Integrand reduction for one-loop calculations and beyond

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Based on work with GoSam Collaboration
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Introduction

- Integrand-Reduction Techniques **evolved** over the past decade.

- The OPP method marked a **beginning**, but then Integrand Reduction grew into a more general and variegated framework. $\rightsquigarrow$ Such **advancements** are due to many **different authors** and groups.

- In this talk I will review some of the stages along this **evolution process**, and underline features that make Integrand Reduction a promising approach to study **multi-loop scattering amplitudes**.

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2. **Integrand Reduction at One Loop**
   - ⇝ Overview of the Integrand-Reduction Approach
   - ⇝ Numerical Implementations and Algorithms

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   - ⇝ Overview of the Integrand-Reduction Approach
   - ⇝ Numerical Implementations and Algorithms
     - ◇ CutTools, 4-dimensional Integrand Reduction
     - ◇ Samurai, d-dimensional Integrand Reduction
     - ◇ Ninja, Integrand Reduction via Laurent Expansion

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   - ⇝ Integrand Reduction via Multivariate Polynomial Division
   - ⇝ Multi-Loop Integrand Reduction
   - ⇝ The Maximum-Cut theorem

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4. **NLO Automation** and **Physical Applications**
   - ⇝ The GoSam Project
   - ⇝ $pp \to t\bar{t}\gamma\gamma$ @ NLO with GoSam + MG5_aMC
   - ⇝ A glance beyond GoSam 2.0

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   - $\rightsquigarrow$ A glance beyond GoSam 2.0 $\rightsquigarrow$ **Talk of Stephen Jones**

Integrand level vs Integral level

Let's consider a two-loop Feynman integral with n denominators:

$$\mathcal{I} = \int dq \int dk \; \mathcal{A}(q,k) = \int dq \int dk \; \frac{\mathcal{N}(q,k)}{D_1 D_2 \dots D_n}$$

◇ Description in terms of Master Integrals $\mathcal{I}_i$ → Integral-Level Approach

$$\mathcal{I} = \int dq \int dk \; \frac{\mathcal{N}(q,k)}{D_1 D_2 \dots D_n} = c_0 \; \mathcal{I}_0 + c_1 \; \mathcal{I}_1 + \dots + C_k \; \mathcal{I}_k$$

⇝ Tensorial Reduction, Lorentz invariance, form factors, IBPs identities

Integrand level vs Integral level

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◇ Analyze the unintegrated $\mathcal{A}(q,k)$ → Integrand-Level Approach

$$\mathcal{A}(q,k) = \frac{\mathcal{N}(q,k)}{D_1 D_2 \dots D_n}$$

⇝ Integrands are ratios of polynomials in the integration variables

⇝ The pole structure is explicit in the integrand

Integrand level vs Integral level

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◇ Description in terms of Master Integrals $\mathcal{I}_i$ → Integral-Level Approach

$$\mathcal{I} = \int dq \int dk \; \frac{\mathcal{N}(q,k)}{D_1 D_2 \ldots D_n} = c_0 \; \mathcal{I}_0 + c_1 \; \mathcal{I}_1 + \ldots + C_k \; \mathcal{I}_k$$

◇ Analyze the unintegrated $\mathcal{A}(q,k)$ → Integrand-Level Approach

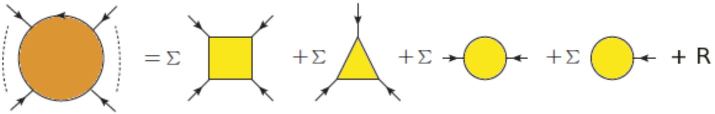
$$\mathcal{A}(q,k) = \frac{\mathcal{N}(q,k)}{D_1 D_2 \ldots D_n}$$

- Can we cast $\mathcal{N}(q,k)$ and $\mathcal{A}(q,k)$ in a simpler form than Feynman Integrals?
- How should we use the knowledge of the set of zeros in the denominators?

One-Loop matching of "master" formulas

◇ At one loop, a general **integral-level** decomposition is well-known

$$\int dq \frac{\mathcal{N}(q)}{D_1 D_2 \ldots D_n} = \sum_{\{i\}} d_i \int \frac{dq}{D_{i_1} D_{i_2} D_{i_3} D_{i_4}} + \sum_{\{i\}} c_i \int \frac{dq}{D_{i_1} D_{i_2} D_{i_3}} + \sum_{\{i\}} b_i \int \frac{dq}{D_{i_1} D_{i_2}} + \sum_i a_i \int \frac{dq}{D_i} + \mathrm{R},$$



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◇ What is the **integrand-level** counterpart?

One-Loop matching of "master" formulas

◇ At one loop, a general **integral-level** decomposition is well-known

$$\int dq \frac{\mathcal{N}(q)}{D_1 D_2 \ldots D_n} = \sum_{\{i\}} d_i \int \frac{dq}{D_{i_1} D_{i_2} D_{i_3} D_{i_4}} + \sum_{\{i\}} c_i \int \frac{dq}{D_{i_1} D_{i_2} D_{i_3}} + \sum_{\{i\}} b_i \int \frac{dq}{D_{i_1} D_{i_2}} + \sum_{i} a_i \int \frac{dq}{D_i} + \mathrm{R},$$

◇ What is the **integrand-level** counterpart? (4-dimensional version)

$$\mathcal{N}(q) = \sum_{\{i\}} \left[d_i + \tilde{d}_i(q)\right] \prod_{j \notin \{i\}} D_j + \sum_{\{i\}} \left[c_i + \tilde{c}_i(q)\right] \prod_{j \notin \{i\}} D_j + \sum_{\{i\}} \left[b_i + \tilde{b}_i(q)\right] \prod_{j \notin \{i\}} D_j + \sum_{i} \left[a_i + \tilde{a}_i(q)\right] \prod_{j \neq i} D_j$$

⇝ All $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ should vanish upon integration (spurious terms)
⇝ The full Rational Term R requires ad-hoc tree-level Feynman rules

del Aguila, Pittau (2005); G.O., Papadopoulos, Pittau (2007)

4-dimensional Integrand Reduction

- The general decomposition can be obtained algebraically by **direct construction** $\rightsquigarrow$ rewrite q in $\mathcal{N}(q)$ in terms of reconstructed denominators $\rightsquigarrow$ the residual q dependence should vanish upon integration
- These vanishing integrands can be classified (by tensorial reduction)

i.e. three-point diagrams depend on two independent momenta p_1 and p_2

Build a basis of momenta ℓ_i in which

$$\mathbf{p_1} = \ell_1 + \alpha_1 \ell_2\,,\ \mathbf{p_2} = \ell_2 + \alpha_2 \ell_1\ ,\ \ {\ell_3}^{\mu} = < \ell_1|\gamma^{\mu}|\ell_2]\,,\ {\ell_4}^{\mu} = < \ell_2|\gamma^{\mu}|\ell_1]$$

Theorems: since $\mathbf{p_{1,2}} \cdot \ell_{3,4} = 0$

$$\int d^4q \frac{q\cdot\ell_3}{D_0 D_1 D_2} = 0\,, \qquad \int d^4q \frac{q\cdot\ell_4}{D_0 D_1 D_2} = 0\,,$$

Then $\tilde{c}(q) = \sum_{j=1}^{j_{max}} \left\{ \tilde{c}_{1j}\ [q\cdot\ell_3]^j + \tilde{c}_{2j}\ [q\cdot\ell_4]^j \right\}$

upgrade: D-dimensional Integrand Reduction

$$\mathcal{N}(\bar{q}) = \sum_{i<<m}^{n-1} \Delta_{ijk\ell m}(\bar{q}) \prod_{h\neq i,j,k,\ell,m}^{n-1} \bar{D}_h + \sum_{i<<\ell}^{n-1} \Delta_{ijk\ell}(\bar{q}) \prod_{h\neq i,j,k,\ell}^{n-1} \bar{D}_h + $$
$$+ \sum_{i<<k}^{n-1} \Delta_{ijk}(\bar{q}) \prod_{h\neq i,j,k}^{n-1} \bar{D}_h + \sum_{i<j}^{n-1} \Delta_{ij}(\bar{q}) \prod_{h\neq i,j}^{n-1} \bar{D}_h + \sum_{i}^{n-1} \Delta_{i}(\bar{q}) \prod_{h\neq i}^{n-1} \bar{D}_h$$

#1 Provide the Functional Form of all the Residues $\Delta_{ij\ldots k\ell}$

- ⇝ They are polynomials in the components of $\bar{q}$
- ⇝ They have a universal, process-independent form ...
- ⇝ ... parametrised by process-dependent coefficients

#2 Extract all coefficients by sampling on the kinematic cuts

- ⇝ Polynomial fitting allows to find all coefficients
- ⇝ we only need Numerator Function evaluated on the cuts

GO, Papadopoulos, Pittau (2007); Ellis, Giele, Kunszt, Melnikov (2008)
Mastrolia, GO, Reiter, Tramontano (2010)

UPGRADE: D-DIMENSIONAL INTEGRAND REDUCTION

$$\mathcal{N}(\bar{q}) = \sum_{i<<m}^{n-1} \Delta_{ijk\ell m}(\bar{q}) \prod_{h\neq i,j,k,\ell,m}^{n-1} \bar{D}_h + \sum_{i<<\ell}^{n-1} \Delta_{ijk\ell}(\bar{q}) \prod_{h\neq i,j,k,\ell}^{n-1} \bar{D}_h + \\ + \sum_{i<<k}^{n-1} \Delta_{ijk}(\bar{q}) \prod_{h\neq i,j,k}^{n-1} \bar{D}_h + \sum_{i<j}^{n-1} \Delta_{ij}(\bar{q}) \prod_{h\neq i,j}^{n-1} \bar{D}_h + \sum_{i}^{n-1} \Delta_{i}(\bar{q}) \prod_{h\neq i}^{n-1} \bar{D}_h$$

3 Recombining with the denominators:

$$\mathcal{A}(\bar{q}) = \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} + \sum_{i<<\ell}^{n-1} \frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell} + \sum_{i<<k}^{n-1} \frac{\Delta_{ijk}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k} + \\ + \sum_{i<j}^{n-1} \frac{\Delta_{ij}(\bar{q})}{\bar{D}_i\bar{D}_j} + \sum_{i}^{n-1} \frac{\Delta_{i}(\bar{q})}{\bar{D}_i} ,$$

- ◇ The multi-pole structure of the integrand is exposed
- ◇ Only Rational Functions ⇝ ratios of polynomials

Explicit Form of $\Delta_{ij\dots k\ell} \to$ D-dim result

- Every one-loop integrand in d dimensions can be decomposed as

$$\mathcal{A} \equiv \frac{\mathcal{N}}{D_0 \dots D_{n-1}} = \sum_{k=1}^{5} \sum_{\{i_1,\cdots,i_k\}} \frac{\Delta_{i_1\cdots i_k}}{D_{i_1}\cdots D_{i_k}}$$

- For any set of denominators $D_{i_1}, \dots, D_{i_k}$, one can choose a 4-dimensional basis of momenta $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$

$$q^\nu = -p_{i_1}^\nu + x_1\ e_1^\nu + x_2\ e_2^\nu + x_3\ e_3^\nu + x_4\ e_4^\nu\ ,\ \bar{q}^2 = q^2 - \mu^2$$

- Numerator and denominators can be written as polynomials in the coordinates $\mathbf{z} \equiv (x_1, x_2, x_3, x_4, \mu^2)$

$$N(\bar{q}) = \mathcal{N}(q, \mu^2) = \mathcal{N}(x_1, x_2, x_3, x_4, \mu^2) = \mathcal{N}(\mathbf{z})$$

EXPLICIT FORM OF $\Delta_{ij\ldots k\ell} \rightarrow$ D-DIM RESULT

$\rightsquigarrow$ The most general parametric form of a residue in a renormalizable theory is

$$\begin{aligned}
\Delta_{i_1 i_2 i_3 i_4 i_5} &= c_0\, \mu^2 \\
\Delta_{i_1 i_2 i_3 i_4} &= c_0 + c_1 x_4 + \mu^2 c_2 + \mu^2 x_4 c_3 + \mu^4 c_4 \\
\Delta_{i_1 i_2 i_3} &= c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 + c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 + \mu^2 c_7 + \mu^2 c_8 x_3 + \mu^2 c_9 x_4 \\
\Delta_{i_1 i_2} &= c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2 \\
\Delta_{i_1} &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4
\end{aligned}$$

$\rightsquigarrow$ We can **"fit-on-the-cut"** and determine all coefficients

Explicit Form of $\Delta_{ij\ldots k\ell} \rightarrow$ D-dim result

⇝ The most general parametric form of a residue in a renormalizable theory is

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\Delta_{i_1 i_2} &= c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2 \\
\Delta_{i_1} &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4
\end{aligned}$$

⇝ Put back the numerator function inside the integral

$$\mathcal{M} \equiv \int \frac{\mathcal{N}}{D_0 \ldots D_{n-1}} = \sum_{k=1}^{5} \sum_{\{i_1, \cdots, i_k\}} \int \frac{\Delta_{i_1 \cdots i_k}}{D_{i_1} \cdots D_{i_k}}$$

⇝ Upon integration, several terms vanish (spurious terms)

Explicit Form of $\Delta_{ij\ldots k\ell} \rightarrow$ D-dim result

⇝ The most general parametric form of a residue in a renormalizable theory is

$$\Delta_{i_1 i_2 i_3 i_4 i_5} = c_0\, \mu^2$$
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$$\Delta_{i_1 i_2} = c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2$$
$$\Delta_{i_1} = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$$

⇝ Terms that do not vanish upon integration become **Master Integrals**.

$$\mathcal{M} = \sum_{\{i_1, i_2, i_3, i_4\}} \left\{ c_0\ I_{i_1 i_2 i_3 i_4} + c_4\ I_{i_1 i_2 i_3 i_4}[\mu^4] \right\} + \sum_{\{i_1, i_2, i_3\}} \left\{ c_0\ I_{i_1 i_2 i_3} + c_7\ I_{i_1 i_2 i_3}[\mu^2] \right\}$$
$$+ \sum_{\{i_1, i_2\}} \left\{ c_0\ I_{i_1 i_2} + c_1\ I_{i_1 i_2}[q \cdot e_2] + c_4\ I_{i_1 i_2}[(q \cdot e_2)^2] + c_9\ I_{i_1 i_2}[\mu^2] \right\} + \sum_{i_1} c_0\ I_{i_1}$$

⇝ The **Rational Term** R arises **naturally** from remaining powers of μ^2

Upgrade: Integrand Reduction via Laurent Expansion

Forde (2007); Badger (2009)
Mastrolia, Mirabella, Peraro (2012)

⇝ If the functional structure of the numerator function is known, the coefficients can be extracting by performing a Laurent expansion with respect to one of the free parameters in the solutions of the cut

- ◇ Laurent series implemented via univariate Polynomial Division
- ◇ Corrections at the coefficient level replace subtractions at the integrand level
- ◇ Lighter reduction algorithm where fewer coefficients are computed
- ◇ Quadruple-cut decoupled from triple-, double-, and single-cut
- ◇ No more "sampling on the cuts" (no need of finding solutions for the cuts)

⇝ **Standalone** and **public** version of the Ninja library Peraro (2014)

Existing interfaces with several codes:

- ◇ GoSam [van Deurzen, Luisoni, Mastrolia, Mirabella, G.O., Peraro (2014)]
- ◇ FormCalc 8.4 [Hahn *et al.* (2014)]
- ◇ More recently: MadLoop within MG5_aMC [Hirschi, Peraro (2016)]

Upgrade: Integrand Reduction via Laurent Expansion ⇝ Ninja

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"Tensor integrand reduction via Laurent expansion" Hirschi, Peraro (2016)

- ◇ It is now possible to interface Ninja to any one-loop matrix element generator that can provide the components of loop numerator tensor
- ◇ The library has been interfaced it to MadLoop, within MG5_aMC
- ◇ Ninja performs better that other reduction algorithms both in speed and numerical stability

Integrand-Level Reduction at Higher Orders

Mastrolia, G.O. (2011)

◇ Let's consider a two-loop integral with n denominators:

$$\int dq\; dk\; \frac{N(q,k)}{D_1 D_2 \dots D_n}$$

◇ As done at one loop, we want to construct an identity for the integrands:

$$\mathcal{N}(q,k) = \sum_{i_1 << i_8}^{n} \Delta_{i_1,\dots,i_8}(q,k) \prod_{h \neq i_1,\dots,i_8}^{n} D_h + \dots + \sum_{i_1 << i_2}^{n} \Delta_{i_1,i_2}(q,k) \prod_{h \neq i_1,i_2}^{n} D_h$$

$$\mathcal{A}(q,k) = \sum_{i_1 << i_8}^{n} \frac{\Delta_{i_1,\dots,i_8}(q,k)}{D_{i_1} D_{i_2} \dots D_{i_8}} + \sum_{i_1 << i_7}^{n} \frac{\Delta_{i_1,\dots,i_7}(q,k)}{D_{i_1} D_{i_2} \dots D_{i_7}} + \dots + \sum_{i_1 << i_2}^{n} \frac{\Delta_{i_1,i_2}(q,k)}{D_{i_1} D_{i_2}}$$

◇ Which terms appear in the above expressions?

◇ What is the general form of the residues $\Delta_{i_1,\dots,i_m}$?

◇ Can we detect the Master Integrals at the Integrand Level?

Integrand-Level Reduction at Higher Orders

Mastrolia, G.O. (2011)

- Let's consider a two-loop integral with n denominators:

$$\int dq\ dk\ \frac{N(q,k)}{D_1 D_2 \ldots D_n}$$

- As done at one loop, we want to construct an identity for the integrands:

$$\mathcal{N}(q,k) = \sum_{i_1 << i_8}^{n} \Delta_{i_1,\ldots,i_8}(q,k) \prod_{h \neq i_1,\ldots,i_8}^{n} D_h + \ldots + \sum_{i_1 << i_2}^{n} \Delta_{i_1,i_2}(q,k) \prod_{h \neq i_1,i_2}^{n} D_h$$

$$\mathcal{A}(q,k) = \sum_{i_1 << i_8}^{n} \frac{\Delta_{i_1,\ldots,i_8}(q,k)}{D_{i_1} D_{i_2} \ldots D_{i_8}} + \sum_{i_1 << i_7}^{n} \frac{\Delta_{i_1,\ldots,i_7}(q,k)}{D_{i_1} D_{i_2} \ldots D_{i_7}} + \ldots + \sum_{i_1 << i_2}^{n} \frac{\Delta_{i_1,i_2}(q,k)}{D_{i_1} D_{i_2}}$$

- Which terms appear in the above expressions?
- What is the general form of the residues $\Delta_{i_1,\ldots,i_m}$?
- Beyond one-loop, there are Irreducible Scalar Products (ISPs) that **do not** integrate to zero! $\rightsquigarrow$ They form additional Master Integrals

Using the language of Algebraic Geometry...

Zhang (2012); Badger, Frellesvig, Zhang (2012)
Mastrolia, Mirabella, G.O., Peraro (2012)

- Set of Multivariate Polynomials $\{\omega_i(z)\}$ where $z = (z_1, z_2, \ldots)$
- **Ideal** $\mathcal{J} = \langle \omega_1(z), \ldots, \omega_s(z) \rangle \rightsquigarrow \mathcal{J} = \left\{ \sum_i \; h_i(z) \; \omega_i(z) \right\}$
- **Multivariate Polynomial Division** of a function $F(\mathbf{z})$ modulo $\{\omega_1, \ldots, \omega_s\}$
  - $\rightsquigarrow$ $F(z) = \sum_i \; h_i(z) \; \omega_i(z) + \mathcal{R}(z)$
  - $\rightsquigarrow$ $h_i(z)$ and $\mathcal{R}(z)$ are **not** unique
- **Gröbner Basis** $\{g_1(z), \ldots, g_r(z)\}$
  - $\rightsquigarrow$ It exists (Buchberger's algorithm) and generates the ideal $\mathcal{J}$
  - $\rightsquigarrow$ Provides a **unique** $\mathcal{R}(z)$
- Hilberts Nullstellensatz
  - $\rightsquigarrow$ $V(\mathcal{J}) \rightarrow$ set of common zeros of $\mathcal{J}$
  - $\rightsquigarrow$ Weak Nullstellensatz: $V(\mathcal{J}) = \mathcal{O} \Leftrightarrow 1 \in \mathcal{J}$

...Integrand Reduction via Multivariate Polynomial Division

$$\mathcal{I}_{i_1\cdots i_n} = \frac{\mathcal{N}_{i_1\cdots i_n}(z)}{D_{i_1}(z)\cdots D_{i_n}(z)}$$

- Ideal: $\mathcal{J}_{i_1\cdots i_n} = \langle D_{i_1}, \ldots, D_{i_n} \rangle$
- Gröbner basis $\mathcal{G}_{i_1\cdots i_n}$: same zero as the denominators
- Multivariate division of $\mathcal{N}_{i_1\cdots i_n}$ modulo $\mathcal{G}_{i_1\cdots i_n}$

$$\mathcal{N}_{i_1\cdots i_n}(z) = \Gamma_{i_1\cdots i_n} + \Delta_{i_1\cdots i_n}(z)$$

- The quotient $\Gamma_{i_1\cdots i_n}$ can be expressed in terms of denominators

$$\Gamma_{i_1\cdots i_n} = \sum_{\kappa=1}^{n} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_n}(z) D_{i_\kappa}(z)$$

- Which provides the Recursive Formula

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{\kappa=1}^{n} \mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1} i_n} + \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}}$$

Mastrolia, G.O., Mirabella, Peraro (2012)

Integrand Reduction at Arbitrary Loops: a Theorem

Let's look at the on-shell conditions, and impose

$$D_1 = D_2 = \ldots = D_n = 0$$

1) There are no solutions → the diagram is **reducible**

- ⇝ The integrand with n denominators can be expressed in terms of integrands with $(n-1)$ denominators
- ⇝ The diagram is fully reducible in terms of lower point functions
  ⇝ i.e. *six-point functions at one-loop*

Integrand Reduction at Arbitrary Loops: a Theorem

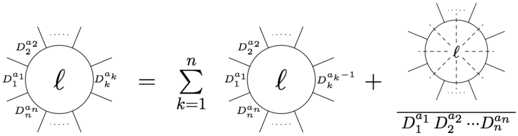
Let's look at the on-shell conditions, and impose

$$D_1 = D_2 = \ldots = D_n = 0$$

1) There are no solutions → **reducible**
2) The cut has solutions → there is a **residue** Δ

Divide the numerator modulo the Gröbner basis of the n-ple cut (a set of polynomials vanishing on the same on-shell cuts of the denominators)

- ⇝ The **remainder** of the division is the **residue** Δ of the n-ple cut.
- ⇝ The **quotients** generate integrands with $(n-1)$ denominators.



Integrand Reduction at Arbitrary Loops: a Theorem

Let's look at the on-shell conditions, and impose

$$D_1 = D_2 = \ldots = D_n = 0$$

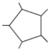
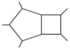
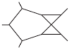
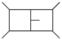
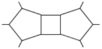
1) There are no solutions $\rightarrow$ **reducible**
2) The cut has solutions $\rightarrow$ **residue Δ**
3) Finite number of solutions n_s $\rightarrow$ **Maximum Cut**
   - ⇝ First term in the integrand decomposition i.e. four-point function at one-loop in 4-dim
   - ⇝ its residue is a **univariate polynomial** parametrized by n_s coefficients
   - ⇝ the corresponding residue **can always be reconstructed at the cut**
   - ⇝ the **residue** is determined as in the previous case, via MPD

Integrand Reduction at Arbitrary Loops: a Theorem

Let's look at the on-shell conditions, and impose

$$D_1 = D_2 = \ldots = D_n = 0$$

1) There are no solutions $\rightarrow$ **reducible**
2) The cut has solutions $\rightarrow$ **residue Δ**
3) Finite number of solutions n_s $\rightarrow$ **Maximum Cut**

diagram	Δ	n_s	diagram	Δ	n_s
	c_0	1		$c_0 + c_1 z$	2
	$\sum_{i=0}^{3} c_i z^i$	4		$\sum_{i=0}^{3} c_i z^{\,i}$	4
	$\sum_{i=0}^{7} c_i z^{\,i}$	8		$\sum_{i=0}^{7} c_i z^{\,i}$	8

Mastrolia, G.O., Mirabella, Peraro

Examples of Applications of MPD

1. Reconstruct **all residues** in the OPP and d-dimensional One-Loop Integrand Decomposition

$$\mathcal{A} \equiv \frac{\mathcal{N}}{D_0 \ldots D_{n-1}} = \sum_{k=1}^{5} \sum_{\{i_1, \cdots, i_k\}} \frac{\Delta_{i_1 \cdots i_k}}{D_{i_1} \cdots D_{i_k}}$$

⇝ Take a rank-k polynomial in $z \equiv (x_1, x_2, x_3, x_4, \mu^2)$ as $\mathcal{N}(z)$

$$\mathcal{N}(z) = \sum_{\vec{j}} \alpha_{\vec{j}}\, z_1^{j_1}\, z_2^{j_2}\, z_3^{j_3}\, z_4^{j_4}\, z_5^{j_5}$$

⇝ Apply the recursion formula:

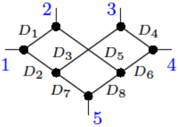
$$\mathcal{I}_{i_1 \cdots i_n} = \sum_{\kappa=1}^{n} \mathcal{I}_{i_1 \cdots i_{\kappa-1} i_{\kappa+1} i_n} + \frac{\Delta_{i_1 \cdots i_n}}{D_{i_1} \cdots D_{i_n}}$$

⇝ Read the residues:

$$\Delta_{i_1 \cdots i_5} = c_0\ ,\ \ \Delta_{i_1 \cdots i_4} = c_0 + c_1 x_4 + \mu^2 (c_2 + c_3 x_4 + \mu^2 c_4)\ ,\ \ldots$$

Examples of Applications of MPD

1. Reconstruct **all residues** in the OPP and d-dimensional One-Loop Integrand Decomposition
2. Examples of Two-loop topologies in $\mathcal{N} = 4$ SYM
   - Example: five-point $\mathcal{N} = 4$ SYM topology



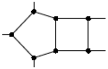
$$\mathcal{I} = \frac{c_0 + c_1(\mathsf{k}_1 \cdot p_5) + c_1(\mathsf{k}_1 \cdot p_1)}{D_1 \cdots D_8} + \sum_{i=1}^{8} \frac{\hat{c}_i}{\prod_{k \neq i} D_k}$$

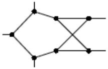
$$\mathcal{N} = \alpha + v_1 \cdot \mathsf{k}_1 + v_2 \cdot \mathsf{k}_2$$ [Carrasco, Johansson '11]

Step 1. Reducing the integrand $\mathcal{I} = \frac{\mathcal{N}}{D_1 \cdots D_8}$

Step 2. Reducing the integrand $\mathcal{I}_{i_1 \cdots i_7} = \frac{\mathcal{N}_{i_1 \cdot i_7}}{D_{i_1} \cdots D_{i_7}}$

- reduction completed after two steps ($\mathcal{N} = 4$) (Checked via the $N = N$ test)
- other topologies & $\mathcal{N} = 8$ SUGRA reduced

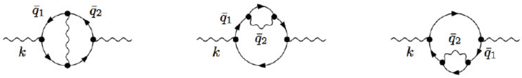




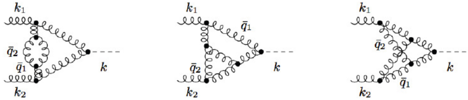
Examples of Applications of MPD

1. Reconstruct **all residues** in the OPP and d-dimensional One-Loop Integrand Decomposition
2. Examples of Two-loop topologies in $\mathcal{N} = 4$ SYM
3. Two-loop diagrams in QED and QCD processes

   ⇝ Two-loop QED corrections to the photon self energy

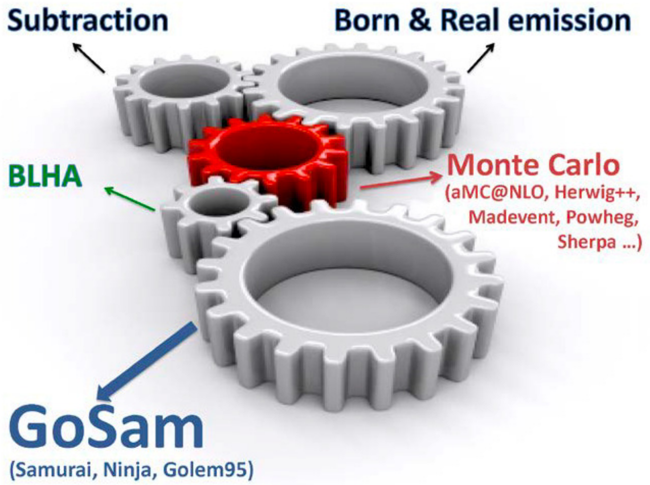


⇝ Two-loop diagrams entering the QCD corrections to $gg \to H$ in the heavy top mass approximation



Mastrolia, Mirabella, GO, Peraro

The GoSam Project



The GoSam Project

GoSam Collaboration (Updated)

N. Greiner, G. Heinrich, S. Jahn, G. Luisoni, P. Mastrolia, G.O., T. Peraro, J.Schlenk, J. F. von Soden-Fraunhofen, F. Tramontano

GoSam 1.0

"Automated One-Loop Calculations with GoSam"
Cullen, Greiner, Heinrich, Luisoni, Mastrolia, G.O., Reiter, Tramontano
Eur.Phys.J. C72 (2012) 1889 [arXiv:1111.2034]

GoSam 2.0

"GOSAM-2.0: a tool for automated one-loop calculations within the Standard Model and beyond"
Cullen, van Deurzen, Greiner, Heinrich, Luisoni, Mastrolia, Mirabella, G.O., Peraro, Schlenk, von Soden-Fraunhofen, Tramontano
Eur.Phys.J. C74 (2014) 3001 [arXiv:1404.7096]

http://gosam.hepforge.org/

Virtual corrections with GoSam

1. Algebraic Generation
    - Amplitudes generated with **Feynman diagrams**
    - Optimization: **grouping of diagrams**, smart **caching**
    - Algebra in **dimension** d, different schemes
    - Suited for QCD, EW, effective Higgs coupling and BSM models$^{(*)}$

GoSam employs a Python “wrapper” which:

$\rightsquigarrow$ generates analytic integrands from **Feynman diagrams** using QGRAF

$\rightsquigarrow$ manipulates and simplifies them with FORM

$\rightsquigarrow$ writes them into FORTRAN95 code for the reduction

Virtual corrections with GoSam

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2. Flexibility in the Reduction
   - Different reduction algorithms available at run-time
   - **Integrand-Level** and/or **Tensorial** Reduction available

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3. Selection of codes for the evaluation of the Master Integrals

OneLoop (**van Hameren**); QCDLoop (**Ellis, Zanderighi**)
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   Golem95C (**Binoth et al.**); LoopTools (**Hahn et al.**)
4. Fully interfaced within several Monte Carlo frameworks

Modular Structure → Ability to incorporate **new ideas** and **techniques**
The GoSam framework is in continuous evolution

Reduction algorithms within GoSam

Ninja

⇝ **Default in GoSam 2.0**, more **Stable** and **Fast**

Integrand-Level Reduction + Laurent Expansion

Mastrolia, Mirabella, Peraro

Golem95

⇝ **Default Rescue System**

Tensorial Reduction

Binoth, Guillet, Heinrich, Pilon, Reiter, von Soden-Fraunhofen

Samurai

⇝ **Default in GoSam 1.0**

$d-$dimensional Integrand-Level Reduction

Mastrolia, G.O., Reiter, Tramontano

GoSam – NLO Pheno Results

- J. Bellm, S. Gieseke, N. Greiner, G. Heinrich, S. Platzer, C. Reuschle and J. F. von Soden-Fraunhofen, "Anomalous coupling, top-mass and parton-shower effects in W^+W^- production," arXiv:1602.05141
- N. Greiner, S. Liebler and G. Weiglein, "Interference contributions to gluon initiated heavy Higgs production in the Two-Higgs-Doublet Model," Eur. Phys. J. C **76**, no. 3, 118 (2016)
- H. van Deurzen, R. Frederix, V. Hirschi, G. Luisoni, P. Mastrolia and G. Ossola, "Spin Polarisation of $t\bar{t}\gamma\gamma$ production at NLO+PS with GoSam interfaced to MG5_aMC," arXiv:1509.02077 [hep-ph], to appear in EPJC
- M. Chiesa, N. Greiner and F. Tramontano, "Automation of electroweak corrections for LHC processes," J. Phys. G **43**, no. 1, 013002 (2016)
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- G. Luisoni, C. Oleari, F. Tramontano, "Wbbj production at NLO with POWHEG+MiNLO", JHEP 1504 (2015) 161
- M. J. Dolan, C. Englert, N. Greiner, M. Spannowsky, "Further on up the road: hhjj production at the LHC", Phys.Rev.Lett. 112 (2014) 101802
- G. Heinrich, A. Maier, R. Nisius, J. Schlenk, J. Winter, "NLO QCD corrections to WWbb production with leptonic decays in the light of top quark mass and asymmetry measurements", JHEP 1406 (2014) 158
- T. Gehrmann, N. Greiner, and G. Heinrich, "Precise QCD predictions for the production of a photon pair in association with two jets", Phys.Rev.Lett. 111 (2013) 222002
- N. Greiner, G. Heinrich, J. Reichel, and J. F. von Soden-Fraunhofen, "NLO QCD corrections to diphoton plus jet production through graviton exchange", JHEP 1311 (2013) 028
- H. van Deurzen, G. Luisoni, P. Mastrolia, E. Mirabella, G.O. and T. Peraro, "NLO QCD corrections to Higgs boson production in association with a top quark pair and a jet", Phys.Rev.Lett. 111 (2013) 171801
- G. Cullen, H. van Deurzen, N. Greiner, G. Luisoni, P. Mastrolia, E. Mirabella, G.O., T. Peraro, and F. Tramontano, "NLO QCD corrections to Higgs boson production plus three jets in gluon fusion," Phys.Rev.Lett. 111 (2013) 131801
- S. Hoeche, J. Huang, G. Luisoni, M. Schoenherr and J. Winter, "Zero and one jet combined NLO analysis of the top quark forward-backward asymmetry," Phys.Rev. D88 (2013) 014040
- G. Luisoni, P. Nason, C. Oleari and F. Tramontano,"HW/HZ + 0 and 1 jet at NLO with the POWHEG BOX interfaced to GoSam and their merging within MiNLO", JHEP 1310 (2013) 083
- M. Chiesa, G. Montagna, L. Barze', M. Moretti, O. Nicrosini, F. Piccinini and F. Tramontano, "Electroweak Sudakov Corrections to New Physics Searches at the CERN LHC," Phys.Rev.Lett. 111 (2013) 121801
- T. Gehrmann, N. Greiner, and G. Heinrich, "Photon isolation effects at NLO in gamma gamma + jet final states in hadronic collisions," JHEP 1306, 058 (2013)
- H. van Deurzen, N. Greiner, G. Luisoni, P. Mastrolia, E. Mirabella, G.O., T. Peraro, J. F. von Soden-Fraunhofen, and F. Tramontano, "NLO QCD corrections to the production of Higgs plus two jets at the LHC," Phys. Lett. B 721, 74 (2013)
- G. Cullen, N. Greiner, and G. Heinrich, "Susy-QCD corrections to neutralino pair production in association with a jet," Eur. Phys. J. C 73, 2388 (2013)
- N. Greiner, G. Heinrich, P. Mastrolia, G. O., T. Reiter and F. Tramontano, "NLO QCD corrections to the production of W+ W- plus two jets at the LHC," Phys. Lett. B 713, 277 (2012)

NLO Automation

$$\sigma_{NLO} = \int_n \left(d\sigma^B + d\sigma^V + \int_1 d\sigma^A \right) + \int_{n+1} \left(d\sigma^R - d\sigma^A \right)$$

Monte Carlo Tools (MC) ⇝

- Tree-level Contributions
- Subtraction Terms
- Integration over phase-space

One-Loop Programs (OLP) ⇝

The values of the Virtual Contributions (at each given phase-space point)

Strategies to full NLO automation:

⇝ **MC** controls the **OLP** via Binoth Les Houches Accord interface (BLHA)

⇝ **OLP** is fully incorporated within the **MC**

Interfaces with external MC

- GoSam + MadGraph+MadDipole+MadEvent
  → ad-hoc interface [Greiner]

- GoSam + Sherpa
  → fully automated [Luisoni, Schönherr, Tramontano]

- GoSam + Powheg
  → fully automated apart from phase space generation [Luisoni, Nason, Oleari, Tramontano]

- GoSam + Herwig++/Matchbox
  → fully automated [Bellm, Gieseke, Greiner, Heinrich, Plätzer, Reuschle, von Soden-Fraunhofen]

- GoSam + mg5_aMC@NLO
  → fully automated [van Deurzen, Frederix, Hirschi, Luisoni, Mastrolia, G.O.]

$pp \to t\bar{t}\gamma\gamma$ @ NLO

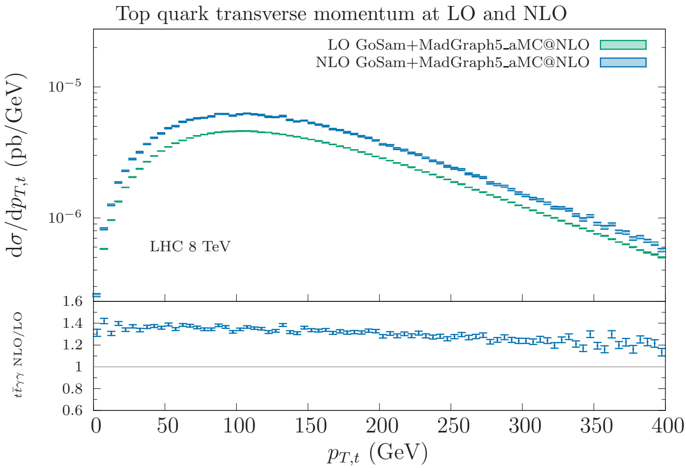
van Deurzen, Frederix, Hirschi, Luisoni, Mastrolia, G.O. (2016)

- Interface between the MG5_aMC and GoSam based on BLHA standards.
- When running the MG5_aMC interactive session, the command

```
$ set OLP GoSam
```

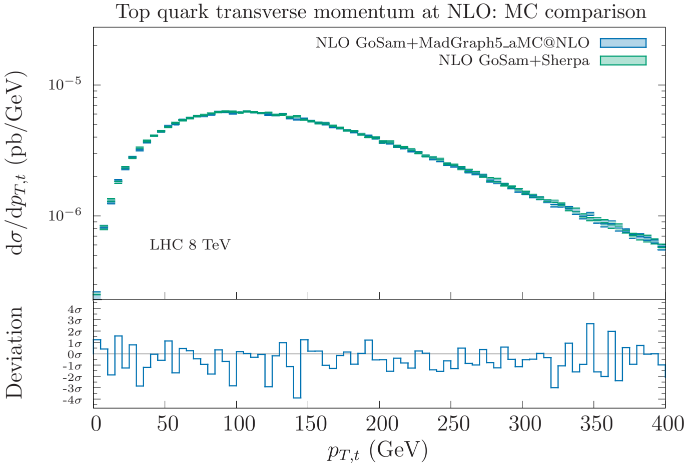
  changes the employed OLP from its default MadLoop to GoSam
- As a first application of this framework, we computed the NLO corrections to $pp \to t\bar{t}H$ and $pp \to t\bar{t}\gamma\gamma$ matched to a parton shower
- We focused on observables sensitive to the polarization of the top quarks

$pp \to t\bar{t}\gamma\gamma$ @ NLO - Validation



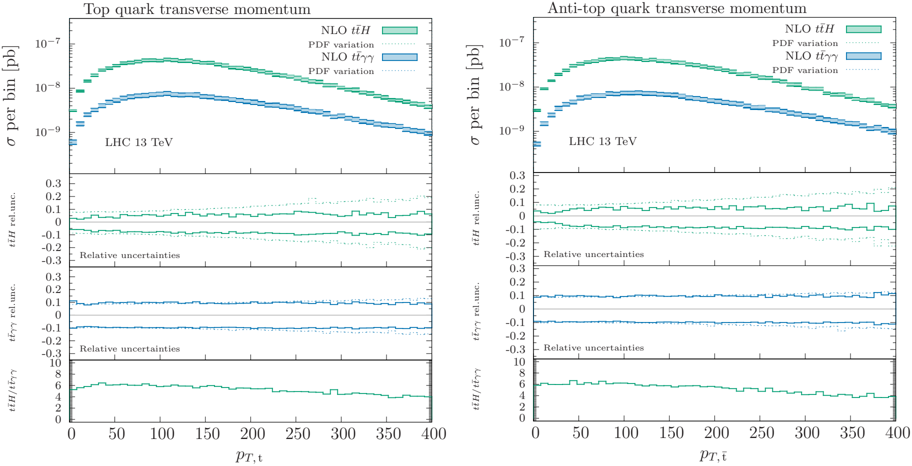
Transverse momentum of the top quark in $pp \to t\bar{t}\gamma\gamma$ for the LHC at 8 TeV LO and NLO distributions

$pp \to t\bar{t}\gamma\gamma$ @ NLO - Validation



NLO comparison between GoSam+MG5_aMC and GoSam+Sherpa

$pp \to t\bar{t}\gamma\gamma$ @ NLO - Distributions



Transverse momentum of the top quark and anti-top quark at 13 TeV.

Approximate NNLO: Hard Functions at NLO with GoSam

Broggio, Ferroglia, Pecjak, Signer, Yang (2015)

- The partonic cross section for **top pair** + **Higgs production** receives potentially large corrections from soft gluon emission
- The resummation of soft emission corrections can be carried out by means of effective field theory methods ⇝ hard functions and soft functions
- The calculation of the NLO hard function requires the evaluation of one loop amplitudes obtained separating out the various color components

$$H_{IJ}^{(1)} = \frac{1}{4} \frac{1}{\langle c_I | c_I \rangle \langle c_J | c_J \rangle} \left[\left\langle c_I \middle| \mathcal{M}_{\text{ren}}^{(0)} \right\rangle \left\langle \mathcal{M}_{\text{ren}}^{(1)} \middle| c_J \right\rangle + \left\langle c_I \middle| \mathcal{M}_{\text{ren}}^{(1)} \right\rangle \left\langle \mathcal{M}_{\text{ren}}^{(0)} \middle| c_J \right\rangle \right]$$

- By default GoSam provides squared amplitudes summed over colors ⇝ to build the hard functions we need to combine color decomposed (complex) amplitudes ⇝ This required ad-hoc modifications of the GoSam code

[see talks of A. Ferroglia at SCET 2015 and A. Broggio at SCET 2016]

A glimpse at the use of GoSam beyond NLO

1. GoSam 2.0 has been already used within NNLO calculations, for the evaluation of the real-virtual contributions...
   - V. Del Duca, C. Duhr, G. Somogyi, F. Tramontano, Z. Trocsanyi, "Higgs boson decay into b-quarks at NNLO accuracy", JHEP 1504 (2015) 036
   - J. Gao, H.X. Zhu, "Top-quark forward-backward asymmetry in e+e- annihilation at NNLO in QCD", Phys.Rev.Lett. 113 (2014) 262001.
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2. ... and for **two-loop virtual** contributions! ⇝ **Talk of Matthias Kerner**
   - S. Borowka, N. Greiner, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, U. Schubert and T. Zirke, "Higgs boson pair production in gluon fusion at NLO with full top-quark mass dependence," arXiv:1604.06447 [hep-ph].

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3. Towards GoSam 2-Loops ⇝ **Talk of Stephen Jones**
   - Extension of the GoSam **generator** to produce all two-loop Feynman diagrams for any process: algebra done automatically (FORM), uses projectors on tensor structures, depicts all contributing diagrams as output on file.
   - Interfaces to Reduze, LiteRed, FIRE.
   - Master Integrals evaluated numerically using the program SecDec-3.0

Conclusions/Outlook

- ◇ Integrand Reduction and OPP do not merely provide a *set of numerical and computational algorithms* to extract coefficients, but a *different approach to scattering amplitudes*, based on the study of the general structure of the integrand of Feynman integrals.
- ◇ At NLO, Integrand Reduction allowed to efficiently compute **Multi-Leg** and **Multi-Scales** Amplitudes. NLO automation is fully realized. There is *still room for improvement*, even at NLO.
- ◇ Will Integrand Reduction be *competitive at NNLO*? **More work is still needed**, but they might provide an **alternative path** for NNLO processes beyond $2 \to 2$. Plenty of activity of multi-loop integrands. Let's wait and see...

Thanks for your Attention!