# Five-point two-loop master integrals in QCD

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Based on work with with Thomas Gehrmann and Johannes Henn

PRL 116, 062001 (2016) [arXiv:1511.05409 [hep-ph] ]

# Introduction

Two-loop NNLO:

- $2 \to 2$ processes calculated recently ($\gamma\gamma$, $ZZ$, $Z\gamma$, $W\gamma$, $WW$, $t\bar{t}$, $Hj$, $Wj$, $jj$)
  [Catani, Cieri, de Florian, Ferrera, Grazzini, Gehrmann, G.-De Ridder, Glover, Boughezal, Focke, Liu,Petriello, Czakon, Fiedler, Mitov, Heymes, Kallweit, Maierhöfer, Rathlev, Pozzorini, Currie, Pires, Chen, Jaquier, Giele, Melnikov, Caola, Schulze, Huss, Morgan, Campbell, Ellis, Li, Williams . . . ]

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- **$2 \to 3$ processes still open**

Among NNLO bottle-necks: **two-loop scattering amplitudes**.

At one-loop Feynman diagrams can be decomposed
into a small set of master integrals (MIs), all of which are known.

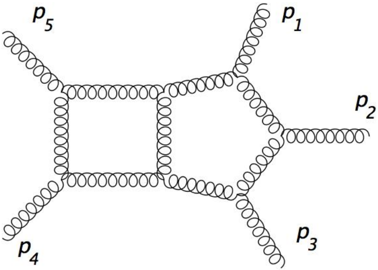
At two-loop much larger set of MIs $\to$ extends to higher multiplicities.

# Introduction

We have computed the full set of planar master integrals at 5-pt

$$G_{\{a_1,\dots,a_{11}\}} = \int \frac{d^D k_1\, d^D k_2}{(i\pi^{D/2})^2} \frac{D_9^{-a_9}\, D_{10}^{-a_{10}}\, D_{11}^{-a_{11}}}{D_1^{a_1}\, D_2^{a_2}\, D_3^{a_3}\, D_4^{a_4}\, D_5^{a_5}\, D_6^{a_6}\, D_7^{a_7}\, D_8^{a_8}}$$

$$\begin{aligned}
D_1 &= -k_1^2\,, \\
D_2 &= -(k_1+p_1)^2\,, \\
D_3 &= -(k_1+p_1+p_2)^2\,, \\
D_4 &= -(k_1+p_1+p_2+p_3)^2\,, \\
D_5 &= -k_2^2\,, \\
D_6 &= -(k_2+p_1+p_2+p_3)^2\,, \\
D_7 &= -(k_2+p_1+p_2+p_3+p_4)^2\,, \\
D_8 &= -(k_1-k_2)^2\,, \\
D_9 &= -(k_1+p_1+p_2+p_3+p_4)^2\,, \\
D_{10} &= -(k_2+p_1)^2\,, \\
D_{11} &= -(k_2+p_1+p_2)^2
\end{aligned}$$




$$s_{ij} = (p_i + p_j)^2$$

$$\vec{x} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{51}\}$$

See also results by [Papadopoulos, Tommasini, Wever]

# Method of Differential Equations

Given $G(a_1, a_2, \ldots, a_n) = \int \prod_{j=1}^{l} \frac{d^D k_j}{i\pi^{D/2}} \frac{1}{D_1^{a_1} \ldots D_n^{a_n}}$ , where $D_i = (k_j - p_i, - \ldots)^2$

Derivaties w.r.t external kinematic invariants, e.g. $s_{12} = (p_1 + p_2)^2$

$$\frac{\partial}{\partial s_{12}} \int \prod_{j=1}^{l} \frac{d^D k_j}{i\pi^{D/2}} \frac{1}{D_1^{a_1} \ldots D_n^{a_n}} = \int \prod_{j=1}^{l} \frac{d^D k_j}{i\pi^{D/2}} \frac{1}{2 s_{12}} \left( (p_1 + p_2)^\mu \frac{\partial}{\partial (p_1 + p_2)^\mu} \right) \frac{1}{D_1^{a_1} \ldots D_n^{a_n}}$$

$\Rightarrow$ first order differential equations [Gehrmann, Remiddi]

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$\Rightarrow$ first order differential equations [Gehrmann, Remiddi]

Integration by part identities $\int \prod_{j=1}^{l} \frac{d^D k_j}{i\pi^{D/2}} \left( \frac{\partial}{\partial k_j^\mu} v^\mu \frac{1}{D_1^{a_1} \ldots D_n^{a_n}} \right) = 0$

relate different integrals $\Longrightarrow$ we can reduce them to MIs.[Laporta alg.]

Fire [Smirnov], Reduze [Studerus, Manteuffel], LiteRed [Lee]

# Method of Differential Equations

MIs basis is not unique. Suitable choice considerably simplifies diff. eqs.:

$\partial_x \vec{f} = A(x,\varepsilon)\vec{f} \longrightarrow \partial_x \vec{f} = \varepsilon A(x)\vec{f}$ can be integrated order by order in $\varepsilon$.

[J. Henn (2013)]

$$\vec{f}(x,\varepsilon) = \vec{f}_0(x) + \varepsilon \vec{f}_1(x) + \varepsilon^2 \vec{f}_2(x) + \ldots \implies \begin{aligned} \vec{f}_0(x) &= \vec{f}_0 \\ \vec{f}_1(x) &= \int dx A(x)\vec{f}_0 \\ \vec{f}_2(x) &= \int dx A(x)\vec{f}_1(x) \\ &\ldots \end{aligned}$$

Starting from $\vec{f}_0(x) = \vec{f}_0$ $\rightarrow$ transcendentality-0 constant

$\Rightarrow$ each order in $\varepsilon$ has **uniform transcendentality** .

# Method of Differential Equations

Further simplification: alphabet $\{\alpha_1, \ldots \alpha_n\}$

$$\partial_x \vec{f} = \varepsilon \sum_k \frac{A_k}{x - x_k} \vec{f} \quad \longrightarrow \quad d\vec{f}(\vec{x}, \varepsilon) = \varepsilon\, d \left[ \sum_k A_k \log \alpha_k(\vec{x}) \right] \vec{f}(\vec{x}, \varepsilon)$$

Solutions expressed in terms of multiple polylogarithms

[Remiddi, Vermaseren; Gehrmann, Remiddi; Goncharov]

$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t),$$

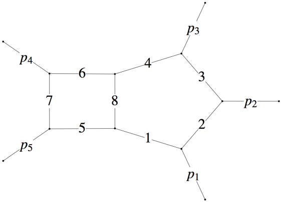
$$\text{with } G(x) = 1, \quad G(0) = 0 \quad \text{and} \quad G(\vec{0}_n; x) = \frac{1}{n!} \log^n x.$$

Simple example: $G(\vec{a}_n; x) = \frac{1}{n!} \log^n \left(1 - \frac{x}{a}\right)$ with $\vec{a}_n = \{a, \ldots, a\}$

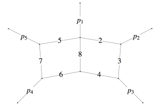
If $a_i \in \{1, -1, 0\}$ $\longrightarrow$ Harmonic Polylogarithms.

# 2-loop five-point planar integrals

61 MIs




{46, G[1, {1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0}], 3}



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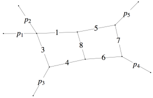

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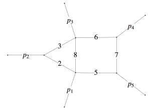

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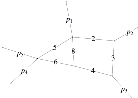

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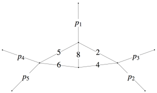
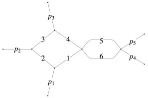


{28, G[1, {0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0}], 1}



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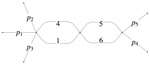

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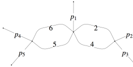

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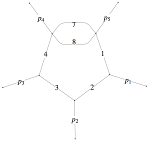

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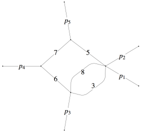
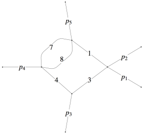
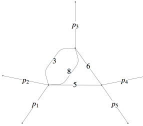


{32, G[1, {0, 0, 1, 0, 1, 1, 1, 1, 0, 0, 0}], 1}



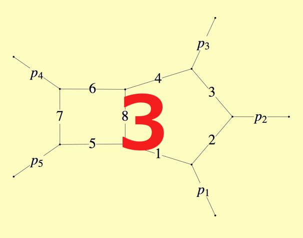
{20, G[1, {1, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0}], 1}



{15, G[1, {0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 0}], 1}

# 2-loop five-point planar integrals

61 MIs , 10 new

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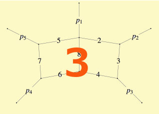

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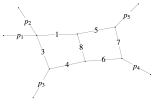

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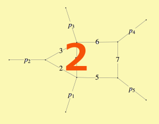

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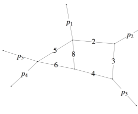

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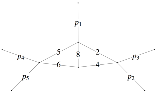

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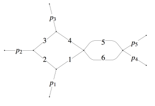

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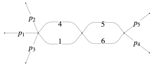

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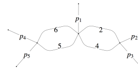

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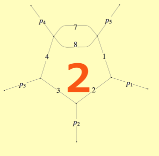

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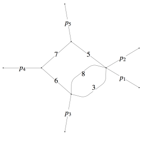

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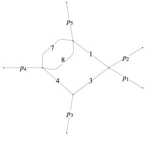

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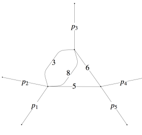

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$\leq 4$ point MIs known

[Gehrmann, Remiddi (2001)]

# 2-loop five-point planar integrals

Alphabet of 24 letter

$$\left\{ v_1 \ , \ v_3+v_4 \ , \ v_1-v_4 \ , \ v_1+v_2-v_4 \ , \ \Delta \ , \ \frac{a-\sqrt{\Delta}}{a+\sqrt{\Delta}} \right\} + \text{cyclic}$$

with $a = v_1 v_2 - v_2 v_3 + v_3 v_4 - v_1 v_5 - v_4 v_5 = \mathrm{tr}[\not p_4 \not p_5 \not p_1 \not p_2] \quad (v_i \equiv s_{i,i+1})$

Gram determinant $\Delta = |2p_i \cdot p_j| = (\mathrm{tr}_5)^2$ with $\mathrm{tr}_5 = \mathrm{tr}[\gamma_5 \not p_4 \not p_5 \not p_1 \not p_2]$ .

With a suitably chosen parametrization, $\Delta \to$ perfect square

$$\begin{aligned}
v_1 &= z_1 \, , \\
v_2 &= z_1 \, z_2 \, z_4 \, , \\
v_3 &= (z_1/z_2) \left[ z_3 \, (z_4 - 1) + z_2 \, z_4 + z_2 \, z_3 \, (z_4 - z_5) \right] , \\
v_4 &= z_1 \, z_2 \, (z_4 - z_5) \, , \\
v_5 &= z_1 \, z_3 \, (1 - z_5)
\end{aligned}$$

obtained by using Momentum Twistor variables [Hodges (2009)]

# Boundary conditions

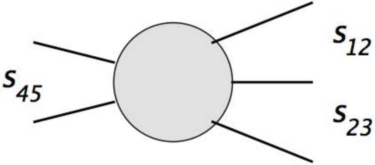
Boundary values can be obtained from physical conditions,
in kinematic limits with **singular diff. eq.** but **regular integrals**.

No singularities in the Euclidean region $s_{i,i+1} < 0$.

Un-physical singularities
appear in the limit

$$s_{45} \to s_{12} + s_{23}$$

and they need to cancel.




$\longrightarrow$ no need to compute any additional integrals.

# Boundary conditions

$\Delta = 0$ defines hypersurface where divergencies need to cancel.

The symmetric point $\vec{x}_{sym} = \{-1,-1,-1,-1,-1\}$

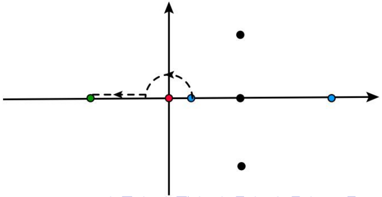
is connected to the $\Delta = 0$ surface by

$$\vec{f}(\vec{x},\varepsilon) = P\exp\left[\varepsilon\int_\gamma dA\right]\vec{f}(\vec{x}_0,\varepsilon)$$

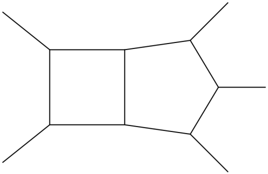
path $\gamma = \{-\frac{y}{(1-y)^2},-1,-1,-1,-1\} \longrightarrow$ reduced alphabet .

$$\text{Sym. pt} \rightarrow y = \frac{3\pm\sqrt{5}}{2}$$

$$\Delta = 0 \rightarrow y = -1$$



# Boundary conditions



$$
\begin{aligned}
f_{59} &= \epsilon^2 \frac{s_{12}s_{23}s_{45}s_{35}}{\sqrt{\Delta}} G[\{1,1,1,1,1,1,1,1,-1,-1,0\}] \\
&+\epsilon^2 s_{12}s_{23}s_{45} \frac{\mathrm{tr}[\not{p}_1\not{p}_2\not{p}_3\not{p}_4]}{4\sqrt{\Delta}} G[\{1,1,1,1,1,1,1,1,-1,0,0\}] \\
&\to \epsilon^3 c_{5,3} + \ldots
\end{aligned}
$$

$$
\begin{aligned}
f_{60} &= \epsilon^2 s_{12}s_{23}s_{45} G[\{1,1,1,1,1,1,1,1,-1,0,0\}] \\
&\to -3 - \frac{11}{6}\pi^2\epsilon^2 + \ldots
\end{aligned}
$$

$$
\begin{aligned}
f_{61} &= \epsilon^2 \frac{s_{12}s_{23}s_{45}s_{35}}{\sqrt{\Delta}} \Big[ s_{45} G[\{1,1,1,1,1,1,1,1,0,-1,0\}] \\
&-G[\{1,1,1,1,1,1,1,1,-1,-1,0\}] \Big] \\
&-\epsilon^2 s_{12}s_{23}s_{45} \frac{\mathrm{tr}[\not{p}_1\not{p}_2\not{p}_3\not{p}_4]}{4\sqrt{\Delta}} G[\{1,1,1,1,1,1,1,1,-1,0,0\}] \\
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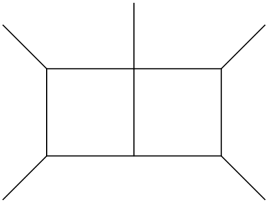
# Boundary conditions

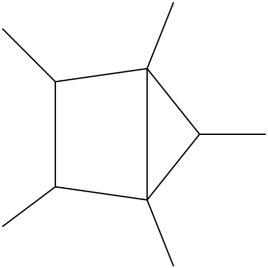


$$f_{56} = \varepsilon^2 s_{12}s_{45}s_{51}G[\{1,1,1,0,1,1,1,1,0,0,0\}] \to -\frac{9}{4}+\frac{29}{24}\pi^2\varepsilon^2+\ldots$$

$$f_{57} = \varepsilon^2 s_{12}s_{45}G[\{1,1,1,0,1,1,1,1,-1,0,0\}] \to -\frac{3}{2}-\frac{3}{4}\pi^2\varepsilon^2+\ldots$$

$$f_{58} = \varepsilon^2\frac{s_{12}s_{45}s_{24}}{\sqrt{\Delta}}G[\{1,1,1,0,1,1,1,1,-1,-1,0\}]+\ldots -s_{12}s_{45}\frac{\mathrm{tr}[\not{p}_1\not{p}_2\not{p}_3\not{p}_4]}{4\sqrt{\Delta}}\Big[s_{51}G[\{1,1,1,0,1,1,1,1,0,0,0\}] +G[\{1,1,1,0,1,1,1,1,-1,0,0\}]\Big]\ldots \to -\varepsilon^3 c_{5,3}+\ldots$$

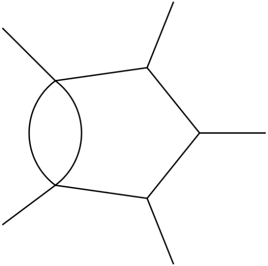
# Boundary conditions



$$f_{46} = \epsilon^2 s_{45}(s_{34} - s_{51}) G[\{0,1,1,0,1,1,1,1,0,0,0\}]$$
$$\to 0 + \ldots$$

$$f_{47} = \epsilon \frac{s_{12}s_{23}s_{34}s_{45}s_{51}}{\sqrt{\Delta}} G[\{0,1,1,0,1,1,1,1,0,0,0\}] +$$
$$+\epsilon^2 \frac{\text{Num}}{\sqrt{\Delta}} G[\{0,1,1,0,1,1,1,2,0,0,0\}] + \ldots$$
$$\to \epsilon^3 c_{5,3} + \ldots$$

# Boundary conditions



$$
\begin{aligned}
f_{37} &= -\varepsilon s_{12}s_{23}\frac{\mathrm{tr}[\not p_1\not p_2\not p_3\not p_4]}{2\sqrt{\Delta}}G[\{1,1,1,1,0,0,2,1,-1,0,0\}] \\
&\quad +\varepsilon\frac{s_{12}s_{23}s_{34}s_{45}s_{51}}{\sqrt{\Delta}}G[\{1,1,1,1,0,0,2,1,0,0,0\}] \\
&\to \frac{3}{2}c_{5,3}+\ldots
\end{aligned}
$$

$$
\begin{aligned}
f_{38} &= \varepsilon s_{12}s_{23}G[\{1,1,1,1,0,0,2,1,-1,0,0\}] \\
&\to -2+\varepsilon^2\pi^2+\varepsilon^3\frac{55}{3}\zeta_3+\ldots
\end{aligned}
$$

# Applications: All-plus amplitude

We have applied our integrals to the **all-plus amplitude** (leading-colour).

$$\mathcal{A}_5\,(1^+,2^+,3^+,4^+,5^+)|_{\text{leading colour}} =$$

$$g_s^7 N_c^2\, c_\Gamma^2 \sum_{\sigma\in S_5} \text{tr}(T^{a_{\sigma(1)}}\, T^{a_{\sigma(2)}}\, T^{a_{\sigma(3)}}\, T^{a_{\sigma(4)}}\, T^{a_{\sigma(5)}}) \sum_{\text{cycl}} A_5^{(2)}\,(\sigma(1)^+,\sigma(2)^+,\sigma(3)^+,\sigma(4)^+,\sigma(5)^+)$$

At one loop we have [Bern, Dixon, Dunbar, Kosower]

$$A^{(1)}(1^+2^+3^+4^+5^+) = \frac{-i\varepsilon(1-\varepsilon)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\Big(2\,(2-\varepsilon)\,\text{tr}_5\, I^{[10-2\varepsilon]}_{[5;12345]}[1]$$
$$+\, s_{12}s_{23}\, I^{[8-2\varepsilon]}_{4;1234}[1] + s_{23}s_{34}\, I^{[8-2\varepsilon]}_{4;2345}[1] + s_{34}s_{45}\, I^{[8-2\varepsilon]}_{4;3451}[1] + s_{45}s_{51}\, I^{[8-2\varepsilon]}_{4;5123}[1]\Big)$$

$$\varepsilon^0 \;\to\; \frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\left(-\frac{1}{6}\,F_5^{(1)}\right) \quad , \qquad F_5^{(1)} = v_1v_2+v_2v_3+v_3v_4+v_4v_5+v_5v_1 + \text{tr}_5$$

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**Parity even part** $\longrightarrow$ at order $\epsilon^2$ yields terms up to weight 3.

**Parity odd part** $\longrightarrow$ at order $\epsilon^2$ yields terms up to weight 4.

# Applications: All-plus amplitude

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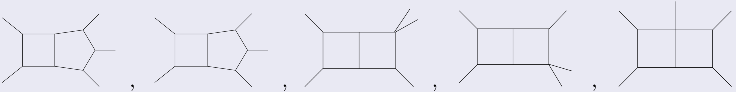
[Badger, Frellesvig, Zhang (2013)]

$$A_5(1^+,2^+,3^+,4^+,5^+) = \frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\Bigg\{ \text{butterfly topologies} +$$



$$\Bigg\}$$

$F_1\,, F_3 \to \quad \left(k_1^{[-2\epsilon]}\right)^2\,, \quad \left(k_2^{[-2\epsilon]}\right)^2 \quad$ and $\quad 2\,k_1^{[-2\epsilon]}\cdot k_2^{[-2\epsilon]} \quad$ polynomials in numerator

$$\int \frac{d^{-2\epsilon}k_1^{[-2\epsilon]}}{(2\pi)^{2\epsilon}} \int \frac{d^{-2\epsilon}k_2^{[-2\epsilon]}}{(2\pi)^{2\epsilon}} \quad \text{translates into} \quad I_A[F_1] \longrightarrow 4\,\epsilon\,(\epsilon-1) \sum_{i,j\in P(A)} \mathbf{i}^+\mathbf{j}^+\, I_A^{[8-2\epsilon]}$$

[Bern, De Freitas, Dixon (2002) ; Badger, Frellesvig, Zhang (2013)]

# Results

The infrared and ultraviolet structure is described by the one-loop amplitude

$$A^{(2)}_{5\,\text{plus}} = A^{(1)}_{5\,\text{plus}} \left[ -\sum_{i=1}^{5} \frac{1}{\varepsilon^2} \left( \frac{\mu^2}{-v_i} \right)^{\varepsilon} \right] + \frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \left( -\frac{1}{6} F^{(2)}_5 \right) + O(\varepsilon)\,,$$

For the finite remainder we find: $(v_i = s_{i,i+1})$ [arXiv:1511.05409 [hep-ph]]

$$F^{(2)}_5 = \frac{5\pi^2}{12} F^{(1)}_5 + \sum_{i=0}^{4} \sigma^i \left\{ \frac{v_5 \mathrm{tr}\left[(1-\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2\right]}{(v_2+v_3-v_5)} I_{23,5} + \frac{1}{6} \frac{\mathrm{tr}\left[(1+\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2\right]^2}{v_1 v_4} + \frac{10}{3} v_1 v_2 + \frac{2}{3} v_1 v_3 \right\}.$$

with $I_{23,5}$ one-loop two-mass easy box function in six dimensions.

$$I_{23,5} = \zeta_2 - \mathrm{Li}_2\left(\frac{v_5 - v_3}{v_2}\right) - \mathrm{Li}_2\left(\frac{v_5 - v_2}{v_3}\right) + \mathrm{Li}_2\left(\frac{(v_5 - v_2)(v_5 - v_3)}{v_2 v_3}\right)$$

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A_5^{(2)}(1^+,2^+,3^+,4^+,5^+) \to & \mathrm{Split}^{P\to 45\,(1)}(P^-,4^+,5^+)A_4^{(1)}(1^+,2^+,3^+,P^+) \\
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& + \mathrm{Split}^{P\to 45\,(0)}(P^-,4^+,5^+)A_4^{(2)}(1^+,2^+,3^+,P^+)\,.
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Recently recomputed by [Dunbar, Perkins (2016)] using on-shell recursions

# Summary and Outlook

- Five-point two-loop MIs (planar) obtained using the Differential-Equation method, with MIs basis that makes the diff. eq. system canonical.
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- Analytic continuation outside Euclidean region ($\to$ physical region).
- Non-planar integrals: in progress.
- Application to other amplitudes.