Numerical integration of massive two-loop Mellin-Barnes integrals in Minkowskian regions

Janusz Gluza (University of Silesia)

in collaboration with DESY Zeuthen group:

Evgen Dubovyk,
Tord Riemann,
Johann Usovitsch

Loops and Legs, Leipzig, 26 April 2016

Outline

1. Introduction
2. Mellin-Barnes representations
3. Efficient Mellin-Barnes numerical integrations in Minkowskian regions, basic ideas
4. Tests and difficult cases
5. Conclusions and Outlook

Mellin-Barnes representations in HEP - method

- "Om definita integraler",
  R. H. Mellin, Acta Soc. Sci. Fenn. 20(7), 1 (1895).

$$\mathcal{M}[f](s) = \int_0^\infty dx x^{s-1} f(x)$$

Mellin-Barnes representations in HEP - method

- "Om definita integraler",
  R. H. Mellin, Acta Soc. Sci. Fenn. 20(7), 1 (1895).

$$\mathcal{M}[f](s) = \int_0^\infty dx x^{s-1} f(x)$$

- "The theory of the gamma function",
  E. W. Barnes Messenger Math. 29(2), 64 (1900).

$$\textit{mathematics} \longrightarrow \frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda+z)\Gamma(-z) \frac{B^z}{A^{\lambda+z}}$$

$$\textit{physics} \longrightarrow \frac{1}{(p^2-m^2)^a} = \frac{1}{\Gamma(a)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(a+z)\Gamma(-z) \frac{(m^2)^z}{(p^2)^{a+z}}$$

Mellin-Barnes representations in HEP - method

- "Om definita integraler",
  R. H. Mellin, Acta Soc. Sci. Fenn. 20(7), 1 (1895).

$$\mathcal{M}[f](s) = \int_0^\infty dx x^{s-1} f(x)$$

- "The theory of the gamma function",
  E. W. Barnes Messenger Math. 29(2), 64 (1900).

$$mathematics \longrightarrow \frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda+z)\Gamma(-z) \frac{B^z}{A^{\lambda+z}}$$

$$physics \longrightarrow \frac{1}{(p^2-m^2)^a} = \frac{1}{\Gamma(a)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(a+z)\Gamma(-z) \frac{(m^2)^z}{(p^2)^{a+z}}$$

- "Perturbation Theory of Scattering Amplitudes at High Energies",
  James D. Bjorken and Tai Tsun Wu Phys. Rev. 130, 1963

Mellin-Barnes representations in HEP - method

- "Om definita integraler",
  R. H. Mellin, Acta Soc. Sci. Fenn. 20(7), 1 (1895).

$$\mathcal{M}[f](s) = \int_0^\infty dx x^{s-1} f(x)$$

- "The theory of the gamma function",
  E. W. Barnes Messenger Math. 29(2), 64 (1900).

$$\textit{mathematics} \longrightarrow \frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda+z)\Gamma(-z) \frac{B^z}{A^{\lambda+z}}$$

$$\textit{physics} \longrightarrow \frac{1}{(p^2-m^2)^a} = \frac{1}{\Gamma(a)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(a+z)\Gamma(-z) \frac{(m^2)^z}{(p^2)^{a+z}}$$

- "Perturbation Theory of Scattering Amplitudes at High Energies",
  James D. Bjorken and Tai Tsun Wu Phys. Rev. 130, 1963

- "Analytical result for dimensionally regularized massless on-shell double box",
  Vladimir Smirnov, PLB, 1999
- "Non-planar massless two-loop Feynman diagrams with four on- shell legs",
  Bas Tausk, PLB, 1999

Mellin-Barnes representations in HEP - developments

https://mbtools.hepforge.org/

- "Automatized analytic continuation of Mellin-Barnes integrals", Michał Czakon, CPC, 2006 $\longrightarrow$ MB.m, MBasymptotics.m
- "On the Resolution of Singularities of Multiple Mellin-Barnes Integrals", A.V. Smirnov, V.A. Smirnov $\longrightarrow$ MBresolve.m
- "AMBRE - a Mathematica package for the construction of Mellin-Barnes representations for Feynman integrals", JG, Krzysztof Kajda, Tord Riemann, CPC, 2007 $\longrightarrow$ AMBRE.m
- "Some Remarks on Non-planar Feynman Diagrams", K. Bielas, I. Dubovyk, JG, T. Riemann, APPB, 2013 $\longrightarrow$ PlanarityTest.m
- $\longrightarrow$ barnesroutines.m : a tool by David Kosower

Mellin-Barnes representations in HEP - developments

https : //mbtools.hepforge.org/

- "Automatized analytic continuation of Mellin-Barnes integrals",
  Michał Czakon, CPC, 2006 $\longrightarrow$ MB.m, MBasymptotics.m
- "On the Resolution of Singularities of Multiple Mellin-Barnes Integrals",
  A.V. Smirnov, V.A. Smirnov $\longrightarrow$ MBresolve.m
- "AMBRE - a Mathematica package for the construction of Mellin-Barnes representations for Feynman integrals",
  JG, Krzysztof Kajda, Tord Riemann, CPC, 2007 $\longrightarrow$ AMBRE.m
- "Some Remarks on Non-planar Feynman Diagrams",
  K. Bielas, I. Dubovyk, JG, T. Riemann, APPB, 2013 $\longrightarrow$ PlanarityTest.m
- $\longrightarrow$ barnesroutines.m : a tool by David Kosower
- "On the Numerical Evaluation of Loop Integrals With Mellin-Barnes Representations",
  Ayres Freitas, Yi-Cheng Huang, JHEP, 2010
- "Angular integrals in d dimensions",
  Gabor Somogyi, J.Math.Phys, 2011
- "Soft triple-real radiation for Higgs production at N3LO",
  C. Anastasiou, C. Duhr, F. Dulat, B. Mistlberger, JHEP, 2013

Direct numerical integrations in Minkowskian regions

- FIESTA, ver. 3.0 [A. V. Smirnov]
- SecDec, ver. 3.0 [S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, T. Zirke]
- NICODEMOS, ver 2.0 [A. Freitas]
- "Direct numerical integration for multi-loop integrals", Sebastian Becker, Stefan Weinzierl, EPJC (2013)

Direct numerical integrations in Minkowskian regions

- FIESTA, ver. 3.0 [A. V. Smirnov]
- SecDec, ver. 3.0 [S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, T. Zirke]
- NICODEMOS, ver 2.0 [A. Freitas]
- "Direct numerical integration for multi-loop integrals", Sebastian Becker, Stefan Weinzierl, EPJC (2013)
- This talk: the MB method

Two steps:

1. AMBRE [K.Kajda (planar, ver.2.2), E.Dubovyk (non-planar, ver3.0)]-PlanarityTest[K.Bielas, E.Dubovyk]
2. MBnumerics [J. Usovitsch, E. Dubovyk] - a completely new software !

Side remarks

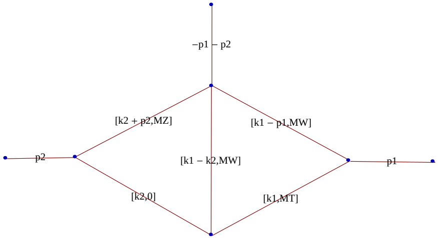
- MB.m[M. Czakon] (cross-checks for analytical calculation of MIs in Euclidean region)
  Used in many projects.
  First application: Bhabha massive QED 2-loop:
  - M. Czakon, JG, T. Riemann,
    "The planar four-point master integrals for massive two-loop Bhabha scattering",
    Nucl. Phys. B751 (2006)
    (MB & expansions)
  - Stefano Actis, Michal Czakon, JG, Tord Riemann
    "Virtual hadronic and leptonic contributions to Bhabha scattering",
    Phys.Rev.Lett. 100 (2008) 131602.
    (MB & dispersion relations)

Side remarks

- MB.m [M. Czakon] (cross-checks for analytical calculation of MIs in Euclidean region)
  Used in many projects.
  First application: Bhabha massive QED 2-loop:
  - M. Czakon, JG, T. Riemann,
    "The planar four-point master integrals for massive two-loop Bhabha scattering",
    Nucl. Phys. B751 (2006)
    (MB & expansions)
  - Stefano Actis, Michal Czakon, JG, Tord Riemann
    "Virtual hadronic and leptonic contributions to Bhabha scattering",
    Phys.Rev.Lett. 100 (2008) 131602.
    (MB & dispersion relations)
- MBnumerics [J. Usovitsch, E. Dubovyk]
  - **First application:** $Z \to b\bar{b}$ project.
    More on this project on Friday, talk by Tord Riemann.
  - **Aim:** complete two-loop electroweak corrections.
  - **Problem:** $M_Z, M_W, m_t, M_H, s(= M_Z^2)$
    $\longrightarrow$ 4 dimensionless scales, thresholds, singularities.

Step 1(a): PlanarityTest

The program determines planarity of diagrams from momenta flaw in propagators.



PlanarityTestv1.2.m,
http://us.edu.pl/∼gluza/ambre/planarity/

Step 1(b): MB representations - AMBRE

Present versions:

- v1.3 - manual approach, useful for testing
- v2.1 - complete, automatic approach

  Basic improvements of these new versions (planars):

  - $$F = F_0 + U \sum_i x_i m_i^2$$
  - if masses from $U \sum x_i m_i^2$ are not needed to form propagators $\Rightarrow U = 1$
  - if some masses from $U \sum x_i m_i^2$ were used, further optimization is also possible

  $$\ldots - k^2 x_1 x_2 + \overbrace{(x_1 + x_2 + x_3 + \ldots)}^{U=1} x_1 m^2 \rightarrow \ldots - (k^2 - m^2) x_1 x_2 + (x_1 + x_3 + \ldots) x_1 m^2$$

  red part is simplified by the 1st Barnes lemma by shifts of variables:

  $$s_1^{\alpha_1 \pm \sum_{i=1} \beta_i z_i} s_2^{\alpha_2 \pm \sum_{i=1} \beta_i z_i} |_{\mathbf{z_1} \rightarrow \mathbf{z_1} - \sum_{\mathbf{i=2}} \beta_\mathbf{i} / \beta_\mathbf{1} \mathbf{z_i}} = s_1^{\alpha_1 \pm \beta_1 \mathbf{z_1}} s_2^{\alpha_2 \pm \beta_1 \mathbf{z_1}}$$ 1st BL for z_i $(i \neq 1)$

- v3.1 - non-planar
  - extended to use with some 3-loop non-planar diagrams

STEP 2 - Numerical Integrations

Mellin-Barnes integrals

BASIC PROBLEMS

I. Bad oscillatory behavior of integrands;
II. Fragile stability for integrations over products and ratios of Γ functions.

QED massive vertex

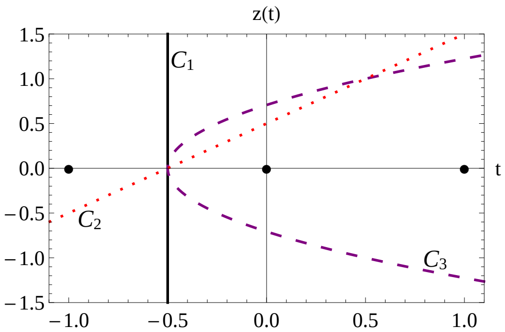
E.g. hep-ph/0511200 [M. Czakon], arXiv:1604.00406 [A. Freitas]

$$V(s) = \frac{e^{\epsilon\gamma_E}}{i\pi^{d/2}} \int \frac{d^d k}{[(k+p_1)^2 - m^2][k^2][(k-p_2)^2 - m^2]}$$
$$= \frac{V_{-1}(s)}{\epsilon} + V_0(s) + \cdots ,$$

$$V_{-1}(s) = -\frac{1}{2s} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{dz}{2\pi i} \underbrace{(-s)^{-z}}_{\textbf{Problem I}} \overbrace{\frac{\Gamma^3(-z)\Gamma(1+z)}{\Gamma(-2z)}}^{\textbf{Problem II}}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{2\arcsin(\sqrt{s}/2)}{\sqrt{4-s}\sqrt{s}},$$

$$z = \Re[z] + i\, y, \quad y \in (-\infty, +\infty)$$



$$z(t) = x_0 + it \;:\quad V_{-1}^{C_1}(s) = \int_{-\infty}^{+\infty} (i)\; dt\; J[z(t)];$$

$$z(t) = x_0 + \theta t + it \;:\quad V_{-1}^{C_2}(s) = \int_{-\infty}^{+\infty} (\theta + i)\; dt\; J[z(t)]$$

$$z(t) = x_0 + at^2 + it \;:\quad V_{-1}^{C_3}(s) = \int_{-\infty}^{+\infty} (2at + i)\; dt\; J[z(t)];.$$

$s = 2, z(t) = \Re[z(t)] + i\, y, \quad y \in (-a, +a)$

$$V_{-1}(2)|_{\text{analyt.}} = 0.785398163397448309 62 = \frac{\pi}{4}$$

$$V_{-1}(2)|^{MB.m}_{\text{Pantis}} = 0.7925 - 0.0225\, i$$

$$V_{-1}(2)|_{C_1,\ a\,=\,15} = 0.7548660085063523 - 0.229985258820015\, i$$

$$V_{-1}(2)|_{C_1,\ a\,=\,10^2} = 0.73479313088852537844 + 0.07490142360293767 6597\, i$$

$$V_{-1}(2)|_{C_1,\ a\,=\,10^3} = 0.84718185073531076915 - 0.09486576064935497 7853\, i$$

$$V_{-1}(2)|_{C_1,\ a\,=\,10^4} = 4.4574554985139977188 + 4.5139812364645122275\, i$$

✓ $$V_{-1}(2)|_{C_2} = 0.785398163^{\checkmark} 3859819 - 5.420140575251864 \cdot 10^{-15\,\checkmark}\, i$$

✓ $$V_{-1}(2)|_{C_3} = 0.785398163^{\checkmark} 2958756 + 2.435551760271437 \cdot 10^{-15\,\checkmark}\, i$$

Similarly above the threshold, $s > 4$. We can get easily more (e.g. 30 digits, C_4,...)

Types of numerical improvements

BASIC METHODS

I. Specific integration methods for oscillating integrands
II. Contour deformations
III. Contour shifts

Types of numerical improvements

BASIC METHODS

I. Specific integration methods for oscillating integrands
II. Contour deformations
III. Contour shifts

- I. and II. limited, though can be quite effective for 1-dim cases (reduction of n-dim MB "difficult" integrals into 1-dim cases?)
- III. **is new, effective for n-dim MB integrations.**

Types of numerical improvements

BASIC METHODS

I. Specific integration methods for oscillating integrands
II. Contour deformations
III. Contour shifts

- I. and II. limited, though can be quite effective for 1-dim cases (reduction of n-dim MB "difficult" integrals into 1-dim cases?)
- III. **is new, effective for n-dim MB integrations.**

Auxiliary tricks.

(a) Contour rotations;
(b) Mappings;
(c) Variables transformations, e.g. $\{z_1, z_2\} \to \{z_1 - z_2, z_2\}$;
(d) Stationary phase method; steepest decent;

Types of numerical improvements

BASIC METHODS

I. Specific integration methods for oscillating integrands
II. Contour deformations
III. Contour shifts

- I. and II. limited, though can be quite effective for 1-dim cases (reduction of n-dim MB "difficult" integrals into 1-dim cases?)
- III. **is new, effective for n-dim MB integrations.**

Auxiliary tricks.

(a) Contour rotations ($\rightarrow$ A. Freitas, here);
(b) Mappings ($\rightarrow$ MB.m, here);
(c) Variables transformations, e.g. $\{z_1, z_2\} \rightarrow \{z_1 - z_2, z_2\}$ ($\rightarrow$ here);
(d) Stationary phase method; steepest decent ($\rightarrow$ Phd by Z. Peng [D. Kosower])

Some chosen issues: Linear deformations (contour rotations)

arXiv:1604.00406 [A. Freitas]:

$$(-p^2)^{(x_0+it)} = (p^2)^{(x_0+it)}(-1-i\epsilon)^{(x_0+it)} = (p^2)^{(x_0+it)} e^{-i\pi x_0} e^{\pi t} \longleftarrow [t \to \infty]$$

$$x_0 + it \to x_0 + (\theta + i)t$$

$$(-p^2)^{(x_0+it)} = (p^2)^{(x_0+it)} e^{-i\pi(x_0+\theta t)} e^{(\pi+\theta \log p^2)t}$$

Drawbacks:

1. θ adjusted for each integral separately;
2. choice depends on kinematics and masses;
3. can be shown that can not work in general in multidimensional case (in some kinematical regions);

Used e.g. in two-loops vertices with fermion subloops by A. Freitas

2-dim rotations

$$J(z_1,z_2)=\frac{(\frac{M_W^2}{M_T^2})^{z_2}s^2(-\frac{s}{M_T^2})^{z_1-z_2}\Gamma[-z_1]\Gamma[z_1]\Gamma[2-z_2]\Gamma[4+z_1-z_2]\Gamma[z_2]\Gamma[-z_2])}{4M_T^4\Gamma[6+z_1-2z_2]}$$

The integral will be integrated over C_1:

$$K^{C_1}(s,M_W,M_T)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(i)^2 J(z_{10}+it_1,z_{20}+it_2)\mathrm{d}t_1\mathrm{d}t_2,\quad z_{10}=0.7,\ z_{20}=-1.2,$$

And C_2:

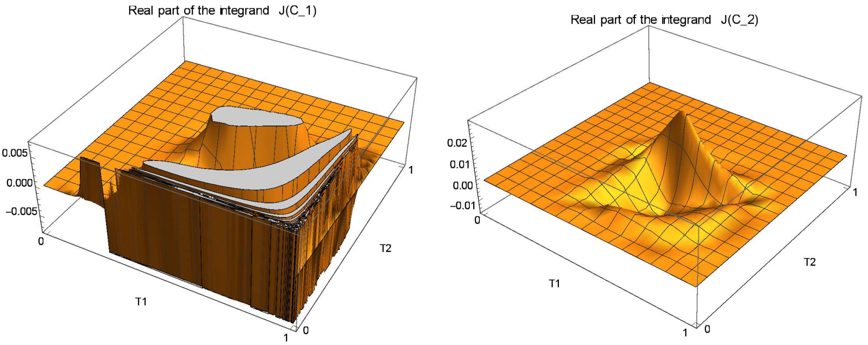
$$K^{C_2}(s,M_W,M_T)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(i+\theta)^2 J(z_{10}+(i+\theta)t_1,z_{20}+(i+\theta)t_2)\mathrm{d}t_1\mathrm{d}t_2,\quad z_{10}=0.7,\ z_{20}=-1.2,$$

Worst asymptotic behavior of the integrand J^{C_1} is given for $t_1\to 0,\ t_2\to-\infty$:

$$J(z_{10}+t_1,z_{20}+t_2)\simeq t_2^{-\frac{3}{2}}$$

- Thus the integral K^{C_1} is very slowly convergent.

2-dim rotations and mappings



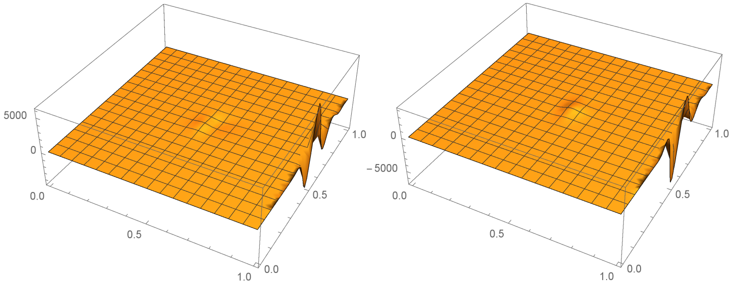
- The asymptotic behavior of the integrand J^{C_2} is like in the euclidean case.
- Both integrals K^{C_1} and K^{C_2} can be evaluated numerically, where $\theta = 0.7$.
- The numerical evaluation of K^{C_2} yields honest high accuracy.
- T_1 and T_2 are integration variables due to a tangents mapping: $t_i = 1/\tan[-\pi T_i]$

Some chosen issues: mappings

$$I = \int_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty} dz_1 \int_{-\frac{2}{3}-i\infty}^{-\frac{2}{3}+i\infty} dz_2 \left(\frac{-s}{M_Z^2}\right)^{-z_1} \frac{\Gamma[-z_1]^3\Gamma[1+z_1]\Gamma[z_1-z_2]\Gamma[-z_2]^3\Gamma[1+z_2]\Gamma[1-z_1+z_2]}{s\ \Gamma[1-z_1]^2\Gamma[-z_1-z_2]\Gamma[1+z_1-z_2]}$$

Logarithmic (in MB.m):

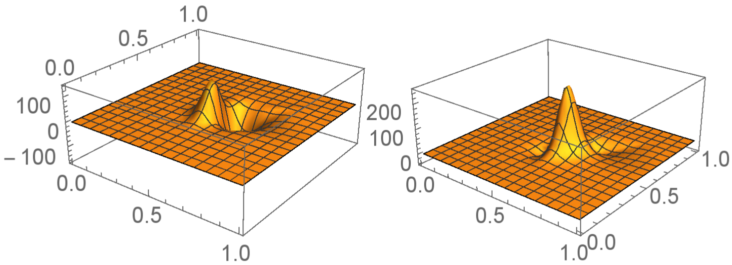
$$z_i = x_i + i\ln\left(\frac{t_i}{1-t_i}\right), \quad t_i \in (0,1), \quad \text{the Jacobians}: \quad J_i(t_i) = \frac{1}{t_i(1-t_i)}.$$



Some chosen issues: mappings

Tangent (in MBnumerics.m):

$$z_i = x_i + i\,\frac{1}{\tan(-\pi t_i)}, \quad t_i \in (0,1), \quad \text{the Jacobians}: \quad J_i = \frac{\pi}{\sin^2[(\pi t_i)]}.$$



In addition, $\Gamma \to e^{\ln \Gamma}$ improves numerical stability considerable, either.

Some chosen issues: Shifts

$$J(z_1,z_2) = \frac{2(-\frac{s}{M_Z^2})^{-z_1-z_2}\Gamma[-1-z_1-2z_2]\Gamma[-z_1-z_2]\Gamma[-z_2]\Gamma[1+z_2]^3\Gamma[1+z_1+z_2]}{s^2\Gamma[1-z_1]}$$

The integral is now a discrete function of the shift n:

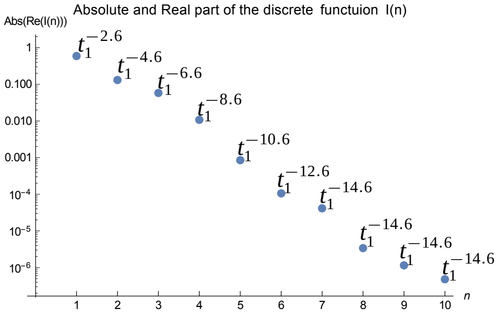
$$I^{C_1}(s,M_Z,n) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(i)^2 J(z_{10}+it_1, z_{20}+n+it_2)\mathrm{d}t_1\mathrm{d}t_2, \quad z_{10}=0,\ z_{20}=-0.7$$

Worst asymptotic behavior for the integrand is for $t_1 \to -\infty,\ t_2 \to 0$:

$$J(z_{10}+it_1, z_{20}+n+it_2) \simeq t_1^{-2-2(z_{20}+n)}$$

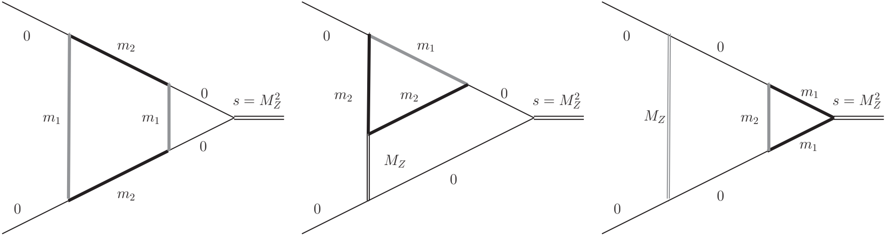
- For $n=0$ and $z_{20}=-0.7$, the integrand $J(z_1,z_2)$ drops of like $t_1^{-0.6}$. This slow convergence is similar to the QED massive vertex, discussed above.
- At the kinematic point $s=1+i\epsilon$, $M_Z=1$, ϵ is arbitrary small no contour deformation C_2 and C_3 will work.

Shifts

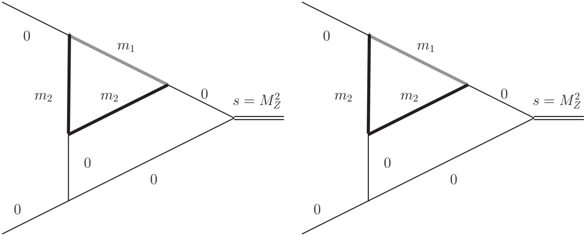


- The shifts improve the asymptotic behavior and reduce the order of magnitude of the integral $I^{C_1}(s, M_Z, n)$.
- The absolute and imaginary part of $I^{C_1}(s, M_Z, n)$ behaves similarly.
- Automatic algorithms for finding the suitable shifts and contour deformations are implemented in MBnumerics.m.

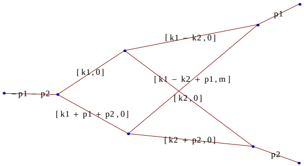
The most difficult cases (for sector decomposition)



Thresholds, $\{m_1, m_2\} = \{m_t, M_W\}$, $3 \div 5$-dim MB



MB vs sector decomposition, see Tord's talk on Friday for more details



Euclidean results (constant part):

Analytical :	-0.4966198306057021
MB(Vegas) :	-0.4969417442183914
MB(Cuhre) :	-0.4966198313219404
FIESTA :	-0.4966184488196595
SecDec :	-0.4966192150541896

Results (constant part):

Analytical :	$-0.778599608979684 - 4.123512593396311 \cdot i$
MBnumerics :	$-0.778599608324769 - 4.123512600516016 \cdot i$
MB(Vegas) :	big error
MB(Cuhre) :	NaN
FIESTA :	big error
SecDec :	big error

Precision and time of evaluation

1. AMBRE: MB tensors of rank ≥ 5, long expressions (many MB integrals), time consuming;
2. MBnumerics:
   (i) time for the optimal contour (grid for thresholds, tails, etc);
   (ii) integration;
3. Cuhre [Thomas Hahn]- not a Monte Carlo $\rightarrow$ decent precision can be obtained;
4. Present situation: max 5-dim MB integrals treatable; 8 digits achieved;

Conclusions

1. MB approach to Feynman integrals reached Minkowskian region;
2. Difficult cases like thresholds are also treatable;
3. Still there is a lot of space for improvements, tests;
4. For the $Z \to bb$, MBnumerics.m turned out to be a very strong and effective tool (see Tord's Friday talk);
5. Many places for potential applications;

Thank you for taking time!

Backup slides

Side remark: my last LL talk

- "Non-planar Feynman integrals, Mellin-Barnes representations, multiple sums",
  J. Blümlein, I. Dubovyk, JG, M. Ochman, C. G. Raab, T. Riemann, C. Schneider
- Package MBsums,
  M. Ochman, T.Riemann, arXiv:1511.01323
- "On convergent series representations of Mellin-Barnes integrals",
  Samuel FRIOT, David GREYNAT, J.Math.Phys, 2012

Complicated sums, no much progress, we changed strategy into direction of numerical solutions

References I

A. V. Kotikov, Differential equations method: New technique for massive Feynman diagrams calculation, Phys. Lett. B254 (1991) 158–164, note.

E. Remiddi, Differential equations for Feynman graph amplitudes, Nuovo Cim. A110 (1997) 1435–1452.
arXiv:hep-th/9711188.

K. Hepp, Proof of the Bogolyubov-Parasiuk theorem on renormalization, Commun. Math. Phys. 2 (1966) 301–326, sector decomposition.

M. Roth, A. Denner, High-energy approximation of one-loop Feynman integrals, Nucl. Phys. B479 (1996) 495–514, sector decomposition.
arXiv:hep-ph/9605420.

T. Binoth, G. Heinrich, Numerical evaluation of multi-loop integrals by sector decomposition, Nucl. Phys. B680 (2004) 375–388.
arXiv:hep-ph/0305234.

A. Denner, S. Pozzorini, An algorithm for the high-energy expansion of multi-loop diagrams to next-to-leading logarithmic accuracyNumerators, sector decomposition etc.
arXiv:hep-ph/0408068.

G. Heinrich, V. Smirnov, Analytical evaluation of dimensionally regularized massive on-shell double boxes, Phys. Lett. B598 (2004) 55–66.
arXiv:hep-ph/0406053.

G. Heinrich, Sector Decomposition, Int.J.Mod.Phys. A23 (2008) 1457–1486.
arXiv:0803.4177, doi:10.1142/S0217751X08040263.

A. Smirnov, M. Tentyukov, Feynman Integral Evaluation by a Sector decomposiTion Approach (FIESTA), Comput.Phys.Commun. 180 (2009) 735–746.
arXiv:0807.4129, doi:10.1016/j.cpc.2008.11.006.

References II

A. Smirnov, V. Smirnov, Hepp and Speer Sectors within Modern Strategies of Sector Decomposition, JHEP 0905 (2009) 004.
arXiv:0812.4700, doi:10.1088/1126-6708/2009/05/004.

N. Usyukina, ON A REPRESENTATION FOR THREE POINT FUNCTION, Teor. Mat. Fiz. 22 (1975) 300–306 (in Russian).

E. Boos, A. Davydychev, A Method of evaluating massive Feynman integrals, Theor. Math. Phys. 89 (1991) 1052–1063.

V. Smirnov, Analytical result for dimensionally regularized massless on-shell double box, Phys. Lett. B460 (1999) 397–404.
arXiv:hep-ph/9905323.

B. Tausk, Non-planar massless two-loop Feynman diagrams with four on- shell legs, Phys. Lett. B469 (1999) 225–234.
arXiv:hep-ph/9909506.

M. Czakon, J. Gluza, T. Riemann, The planar four-point master integrals for massive two-loop Bhabha scattering, Nucl. Phys. B751 (2006) 1–17.
arXiv:hep-ph/0604101.

M. Czakon, Automatized analytic continuation of Mellin-Barnes integrals, Comput. Phys. Commun. 175 (2006) 559–571.
arXiv:hep-ph/0511200, doi:10.1016/j.cpc.2006.07.002.

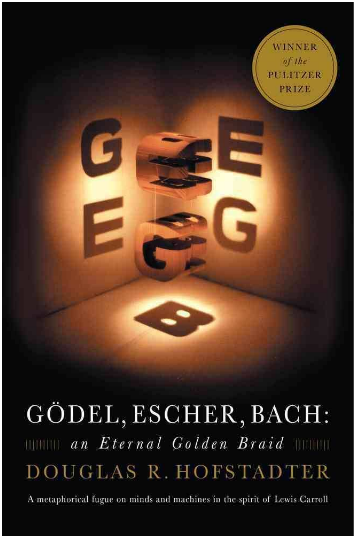
J. Gluza, K. Kajda, T. Riemann, AMBRE - a Mathematica package for the construction of Mellin-Barnes representations for Feynman integrals, Comput. Phys. Commun. 177 (2007) 879–893.
arXiv:0704.2423, doi:10.1016/j.cpc.2007.07.001.

J. Gluza, K. Kajda, T. Riemann, V. Yundin, News on Ambre and CSectors, Nucl.Phys.Proc.Suppl. 205-206 (2010) 147–151.
arXiv:1006.4728, doi:10.1016/j.nuclphysbps.2010.08.034.

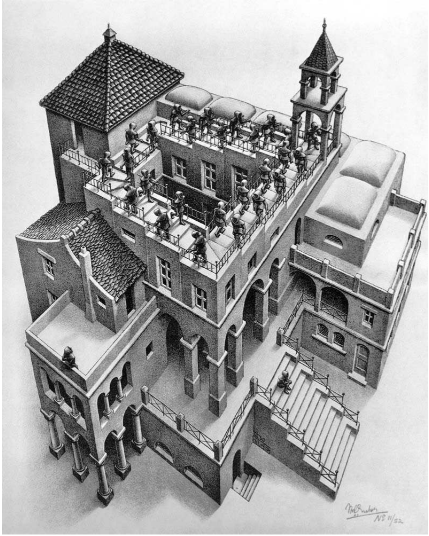
A. V. Smirnov, V. A. Smirnov, On the Resolution of Singularities of Multiple Mellin- Barnes Integrals, Eur. Phys. J. C62 (2009) 445.
arXiv:0901.0386.

Bach in Leipzig

"Strange Loops is the concept of infinity, since what else is a loop but a way of representing an endless process in a finite way?"



Strange Loops, Escher: Ascending and descending



Strange Loops: The Royal Theme

"...Somehow Bach has contrived to modulate (change keys) right under the listener's nose. And it is so constructed that this "ending" ties smoothly onto the beginning again; thus one can repeat the process and return in the key of E, only to join again to the beginning. These successive modulations lead the ear to increasingly remote provinces of tonality, so that after several of them, one would expect to be hopelessly far away from the starting key. And yet magically, after exactly six such modulations, the original key of C minor has been restored!"