2}$$
$$+ \Delta_\lambda \Big[\Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu\Big]$$
$$\times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j}$$
$$+ \left\{ {p_1 \to p_2 \to p_3 \to p_1 \atop \mu \to \nu \to \lambda \to \mu} \right\} + \left\{ {p_1 \to p_3 \to p_2 \to p_1 \atop \mu \to \lambda \to \nu \to \mu} \right\} \Bigg)\ , \quad N \geq 2$$

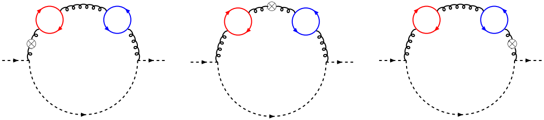


$$g^2 \frac{1+(-1)^N}{2} \Bigg(f^{abe} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4)$$
$$+ f^{ace} f^{bde} O_{\mu\lambda\nu\sigma}(p_1, p_3, p_2, p_4) + f^{ade} f^{bce} O_{\mu\sigma\nu\lambda}(p_1, p_4, p_2, p_3) \Bigg)\ ,$$
$$O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_\nu \Delta_\lambda \Bigg\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2}$$
$$+ [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i}$$
$$- [p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i}$$
$$+ [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma]$$
$$\times \sum_{i=0}^{N-4} \sum_{j=0}^{i} (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \Bigg\}$$
$$- \left\{ {p_1 \leftrightarrow p_2 \atop \mu \leftrightarrow \nu} \right\} - \left\{ {p_3 \leftrightarrow p_4 \atop \lambda \leftrightarrow \sigma} \right\} + \left\{ {p_1 \leftrightarrow p_2,\ p_3 \leftrightarrow p_4 \atop \mu \leftrightarrow \nu,\ \lambda \leftrightarrow \sigma} \right\}\ , \quad N \geq 2$$

Diagrams for $A_{qq}^{(3),\mathrm{NS}}$



Diagrams for $A_{gq}^{(3)}$

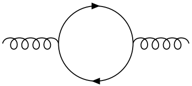


The diagrams are generated using QGRAF [Nogueira 1993 J. Comput. Phys].

	$A_{qq,Q}^{(3),\mathrm{NS}}$	$A_{gq,Q}^{(3)}$	$A_{Qq}^{(3),\mathrm{PS}}$	$A_{gg,Q}^{(3)}$	$A_{Qg}^{(3)}$
No. diagrams	6	6	16	72	256

The NS and gq contributions at general values of N

One massive fermion loop insertion is effectively rendered massless via a Mellin-Barnes representation:



$$= a_s T_F \frac{4}{\pi} (4\pi)^{-\varepsilon/2} \left(k_\mu k_\nu - k^2 g_{\mu\nu}\right) \times \int_{-i\infty}^{+i\infty} d\sigma \left(\frac{m^2}{\mu^2}\right)^{\sigma} \left(-k^2\right)^{\varepsilon/2-\sigma} \times \frac{\Gamma(\sigma-\varepsilon/2)\Gamma^2(2-\sigma+\varepsilon/2)\Gamma(-\sigma)}{\Gamma(4-2\sigma+\varepsilon)}$$

The Introduction of Feynman parameters then leads to an expression for the integrals of the form

$$I \propto C(\varepsilon, N) \int_{-i\infty}^{+i\infty} d\xi\, \eta^{\xi} \Gamma \left[\begin{matrix} g_1(\varepsilon)+\xi, g_2(\varepsilon)+\xi, g_3(\varepsilon)+\xi, g_4(\varepsilon)-\xi, g_5(\varepsilon)-\xi \\ g_6(\varepsilon)+\xi, g_7(\varepsilon)-\xi \end{matrix}\right]$$

where the g_j are linear functions in ε and $\eta = m_1/m_2$.

After closing the contour and collecting the residues a linear combination of generalized hypergeometric ${}_4F_3$–functions is obtained

$$I = \sum_j C_j\,(\varepsilon, N)\,{}_4F_3\left[\begin{matrix} a_1(\varepsilon), a_2(\varepsilon), a_3(\varepsilon), a_4(\varepsilon) \\ b_1(\varepsilon), b_2(\varepsilon), b_3(\varepsilon) \end{matrix}, \eta\right].$$

For $A_{qq}^{(3),\mathrm{NS}}$ and $A_{gq}^{(3)}$ the arguments of the hypergeometric ${}_PF_Q$-function are completely independent of the Mellin variable N
$\rightarrow$ the N and $\eta = m_1/m_2$ dependendence factorize!

The ε expansion can be done using `HypExp 2`.
The results are given in terms of the following (poly)logarithmic functions:

$$\{\ln(\eta),\ \ln(1 \pm \eta),\ \ln\left(1 \pm \sqrt{\eta}\right),\ \mathrm{Li}_2\left(\pm\sqrt{\eta}\right),\ \mathrm{Li}_2\left(\pm\eta\right),\ \mathrm{Li}_3\left(\pm\sqrt{\eta}\right)\}$$

The pre-factor $C_j\,(\varepsilon, N)$ may contain a sum stemming from the operator insertion on the vertex. This sum is evaluated in terms of harmonic sums using the summation package `Sigma` (see C. Schneider's talk).

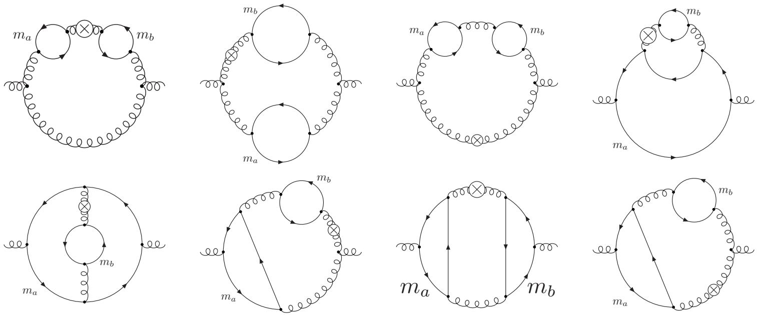
The flavor non-singlet contribution

$$
\begin{aligned}
\tilde{\tilde{a}}_{qq,Q}^{(3),\mathrm{NS}} = {} & C_F T_F^2 \Bigg\{ \left(\frac{32}{27} S_1 - \frac{8\left(3N^2+3N+2\right)}{27N(N+1)} \right) \ln^3(\eta) + \Bigg[- \frac{R_1}{18N^2(N+1)^2\eta} \\
& + \Bigg[\frac{\left(3N^2+3N+2\right)(\eta+1)\left(5\eta^2+22\eta+5\right)}{36N(N+1)\eta^{3/2}} - \frac{(\eta+1)\left(5\eta^2+22\eta+5\right)}{9\eta^{3/2}} S_1 \Bigg] \ln\left(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\right) \\
& + \frac{2\left(5\eta^2+2\eta+5\right)}{9\eta} S_1 + \ln(1-\eta) \left(\frac{16\left(3N^2+3N+2\right)}{9N(N+1)} - \frac{64}{9} S_1 \right) + \frac{32}{9} S_2 \Bigg] \ln^2(\eta) + \Bigg[\frac{40(\eta-1)(\eta+1)}{9\eta} S_1 \\
& - \frac{10\left(3N^2+3N+2\right)(\eta-1)(\eta+1)}{9N(N+1)\eta} + \frac{(\eta+1)\left(5\eta^2+22\eta+5\right)}{9\eta^{3/2}} \Bigg[8S_1 - \frac{2\left(3N^2+3N+2\right)}{N(N+1)} \Bigg] \mathrm{Li}_2\left(\sqrt{\eta}\right) \\
& + \frac{(\sqrt{\eta}+1)^2\left(-10\eta^{3/2}+5\eta^2+42\eta-10\sqrt{\eta}+5\right)}{9\eta^{3/2}} \left[\frac{\left(3N^2+3N+2\right)}{2N(N+1)} - 2S_1 \right] \mathrm{Li}_2(\eta) \Bigg] \ln(\eta) \\
& + \frac{16\left(3N^4+6N^3+47N^2+20N-12\right)\zeta_2}{27N^2(N+1)^2} + \frac{(\eta+1)\left(5\eta^2+22\eta+5\right)}{9\eta^{3/2}} \Bigg[\frac{4\left(3N^2+3N+2\right)}{N(N+1)} \\
& - 16S_1 \Bigg] \mathrm{Li}_3\left(\sqrt{\eta}\right) + \frac{(\sqrt{\eta}+1)^2\left(-10\eta^{3/2}+5\eta^2+42\eta-10\sqrt{\eta}+5\right)}{9\eta^{3/2}} \Bigg[2S_1 \\
& - \frac{\left(3N^2+3N+2\right)}{2N(N+1)} \Bigg] \mathrm{Li}_3(\eta) + \left[\frac{16\left(405\eta^2-3238\eta+405\right)}{729\eta} + \frac{256\zeta_3}{27} - \frac{640\zeta_2}{27} \right] S_1 \\
& + \left[\frac{128\zeta_2}{9} + \frac{3712}{81} \right] S_2 - \frac{1280}{81} S_3 + \frac{256}{27} S_4 - \frac{64\left(3N^2+3N+2\right)\zeta_3}{27N(N+1)} - \frac{4R_2}{729N^4(N+1)^4\eta} \Bigg\}
\end{aligned}
$$

The A_{gq} contribution

$$
\begin{aligned}
\tilde{\tilde{a}}_{gq,Q}^{(3)} = \; & C_F T_F^2 \Bigg\{ -\frac{64(N^2+N+2)S_1^3}{27(N-1)N(N+1)} - \frac{128(N^2+N+2)S_3}{27(N-1)N(N+1)} \\
& + \frac{64(8N^3+13N^2+27N+16)S_1^2}{27(N-1)N(N+1)^2} + \frac{64(8N^3+13N^2+27N+16)S_2}{27(N-1)N(N+1)^2} \\
& - \frac{4(\sqrt{\eta}+1)^2(N^2+N+2)R_8}{3\eta^{3/2}(N-1)N(N+1)} \mathrm{Li}_3(-\sqrt{\eta}) + \frac{4(\sqrt{\eta}-1)^2(N^2+N+2)R_9}{3\eta^{3/2}(N-1)N(N+1)} \mathrm{Li}_3(\sqrt{\eta}) \\
& - \frac{8R_{12}}{243\eta(N-1)N(N+1)^4} + \Bigg[-\frac{20(N^2+N+2)(\eta^2-1)}{3\eta(N-1)N(N+1)} \\
& + \frac{2(\sqrt{\eta}+1)^2(N^2+N+2)R_8}{3\eta^{3/2}(N-1)N(N+1)} \mathrm{Li}_2(-\sqrt{\eta}) - \frac{2(\sqrt{\eta}-1)^2(N^2+N+2)R_9}{3\eta^{3/2}(N-1)N(N+1)} \mathrm{Li}_2(\sqrt{\eta}) \Bigg] \ln(\eta) \\
& + \Bigg[-\frac{16(N^2+N+2)S_1}{3(N-1)N(N+1)} + \frac{(\sqrt{\eta}+1)^2(N^2+N+2)R_8}{6\eta^{3/2}(N-1)N(N+1)} \ln(1+\sqrt{\eta}) \\
& - \frac{(\sqrt{\eta}-1)^2(N^2+N+2)R_9}{6\eta^{3/2}(N-1)N(N+1)} \ln(1-\sqrt{\eta}) - \frac{R_{10}}{3\eta(N-1)N(N+1)^2} \Bigg] \ln^2(\eta) \\
& + \Bigg(-\frac{64R_{11}}{27(N-1)N(N+1)^3} - \frac{64(N^2+N+2)}{9(N-1)N(N+1)} S_2 - \frac{64(N^2+N+2)\zeta_2}{3(N-1)N(N+1)} \Bigg) S_1 \\
& + \frac{64(8N^3+13N^2+27N+16)\zeta_2}{9(N-1)N(N+1)^2} - \frac{16(N^2+N+2)}{9(N-1)N(N+1)} \ln^3(\eta) - \frac{128(N^2+N+2)\zeta_3}{9(N-1)N(N+1)} \Bigg\}
\end{aligned}
$$

Scalar $A_{gg,Q}$ diagrams with $m_1 \neq m_2$



The strategy:

- Introduce Feynman parameters and do the momentum integration for one of the closed fermion lines $\rightarrow$ effective propagator.
- Detach mass using the Mellin-Barnes representation

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\xi \frac{B^{\xi}}{A^{\lambda+\xi}} \Gamma(\lambda+\xi)\Gamma(-\xi)$$

- Perform the remaining momentum integrals and the Feynman parameter integrals (except the one where both ξ and N appear)

$$\to \quad C(N, m_1, m_2, \varepsilon)\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\int_0^1 dX\, \eta^\xi X^{\xi+N+\alpha\varepsilon+\beta}(1-X)^{-\xi+\gamma\varepsilon+\delta}$$

$$\times\Gamma\begin{bmatrix} a_1+b_1\varepsilon+c_1\xi,\dots,a_i+b_i\varepsilon+c_i\xi \\ d_1+e_1\varepsilon+f_1\xi,\dots,d_j+e_j\varepsilon+f_j\xi \end{bmatrix}$$

$a_k,\ d_k,\ \beta,\ \delta\in\mathbb{Z}$, $b_k,\ e_k,\ \alpha,\ \gamma\in\mathbb{Z}/2$, $c_k\in\{-1,1\}$ and $f_k\in\{-2,-1,1,2\}$, with $\sum_{k=1}^{i} c_k=\sum_{k=1}^{j} f_k$

- Split integration range and remap to $[0,1]$ using

$$\int_{-i\infty}^{+i\infty}d\xi\int_0^1 dX f(\xi,X)\left(\frac{\eta X}{1-X}\right)^\xi = \left(\int_0^{\frac{1}{1+\eta}}dX+\int_{\frac{1}{1+\eta}}^1 dX\right)f(\xi,X)\left(\frac{\eta X}{1-X}\right)^\xi$$

$$= \int_{-i\infty}^{+i\infty}d\xi\int_0^1 dT\left[\frac{\eta}{(\eta+T)^2}f\left(\xi,\frac{T}{\eta+T}\right)T^\xi + \frac{1}{(1+\eta T)^2}f\left(\xi,\frac{1}{1+\eta T}\right)T^{-\xi}\right]$$

- Regulate poles and expand in ε.
- Take residues and sum using Sigma (see C. Schneider's talk) and HarmonicSums (see J. Ablinger's talk). Results in terms of GHPLs, i.e., iterated integrals over the alphabet

$$\left\{\frac{1}{\tau}, \frac{1}{\tau+T}, \frac{1}{1+T\tau^2}\right\}$$

- Rewrite GHPLs so that T appears only in the argument.
- Absorb rational, N-dependent factors into the integrals using

$$\begin{aligned}
N\int_0^1 dx g(x)^N f(x) &= g(x)^{N+1}\frac{f(x)}{g'(x)}\Big|_0^1 \\
&\quad -\int_0^1 dx\,(g(x))^N \frac{d}{dx}\left[\frac{f(x)g(x)}{g'(x)}\right] \\
\frac{1}{(N+a)}\int_0^1 dx g(x)^N f(x) &= \frac{1}{(N+a)} g(x)^{N+a}\left(\int_0^x dy \frac{f(y)}{g(y)^a}\right)\Big|_{x=0}^1 \\
&\quad -\int_0^1 dx g(x)^{N+a-1}\frac{dg(x)}{dx}\left(\int_0^x dy \frac{f(y)}{g(y)^a}\right)
\end{aligned}$$

- Rewrite the remaining integral doing the change of variable

$$g(x,\eta) \to x'$$

- Final result in *z*-space in terms of generalized iterated integrals

$$G\left(\{f_1(\tau), f_2(\tau), \cdots, f_n(\tau)\}, z\right) = \int_0^z d\tau_1\; f_1(\tau_1) G\left(\{f_2(\tau), \cdots, f_n(\tau)\}, \tau_1\right)$$

with

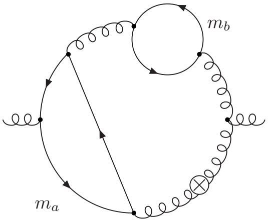
$$G\left(\Bigg\{\underbrace{\frac{1}{\tau}, \frac{1}{\tau}, \cdots, \frac{1}{\tau}}_{\text{n times}}\Bigg\}, z\right) = \frac{1}{n!}\mathrm{H}_0(z)^n \equiv \frac{1}{n!}\ln^n(z)\;.$$

and in *N*-space in terms of generalized harmonic sums

$$S_{b,\vec{a}}(c,\vec{d};N) = \sum_{k=1}^{N} \frac{c^k}{k^b} S_{\vec{a}}(\vec{d};k), c, d_i \in \mathbb{R}\backslash\{0\}; \quad b, a_i \in \mathbb{N}\backslash\{0\},$$

and other generalizations including inverse binomial sums.

Example:



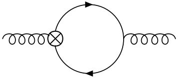
$$
\begin{aligned}
D_{8a}^{(+)}(z) \quad = \quad & \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{-3+\varepsilon}\Bigg\{-\frac{1}{\varepsilon}\frac{1}{90(1-z)}-\frac{1}{450(1-z)}+\frac{1}{180(1-z)}\mathrm{H}_1(z) \\
& +\frac{25+(63\eta-100)(1-z)}{3360\eta(1-z)^{3/2}}\sqrt{z}\Bigg[\left(\eta-1-\frac{1+\eta}{2}\ln(\eta)\right)G\left(\left\{\sqrt{1-\tau}\sqrt{\tau}\right\},z\right) \\
& +\frac{(1-\eta)^2}{8}\Bigg[-\ln(\eta)\,G\left(\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\right\},z\right)+G\left(\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right) \\
& +G\left(\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right)\Bigg]+\frac{1+\eta}{2}\Bigg[G\left(\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\right\},z\right) \\
& +G\left(\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\right\},z\right)\Bigg]\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
D_{8a}(N) \quad = \quad & \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{-3+\varepsilon}\left[\frac{1+(-1)^N}{2}\right]\Biggl\{-\frac{N+2}{45\varepsilon^2(N+1)} \\
& +\frac{1}{\varepsilon}\left[\frac{(N+2)S_1(N)}{90(N+1)}-\frac{8N^3+(4-25\eta)N^2-(25\eta+24)N+20}{1800N(N+1)^2}\right] \\
& -\frac{\left(7N\left(N^2+3N+2\right)-3\eta^3\right)S_1^2(N)}{2520N(N+1)^2}+\frac{2^{-2N-8}\binom{2N}{N}P_{45}}{105\sqrt{\eta}(N+1)^2}\left[\mathrm{H}_{-1,0,0}\left(\sqrt{\eta}\right)+\mathrm{H}_{1,0,0}\left(\sqrt{\eta}\right)\right] \\
& +\frac{2^{-2N}\binom{2N}{N}P_{45}}{53760(\eta-1)\eta(N+1)^2}\sum_{i_1=1}^{N}\frac{2^{2i_1}\left(\frac{\eta}{-1+\eta}\right)^{i_1}\left[S_2\left(\frac{-1+\eta}{\eta},i_1\right)-S_{1,1}\left(\frac{-1+\eta}{\eta},1,i_1\right)\right]}{\binom{2i_1}{i_1}} \\
& +\ln^2(\eta)\Biggl[\frac{\eta^3}{840N(N+1)^2}-\frac{(\eta-1)^{-N-1}\eta^N}{53760N(N+1)^2}P_{44}-\frac{2^{-2N-10}\binom{2N}{N}P_{45}}{105(\eta-1)(N+1)^2} \\
& -\frac{2^{-2N-10}\binom{2N}{N}P_{45}}{105(\eta-1)\eta(N+1)^2}\sum_{i_1=1}^{N}\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}}{\binom{2i_1}{i_1}}\Biggr]+\frac{P_{48}}{9072000\eta N^2(N+1)^3} \\
& -\frac{P_{47}S_1(N)}{403200\eta(N+1)^2}-\frac{\left(3\eta^3+7N\left(N^2+3N+2\right)\right)S_2(N)}{2520N(N+1)^2}+\ln(\eta)\Biggl[-\frac{S_1(N)\,\eta^3}{420N(N+1)^2} \\
& -\frac{2^{-2N-8}\binom{2N}{N}P_{45}}{105\eta(N+1)^2}+\frac{2^{-2N-9}\binom{2N}{N}P_{45}}{105(\eta-1)\eta(N+1)^2}\sum_{i_1=1}^{N}\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}S_1\left(\frac{-1+\eta}{\eta},i_1\right)}{\binom{2i_1}{i_1}} \\
& +\frac{(\eta-1)^{-N-1}\eta^N P_{44}}{26880N(N+1)^2}S_1\left(\frac{\eta-1}{\eta},N\right)+\frac{P_{46}}{80640(N+1)^2\eta}\Biggr]-\frac{2^{-2N-7}\binom{2N}{N}P_{45}}{105\eta(N+1)^2} \\
& +\frac{(\eta-1)^{-N-1}\eta^N P_{44}}{26880N(N+1)^2}\left[S_2\left(\frac{\eta-1}{\eta},N\right)-S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)\right]-\frac{(N+2)\zeta_2}{120(N+1)}\Biggr\}
\end{aligned}
$$

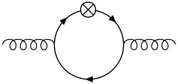
The PS contribution at general values of N

We use the same trick we used before for A_{qq}^{NS} and A_{gq} to decouple the mass coming from the fermion loop without operator insertion.

For the fermion loop with operator insertion we use



$$= 16\delta_{ab} T_F g_s^2 \frac{(\Delta.k)^{N-2}}{(4\pi)^{D/2}} \Gamma(2-D/2) \times \int_0^1 dx\ x^N (1-x) \frac{(\Delta.k)\Delta_\mu k_\nu - k^2 \Delta_\mu \Delta_\nu}{(m^2 - x(1-x)k^2)^{2-D/2}}$$



$$\begin{aligned} = {} & 4\delta_{ab} T_F g_s^2 \frac{(\Delta.k)^{N-2}}{(4\pi)^{D/2}} \int_0^1 dx\ x^{N-2}(1-x) \Big[\\ & - \left(x(1-x)(g_{\mu\nu} k^2 - 2k_\mu k_\nu) + 2m^2 g_{\mu\nu} \right) \frac{x^2 \Gamma(3-D/2)(\Delta.k)^2}{(m^2 - x(1-x)k^2)^{3-D/2}} \\ & + \Gamma(2-D/2)(2Nx + 1 - N) \frac{x(k_\mu \Delta_\nu + k_\nu \Delta_\mu)(\Delta.k)}{(m^2 - x(1-x)k^2)^{2-D/2}} \\ & + \Gamma(2-D/2)((N-1)(1-2x) - Dx) \frac{x g_{\mu\nu} (\Delta.k)^2}{(m^2 - x(1-x)k^2)^{2-D/2}} \\ & - \Gamma(1-D/2) \frac{N-1}{1-x} (N(1-x) - 1) \frac{\Delta_\mu \Delta_\nu}{(m^2 - x(1-x)k^2)^{1-D/2}} \Big] \end{aligned}$$

The diagrams end up being expressed as a linear combination of integrals of the form

$$I_1 = C_1(N, m_1, m_2, \varepsilon)\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\xi \int_0^1 dx\, \eta^{\xi} x^{\xi+N+\alpha\varepsilon+\beta}(1-x)^{\xi+\gamma\varepsilon+\delta} \times \Gamma\begin{bmatrix} a_1 + b_1\varepsilon + c_1\xi, \dots, a_i + b_i\varepsilon + c_i\xi \\ d_1 + e_1\varepsilon + f_1\xi, \dots, d_j + e_j\varepsilon + f_j\xi \end{bmatrix}$$

or

$$I_2 = C_2(N, m_1, m_2, \varepsilon)\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\xi \int_0^1 dx\, \eta^{\xi} x^{-\xi+N+\alpha'\varepsilon+\beta'}(1-x)^{-\xi+\gamma'\varepsilon+\delta'} \times \Gamma\begin{bmatrix} a'_1 + b'_1\varepsilon + c'_1\xi, \dots, a'_i + b'_i\varepsilon + c'_i\xi \\ d'_1 + e'_1\varepsilon + f'_1\xi, \dots, d'_j + e'_j\varepsilon + f'_j\xi \end{bmatrix}$$

Notice the difference with $A_{gg}^{(3)}$, where the relevant variable was

$$\frac{\eta x}{1-x}.$$

Now the relevant variables are

$$\eta x(1-x) \text{ for } I_1, \quad \text{and} \quad \frac{\eta}{x(1-x)} \text{ for } I_2$$

The calculation of I_2 is particularly cumbersome since

$$\begin{aligned}
\frac{\eta}{x(1-x)} &< 1, \quad \text{for} \quad x \in \left(\frac{1}{2}\left(1-\sqrt{1-4\eta}\right), \frac{1}{2}\left(1+\sqrt{1-4\eta}\right)\right) \\
\frac{\eta}{x(1-x)} &> 1, \quad \text{for} \quad x \in \left(0, \frac{1}{2}\left(1-\sqrt{1-4\eta}\right)\right) \\
& \qquad\quad \text{or} \quad x \in \left(\frac{1}{2}\left(1+\sqrt{1-4\eta}\right), 1\right)
\end{aligned}$$

We can, however, perform the contour integrals for both, I_1 and I_2 by taking residues and summing them with `Sigma` and `HarmonicSums`. The results are expressed in terms of GHPLs.

The problem is that now, unlike the case of $A_{gg}^{(3)}$, the parameters and arguments of the GHPLs depend on both x and η, so it's not so easy to absorb the rational factors of N coming from $C_1(N, m_1, m_2, \varepsilon)$ and $C_2(N, m_1, m_2, \varepsilon)$.

In the case of the I_1 integrals we proceed as follows:

- First we do the remaining Feynman parameter integral in x.
- We then use the MB package to analytically continue and expand the integrals around $\varepsilon = 0$.
- After the expansion, we express the remaining contour integral as a sum of residues.

For example, during the calculation of one of the diagrams the following contour integral appears

$$I = \frac{1}{N+1}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma \;\; \eta^{-\sigma}\frac{\Gamma(N-\sigma)\Gamma(-\sigma)^2\Gamma(\sigma)^2\Gamma(2-\sigma)\Gamma(2+\sigma)^2}{\Gamma(N+2-2\sigma)\Gamma(4+2\sigma)}$$

for $\eta \sim 0.1$, we can close the contour to the right and take residues.

We obtain

$$
\begin{aligned}
I \quad \to \quad & \sum_{k=2}^{\infty} \eta^k B(k+N, k+2) \frac{\Gamma(k)^2 \Gamma(2k-3)}{\Gamma(k-1)^2 \Gamma(k+1)^2} \Big\{ \ln^2(\eta) \\
& -2\ln(\eta) \big[-S_1(k+N-1) + 2S_1(2k+N+1) + 2S_1(k-2) \\
& -2S_1(k-1) + 2S_1(k) - S_1(k+1) - 2S_1(2k-4) \big] \\
& +S_1(k+N-1)^2 + 4S_1(2k+N+1)^2 - 4S_1(k-2)S_1(k+N-1) \\
& +4S_1(k-1)S_1(k+N-1) - 4S_1(k)S_1(k+N-1) \\
& +2S_1(k+1)S_1(k+N-1) + 4S_1(2k-4)S_1(k+N-1) \\
& +8S_1(k-2)S_1(2k+N+1) - 8S_1(k-1)S_1(2k+N+1) \\
& +8S_1(k)S_1(2k+N+1) - 4S_1(k+1)S_1(2k+N+1) \\
& -8S_1(2k-4)S_1(2k+N+1) - 4S_1(k+N-1)S_1(2k+N+1) \\
& -S_2(k+N-1) + 4S_2(2k+N+1) + 4S_1(k-2)^2 + 4S_1(k-1)^2 \\
& +4S_1(k)^2 + S_1(k+1)^2 + 4S_1(2k-4)^2 - 8S_1(k-2)S_1(k-1) \\
& +8S_1(k-2)S_1(k) - 8S_1(k-1)S_1(k) - 4S_1(k-2)S_1(k+1) \\
& +4S_1(k-1)S_1(k+1) - 4S_1(k)S_1(k+1) - 8S_1(k-2)S_1(2k-4) \\
& +8S_1(k-1)S_1(2k-4) - 8S_1(k)S_1(2k-4) + 4S_1(k+1)S_1(2k-4) \\
& +2S_2(k-2) - 2S_2(k-1) + 2S_2(k) - S_2(k+1) - 4S_2(2k-4) \Big\}
\end{aligned}
$$

Unfortunately, this sum cannot be done using `Sigma`.

We proceed as follows

- First we reintroduce an integral in x using

$$
\begin{aligned}
\int_0^1 dx\, x^{k+N-1}(1-x)^{k+1} &= B(k+N,k+2) \\
\int_0^1 dx\, x^{k+N-1}(1-x)^{k+1}\ln(x) &= B(k+N,k+2)\left(S_1(k+N-1)-S_1(2k+N+1)\right) \\
\int_0^1 dx\, x^{k+N-1}(1-x)^{k+1}\ln(1-x) &= B(k+N,k+2)\left(S_1(k+1)-S_1(2k+N+1)\right) \\
\int_0^1 dx\, x^{k+N-1}(1-x)^{k+1}\ln(x)^2 &= B(k+N,k+2)\big[\left(S_1(k+N-1)-S_1(2k+N+1)\right)^2 \\
&\quad -S_2(k+N-1)+S_2(2k+N+1)\big] \\
\int_0^1 dx\, x^{k+N-1}(1-x)^{k+1}\ln(1-x)^2 &= B(k+N,k+2)\big[\left(S_1(k+1)-S_1(2k+N+1)\right)^2 \\
&\quad -S_2(k+1)+S_2(2k+N+1)\big] \\
\int_0^1 dx\, x^{k+N-1}(1-x)^{k+1}\ln(x)\ln(1-x) &= B(k+N,k+2)\big[-\zeta(2)+S_2(2k+N+1) \\
&\quad \left(S_1(k+1)-S_1(2k+N+1)\right) \\
&\quad \times\left(S_1(k+N-1)-S_1(2k+N+1)\right)\big]
\end{aligned}
$$

$$
\begin{aligned}
I \quad \rightarrow \quad & \sum_{k=3}^{\infty} \eta^k \int_0^1 dx\, (1-x)^{k+1} x^{k+N-1} \frac{\Gamma(k)^2 \Gamma(2k-3)}{\Gamma(k-1)^2 \Gamma(k+1)^2} \Big\{ \ln^2(\eta) \\
& +2\ln(\eta)\big[-2S_1(k-2)+2S_1(k-1)-2S_1(k)+2S_1(2k-4) \\
& +\ln(1-x)+\ln(x)\big] - 4\big[S_1(k-2)-S_1(k-1)+S_1(k) \\
& -S_1(2k-4)\big](\ln(1-x)+\ln(x)) + 4S_1(k-2)^2 + 4S_1(k-1)^2 \\
& +4S_1(k)^2 + 4S_1(2k-4)^2 - 8S_1(k-2)S_1(k-1) + 2\zeta(2) \\
& +8S_1(k-2)S_1(k) - 8S_1(k-1)S_1(k) - 8S_1(k-2)S_1(2k-4) \\
& +8S_1(k-1)S_1(2k-4) - 8S_1(k)S_1(2k-4) + 2S_2(k-2) \\
& -2S_2(k-1) + 2S_2(k) - 4S_2(2k-4) + (\ln(1-x)+\ln(x))^2 \Big\}
\end{aligned}
$$

– Now we do the binomial expansion of the term $(1-x)^{k+1}$ and use

$$
\frac{1}{N+a} \int_0^1 dx\, x^N f(x) = \int_0^1 dx\, x^{N+a-1} \left(\int_0^1 dy\, y^{-a} f(y) - \int_0^x dy\, y^{-a} f(y) \right)
$$

in order to absorb the $\frac{1}{N+1}$ factor in I.

– The integrals from 0 to x can be done using

$$\int_0^x dy\; y^n \ln(y) = \frac{x^{n+1}\ln(x)}{n+1} - \frac{x^{n+1}}{(n+1)^2}$$

$$\int_0^x dy\; y^n \ln(1-y) = -\frac{S_1(\{x\},n)}{n+1} - \frac{x^{n+1}}{(n+1)^2} + \frac{\left(x^{n+1}-1\right)\ln(1-x)}{n+1}$$

$$\int_0^x dy\; y^n \ln(y)^2 = \frac{2x^{n+1}}{(n+1)^3} + \frac{x^{n+1}\ln^2(x)}{n+1} - \frac{2x^{n+1}\ln(x)}{(n+1)^2}$$

$$\begin{aligned}\int_0^x dy\; y^n \ln(1-y)^2 = {} & \frac{\ln^2(x)}{2(n+1)}\Bigg[4\left(\frac{S_1(\{x\},n)}{n+1} + S_{1,1}(\{1,x\},n) + \frac{x^{n+1}}{(n+1)^2} - x\right) \\ & -4\ln(1-x)\left(S_1(\{x\},n) + \frac{x^{n+1}}{n+1} - x\right) + 4\Big(S_1(n+1) - 1\Big)\ln(1-x) \\ & -2x\left(1-x^n\right)\ln^2(1-x) + 4x + (1-x)\Big(4\ln(1-x) - 2\ln^2(1-x)\Big)\Bigg]\end{aligned}$$

$$\begin{aligned}\int_0^x dy\; y^n \ln(y)\ln(1-y) = {} & \frac{S_2(\{x\},n) - \frac{x^n}{n^2}}{n+1} + \frac{S_1(\{x\},n) - \frac{x^n}{n}}{(n+1)^2} - \frac{\ln(x)\left(S_1(\{x\},n) - \frac{x^n}{n}\right)}{n+1} \\ & + \frac{(2n+1)x^n}{n^2(n+1)^2} - \frac{\mathrm{Li}_2(x)}{n+1} + \frac{2x^{n+1}}{(n+1)^3} - \frac{x^{n+1}\ln(x)}{(n+1)^2} \\ & - \frac{\left(x^{n+1}-1\right)\ln(1-x)}{(n+1)^2} + \frac{\left(x^{n+1}-1\right)\ln(1-x)\ln(x)}{n+1} - \frac{x^n\ln(x)}{n(n+1)}\end{aligned}$$

$$
\begin{aligned}
I \;=\; & \sum_{k=2}^{\infty}\sum_{i=0}^{k+1}(-1)^i\eta^k\binom{k+1}{i}\frac{\Gamma(k)^2\Gamma(2k-3)}{\Gamma(k-1)^2\Gamma(k+1)^2}\Bigg\{2\ln(\eta)\left(-\frac{S_1(\{1\},i+k-2)}{i+k-1}-\frac{1}{(i+k-1)^2}\right)\\
& -4S_1(k-2)\left(-\frac{S_1(\{1\},i+k-2)}{i+k-1}-\frac{1}{(i+k-1)^2}\right)+4S_1(k-1)\left(-\frac{S_1(\{1\},i+k-2)}{i+k-1}\right.\\
& \left.-\frac{1}{(i+k-1)^2}\right)-4S_1(k)\left(-\frac{S_1(\{1\},i+k-2)}{i+k-1}-\frac{1}{(i+k-1)^2}\right)\\
& +4S_1(2k-4)\left(-\frac{S_1(\{1\},i+k-2)}{i+k-1}-\frac{1}{(i+k-1)^2}\right)+2\left(\frac{S_1(\{1\},i+k-2)-\frac{1}{i+k-2}}{(i+k-1)^2}\right.\\
& \left.+\frac{S_2(\{1\},i+k-2)-\frac{1}{(i+k-2)^2}}{i+k-1}+\frac{2(i+k-2)+1}{(i+k-2)^2(i+k-1)^2}+\frac{2}{(i+k-1)^3}-\frac{\pi^2}{6(i+k-1)}\right)\\
& +\frac{2\left(\frac{S_1(\{1\},i+k-2)}{i+k-1}+S_{1,1}(\{1,1\},i+k-2)+\frac{1}{(i+k-1)^2}\right)}{(i+k-1)}+\frac{\ln^2(\eta)}{i+k-1}\\
& -\ln(\eta)\left[\frac{2}{(i+k-1)^2}+\frac{4}{i+k-1}\left(-S_1(k-2)+S_1(k-1)-S_1(k)+S_1(2k-4)\right)\right]\\
& +\frac{4}{i+k-1}\left[S_1(k-2)^2+S_1(k-1)^2+S_1(k)^2+S_1(2k-4)^2-2S_1(k-2)S_1(k-1)\right.\\
& +2S_1(k-2)S_1(k)-2S_1(k-1)S_1(k)+S_1(k)-2S_1(k-2)S_1(2k-4)\\
& \left.+2S_1(k-1)S_1(2k-4)-2S_1(k)S_1(2k-4)-2S_1(k)S_1(2k-4)-S_2(2k-4)+2\zeta(2)\right]\\
& +\frac{2S_2(k-2)}{i+k-1}-\frac{2S_2(k-1)}{i+k-1}+\frac{2S_2(k)}{i+k-1}+\frac{4}{(i+k-1)^2}\left[S_1(k-2)-S_1(k-1)\right.\\
& \left.-S_1(2k-4)\right]+\frac{2}{(i+k-1)^3}+\cdots\Bigg\}
\end{aligned}
$$

We can do this double sum using Sigma and HarmonicSums. We get

$$
\begin{aligned}
I \quad = \quad & \int_0^1 dx\, x^N \Bigg\{ \frac{1}{288}\ln^4(1-x) + \left(\frac{12x^2-x+6}{216x} + \frac{\ln(x)}{72} + \frac{\ln(\eta)}{72}\right)\ln^3(1-x) + \Bigg[\frac{\ln^2(x)}{48} \\
& + \frac{\ln^2(\eta)}{48} + \frac{1}{144}\sqrt{\eta}(5\eta-27)\ln\left(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\right) - \frac{11}{72}\ln(1-\eta) + \frac{\mathrm{Li}_2(\eta)}{24}\Bigg]\ln^2(1-x) \\
& + \Bigg[\frac{\ln^3(\eta)}{72} - \frac{11}{18}\ln(1-\eta)\left(2x-1+\frac{\ln(\eta)}{2}\right) - \frac{1}{18}\sqrt{\eta}(5\eta-27)\mathrm{Li}_2\left(\sqrt{\eta}\right) \\
& + \frac{1}{72}\left(5\eta^{3/2} - 27\sqrt{\eta} + 24x - 34\right)\mathrm{Li}_2(\eta) - \frac{\mathrm{Li}_3(\eta)}{6}\Bigg]\ln(1-x) + \frac{\ln^4(x)}{288} + \frac{\ln^4(\eta)}{288} \\
& - \frac{1}{24} G\left(\left\{\frac{1}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}\right\}, \eta x(1-x)\right)^2 - \frac{\pi^2}{162x}\left(15x^2-8x+3\right)\left(1-\eta x(1-x)\right)^{3/2} \\
& + \frac{11}{36} G\left(\left\{\frac{\sqrt{1-4\tau}\sqrt{1-4\eta\tau}}{\tau}\right\}, x(1-x)\right)\left[G\left(\left\{\frac{1}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}\right\}, \eta x(1-x)\right) + \zeta_2\right] \\
& + \ln^3(x)\left(\frac{12x^2-x+6}{216x} + \frac{\ln(\eta)}{72}\right) + \frac{20}{9}\eta G\left(\left\{\frac{1}{\tau}, \frac{\sqrt{1-4\eta\tau}}{\tau}, \sqrt{1-4\tau}\sqrt{1-4\eta\tau}\right\}, x(1-x)\right) \\
& - \frac{12x^2-7x+6}{36x} G\left(\left\{\frac{\sqrt{4\tau-1}}{\tau}, \frac{1}{\tau}, \frac{\sqrt{4\tau-1}}{\tau}\right\}, \eta x(1-x)\right) - \frac{27}{36}\sqrt{\eta}\ln(x)\ln\left(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\right) \\
& + \frac{1}{6}\ln\left(1-\sqrt{\eta}\right)\mathrm{Li}_3(\eta) + \frac{\mathrm{Li}_4(\eta)}{2} + \frac{1}{12} G\left(\left\{\frac{1}{\tau}, \frac{\sqrt{1-4\eta\tau}}{\tau}, \frac{\sqrt{1-4\eta\tau}}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}\right\}, x(1-x)\right) \\
& - \frac{1}{12} G\left(\left\{\frac{\sqrt{1-4\tau}}{\tau}, \frac{1}{\tau}, \frac{1}{\tau}, \frac{1}{\tau}\right\}, x(1-x)\right) + \frac{1}{36}\left(-5\eta^{3/2} + 27\sqrt{\eta} - 24x + 22\right)\mathrm{Li}_3(\eta) \\
& + \cdots \Bigg\}
\end{aligned}
$$

Summary

- In the calculation of 3-loop heavy flavor corrections to DIS Wilson coefficients we need to consider the contribution from diagrams with two different masses since $m_c^2/m_b^2 \sim 0.1$.
- We have computed the 3-loop 2-mass contributions to $A_{qq}^{(3),\mathrm{NS}}$ and $A_{gq}^{(3)}$ for general values of the Mellin variable N in analytic form.
- We have calculated scalar diagrams for $A_{gg}^{(3)}$, witnessing the appearance of generalized harmonic sums and generalized harmonic polylogarithms.
- Half of the diagrams for $A_{Qq}^{(3),\mathrm{PS}}$ are by now computed. The results are expressed in terms of 18 GHPLs with the alphabet

$$\left\{\frac{1}{\tau},\ \frac{\sqrt{1-4\tau}}{\tau},\ \frac{\sqrt{1-4\eta\tau}}{\tau},\ \sqrt{1-4\tau}\sqrt{1-4\eta\tau},\ \frac{\sqrt{1-4\tau}\sqrt{1-4\eta\tau}}{\tau},\ \sqrt{\tau}\sqrt{1-4\tau}\right\}$$

- Different new Computer-algebra and mathematical technologies have been and continue to be developed.