

THRESHOLD LOGARITHMS BEYOND LEADING POWER

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Outline

- Introduction
- Threshold resummations from LP to NLP
- Next-to-soft approximation
- Hard collinear issues
- Non-abelian intricacies
- Outlook

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Stacey Melville
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Leonardo Vernazza

INTRODUCTION



Factorization

We will **focus** on processes involving **parton annihilation** into **electroweak final states** (**Drell-Yan, Higgs, di-boson** final states): very well **understood** at **LP**, **simpler** at **NLP**.

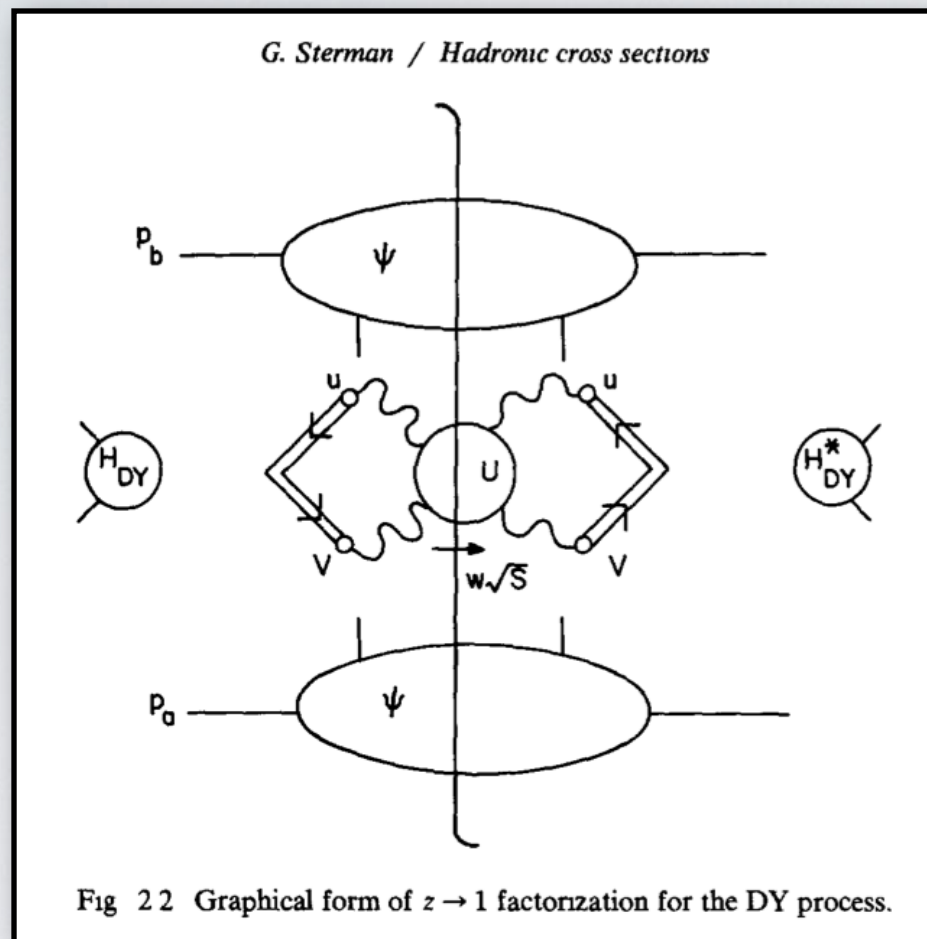
- **LP** threshold **resummation** is based on **factorization**: the **Mellin-space** partonic cross section reads

$$\omega(N, \epsilon) = |H_{\text{DY}}|^2 \psi(N, \epsilon)^2 U(N) + \mathcal{O}\left(\frac{1}{N}\right).$$

- **Collinear** poles can be **subtracted** with suitable parton distributions,

$$\hat{\omega}_{\overline{\text{MS}}}(N) \equiv \frac{\omega(N, \epsilon)}{\phi_{\overline{\text{MS}}}(N, \epsilon)^2}$$

- Each factor in ω obeys **evolution equations** near **threshold**, leading to **exponentiation**.



The original factorization near threshold

- **Real** and **virtual** contributions can be treated **separately**.

$$\psi_R(N, \epsilon) = \exp \left\{ \int_0^1 dz \frac{z^{N-1}}{1-z} \int_z^1 \frac{dy}{1-y} \kappa_\psi(\bar{\alpha}((1-y)^2 Q^2), \epsilon) \right\}.$$

Exponentiation

A well-established formalism exists for **distributions** in processes that are **electroweak at tree level** (Gardi, Grunberg 07). For an observable **r** vanishing in the two-jet limit

$$\frac{d\sigma}{dr} = \delta(r) [1 + \mathcal{O}(\alpha_s)] + C_R \frac{\alpha_s}{\pi} \left\{ \left[-\frac{\log r}{r} + \frac{b_1 - d_1}{r} \right]_+ + \mathcal{O}(r^0) \right\} + \mathcal{O}(\alpha_s^2)$$

The Mellin (Laplace) transform, $\sigma(N) = \int_0^1 dr (1-r)^{N-1} \frac{d\sigma}{dr}$

exhibits **log N** singularities that can be organized in **exponential form**

$$\sigma(\alpha_s, N, Q^2) = H(\alpha_s) \mathcal{S}(\alpha_s, N, Q^2) + \mathcal{O}(1/N)$$

where the exponent of the '**Sudakov factor**' is in turn a Mellin transform

$$\mathcal{S}(\alpha_s, N, Q^2) = \exp \left\{ \int_0^1 \frac{dr}{r} \left[(1-r)^{N-1} - 1 \right] \mathcal{E}(\alpha_s, r, Q^2) \right\}$$

and the general form of the **kernel** is

$$\mathcal{E}(\alpha_s, r, Q^2) = \int_{r^2 Q^2}^{r Q^2} \frac{d\xi^2}{\xi^2} A(\alpha_s(\xi^2)) + B(\alpha_s(r Q^2)) + D(\alpha_s(r^2 Q^2))$$

where **A** is the **cusp** anomalous dimension, and **B** and **D** have **distinct physical characters**.

Resummation

A classic way to **organize** threshold logarithms in terms of the **Mellin (Laplace) transform** of the momentum space cross section (**Catani et al. 93**) is to write

$$\begin{aligned} d\sigma(\alpha_s, N) &= \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{k=0}^{2n} c_{nk} \log^k N + \mathcal{O}(1/N) \\ &= H(\alpha_s) \exp \left[\log N g_1(\alpha_s \log N) + g_2(\alpha_s \log N) + \alpha_s g_3(\alpha_s \log N) + \dots \right] + \mathcal{O}(1/N) \end{aligned}$$

This displays the main **features of threshold resummation**

- 🎤 **Predictive:** a **k**-loop calculation determines **g_k** and thus a whole **tower** of logarithms to all orders in perturbation theory.
- 🎤 **Effective:**
 - the **range of applicability** of perturbation theory is **extended** (finite order: **α_s log²N** small. NLL resummed: **α_s** small);
 - the renormalization **scale dependence** is naturally **reduced**.
- 🎤 **Theoretically interesting:** resummation **ambiguities** related to the **Landau pole** give access to non-perturbative **power-suppressed corrections**.
- 🎤 **Well understood:**
 - **NLL** Sudakov resummations **exist** for most **inclusive** observables at hadron colliders, **NNLL** and approximate **N³LL** in simple cases.

FROM LEADING
TO NEXT-TO-LEADING POWER

FROM LEADING TO NEXT-TO-LEADING POWER



More logarithms

- **Threshold logarithms** are associated with kinematic variables ξ that **vanish** at **Born level** and get **corrections** that are **enhanced** because **phase space** for real radiation is **restricted** near **partonic** threshold: examples are $1 - T$, $1 - M^2/\hat{s}$, $1 - x_{BJ}$.
- At **leading power** in the threshold variable ξ logarithms are **directly related** to **soft** and **collinear divergences**: real radiation is proportional to factors of

$$\frac{1}{\xi^{1+p\epsilon}} = -\frac{1}{p\epsilon} \delta(\xi) + \left(\frac{1}{\xi}\right)_+ - p\epsilon \left(\frac{\log \xi}{\xi}\right)_+ + \dots$$

Cancels virtual IR poles

Leading power threshold logs

- **Beyond** the **leading power**, $1/\xi$, the perturbative cross section takes the form

$$\frac{d\sigma}{d\xi} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{2n-1} \left[c_{nm}^{(-1)} \left(\frac{\log^m \xi}{\xi}\right)_+ + c_n^{(\delta)} \delta(\xi) + c_{nm}^{(0)} \log^m \xi + \dots \right]$$

Resummed to high accuracy

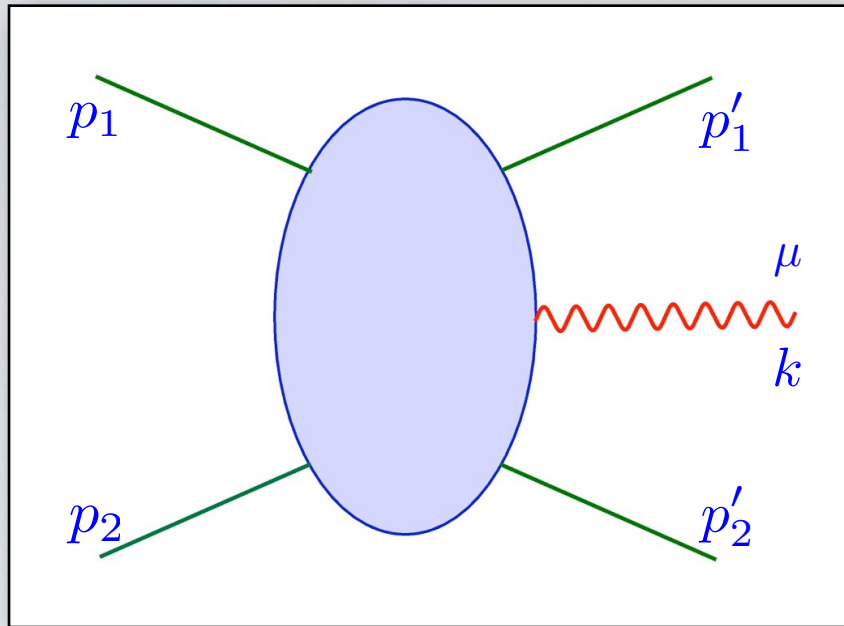
All-order structure in some cases

NLP threshold logs

- The **structure** of **NLP** threshold logarithms may be understood to **all orders**.

The LBKD Theorem

The **earliest evidence** that infrared effects can be **controlled** at **NLP** is **Low's theorem** (**Low 58**)



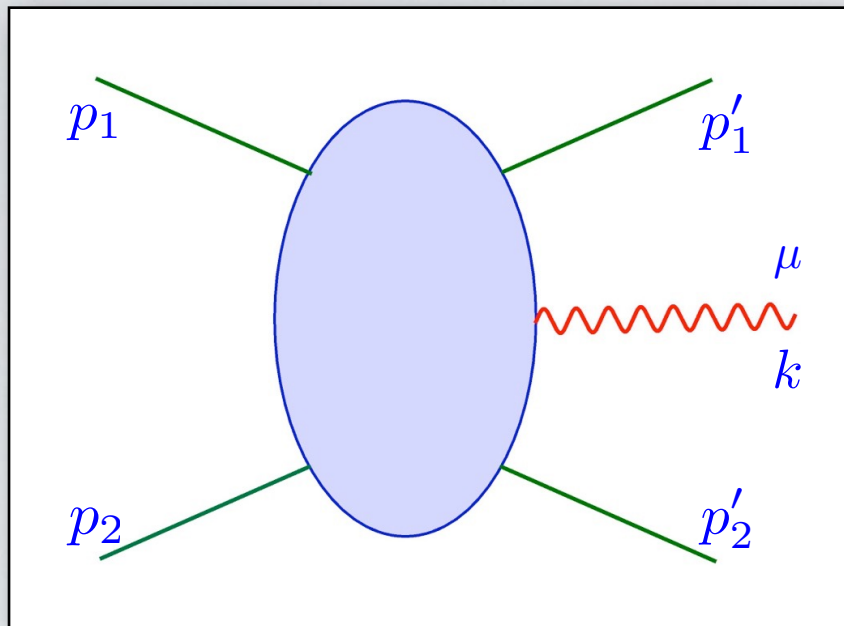
A radiative matrix element

$$M_\mu = e \left(\frac{p_{1\mu}'}{p_1' \cdot k} - \frac{p_{1\mu}}{p_1 \cdot k} \right) T(\nu, \Delta) \\ + e \left(\frac{p_{1\mu}' p_{2\mu}' \cdot k}{p_1' \cdot k} - p_{2\mu}' + \frac{p_{1\mu} p_{2\mu} \cdot k}{p_1 \cdot k} - p_{2\mu} \right) \frac{\partial T(\nu, \Delta)}{\partial \nu} + O(k),$$

Low's original expression for the radiative matrix element

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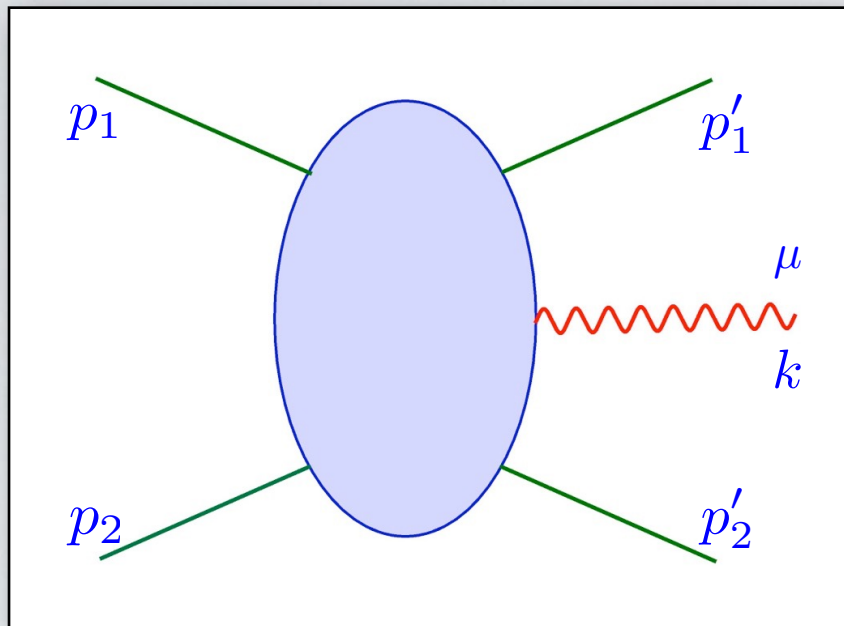
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Next-to-eikonal contribution

The **radiative matrix element** for the emission of a (next-to-) soft photon is **determined** by the **Born amplitude** **T** and its **first derivative** with respect to external momenta.

- Low's result established for a **single charged scalar** particle, follows from **gauge invariance**.
- It **generalizes** the well known properties of soft emissions in the **eikonal approximation**.
- The theorem was **extended** by (Burnett, Kroll 68) to particles with **spin**.
- The **LBK** theorem applies to **massive particles** and uses the **mass** as a **collinear cutoff**.
- It was **extended** to **massless** particles by (Del Duca 90), as discussed **below**.

Modified DGLAP

An **important source** of known **NLP** logarithms is the **DGLAP anomalous dimension**. Non-trivial **connections** between **LP** and **NLP** logarithms in **DGLAP** were **uncovered** (**Moch, Vermaseren, Vogt 08**) and made **systematic** (**Dokshitzer, Marchesini, Salam 08**).

Conventional DGLAP for a quark distribution reads

$$\mu^2 \frac{\partial}{\partial \mu^2} q(x, \mu^2) = \int_x^1 \frac{dz}{z} q\left(\frac{x}{z}, \mu^2\right) P_{qq}(z, \alpha_s(\mu^2)).$$



$$\mu^2 \frac{\partial}{\partial \mu^2} \tilde{q}(N, \mu^2) = \gamma_N(\alpha_s(\mu^2)) \tilde{q}(N, \mu^2),$$

The **large-N** behavior of the anomalous dimension is **single-logarithmic** in the MS scheme. **NLP** terms **suppressed** by **N** are **related** to **LP**

$$\begin{aligned} \gamma_N(\alpha_s) = & -A(\alpha_s) \ln \bar{N} + B_\delta(\alpha_s) \\ & - C_\gamma(\alpha_s) \frac{\ln \bar{N}}{N} + D_\gamma(\alpha_s) \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right), \end{aligned}$$

MVV relations

$$C_1 = 0, \quad C_2 = 4C_F A_1, \quad C_3 = 8C_F A_2. \quad (3.12)$$

Especially the relation for C_3 is very suggestive and seems to call for a structural explanation.

These relation **extend** to the function **D**, and recursively **to all orders: modified** splitting functions can be **defined** which **vanish** at **large x beyond one loop**.



Modified DGLAP

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Modified DGLAP

DMS, with refinements implied by (Basso, Korchemsky 06) propose to **modify DGLAP** as

$$\mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) = \int_x^1 \frac{dz}{z} \psi\left(\frac{x}{z}, z^\sigma \mu^2\right) \mathcal{P}\left(z, \alpha_s\left(\frac{\mu^2}{z}\right)\right).$$

applying to **both PDF's** and **fragmentation**, with $\sigma = \pm 1$ respectively, and the **same kernel** (Gribov-Lipatov reciprocity). The resulting kernel \mathcal{P} and is claimed to **vanish** as $z \rightarrow 1$ **beyond one loop** in the “physical” **MC** scheme where $\alpha_s = \gamma_{\text{cusp}}$. Therefore

$$\mathcal{P}(z, \alpha_s) = \frac{A(\alpha_s)}{(1-z)_+} + B_\delta(\alpha_s)\delta(1-z) + \mathcal{O}(1-z).$$

The modified equation **cannot be diagonalized** by **Mellin** transform: it must be solved **by iteration**, using a formal translation operator

$$\begin{aligned} \mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) &= \int_x^1 \frac{dz}{z} e^{-\ln z \left(\beta(\alpha_s) \frac{\partial}{\partial \alpha_s} - \sigma \frac{\partial}{\partial \ln \mu^2} \right)} \\ &\quad \times \psi\left(\frac{x}{z}, \mu^2\right) \mathcal{P}(z, \alpha_s(\mu^2)), \end{aligned}$$

In practice, this procedure **constructs** a **modified kernel** where high-order terms are **generated by shifts** of lower-orders

Towards systematics

The problem of **NLP** threshold logarithms has been of interest for a **long time**, and several **different approaches** have been proposed. Recent years have seen a **resurgence of interest**, both from a **theoretical** point of view and for **phenomenology**.

- 🔊 **Early attempts** include a study of the impact of **NLP** logs on the **Higgs** cross section by **Kraemer, Laenen, Spira (98)**; work on **F_L** by **Akhoury and Sterman (99)** (logs without plus distributions are however **leading**) and work by **Grunberg et al. (07-09)** on **DIS**.
- 🔊 **Important results** can be obtained by using **physical kernels** (**Vogt et al. 09-14**) which are conjectured to be **single-logarithmic** at large **z** , which poses **constraints** on their **factorized** expression. Note in particular a **recent application** to **Higgs** production by **De Florian, Mazzitelli, Moch, Vogt (14)**.
- 🔊 **Useful approximations** can be obtained by combining **constraints** from **large N** with **high-energy** constraints for **$N \sim 1$** and **analyticity** (**Ball, Bonvini, Forte, Marzani, Ridolfi, 13**), together with **phase space** refinements.
- 🔊 **SCET techniques** are **well-suited** to the problem: see thorough **one-loop** analysis in (**Larkoski, Neill, Stewart, 15**); see also (**Kolodrubetz, Moult, Neill, Stewart, 16**).
- 🔊 A lot of recent **formal work** on the behavior of **gauge** and **gravity** scattering **amplitudes beyond the eikonal** limit was triggered by a link to **asymptotic symmetries** of the **S** matrix (many authors from **A(ndy Strominger)** to **Z(vi Bern)**, **14-15**).

ISOLDE

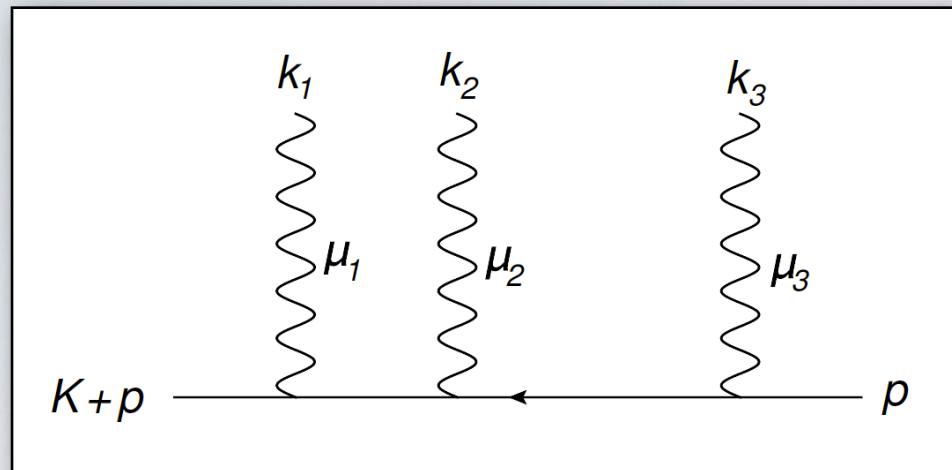


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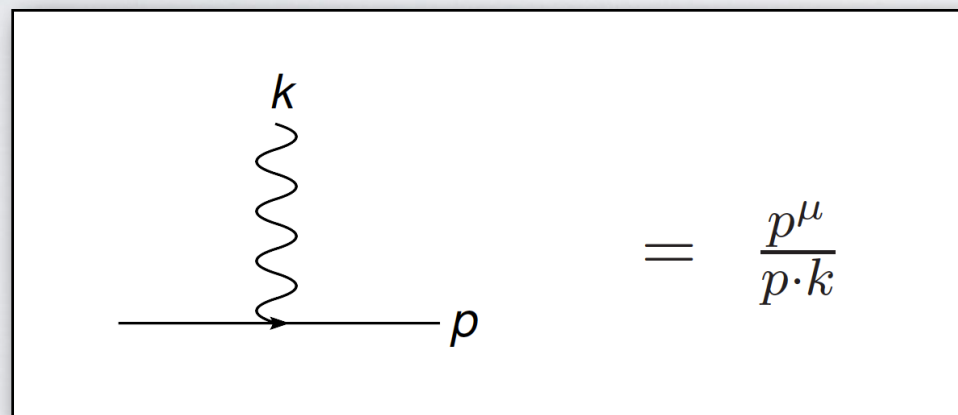


Mild und leise wie er lächelt ...

On the soft approximation



A fast particle emitting soft photons



Eikonal Feynman rule

- Taking the **soft approximation** at **leading power** on emissions from an **energetic** (or very **massive**) particle yields a set of **simplified Feynman rules**.
- These rules correspond to emissions from a **Wilson line** oriented **along the trajectory** of the energetic particle, in the **same color** representation.

$$\Phi_n(\lambda_2, \lambda_1) = P \exp \left[ig \int_{\lambda_1}^{\lambda_2} d\lambda n \cdot A(\lambda n) \right] .$$

- The results **do not depend on the energy** and **spin** of the emitter, only on its **direction** and **color charge**.
- **Physically**, we are **neglecting** the **recoil** of the emitter: the only effect of interaction with soft radiation is that the **emitter acquires a phase**.
- The **soft** limit of a **multi-particle** amplitude is a **correlator of Wilson lines**

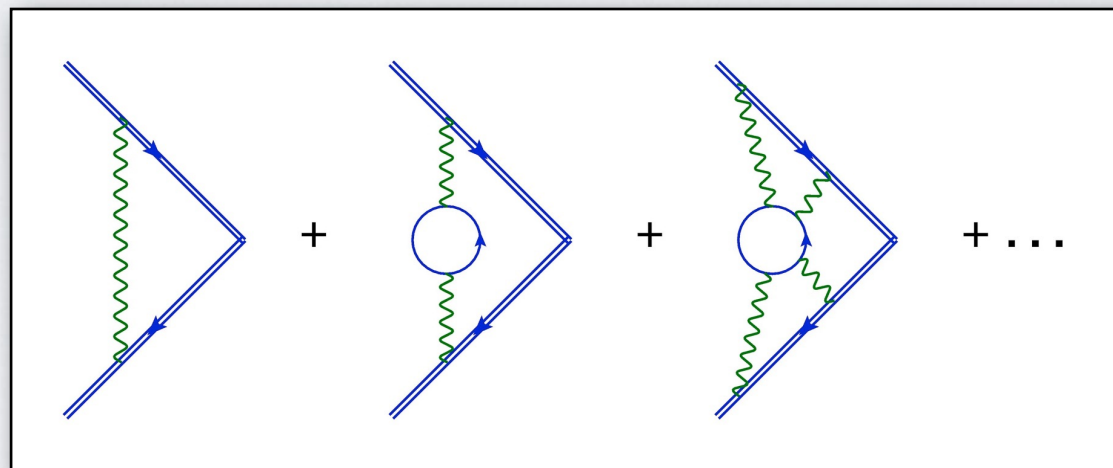
Infrared exponentiation

All correlators of Wilson lines, regardless of shape, resum in exponential form.

$$S_n \equiv \langle 0 | \Phi_1 \otimes \dots \otimes \Phi_n | 0 \rangle = \exp(\omega_n)$$

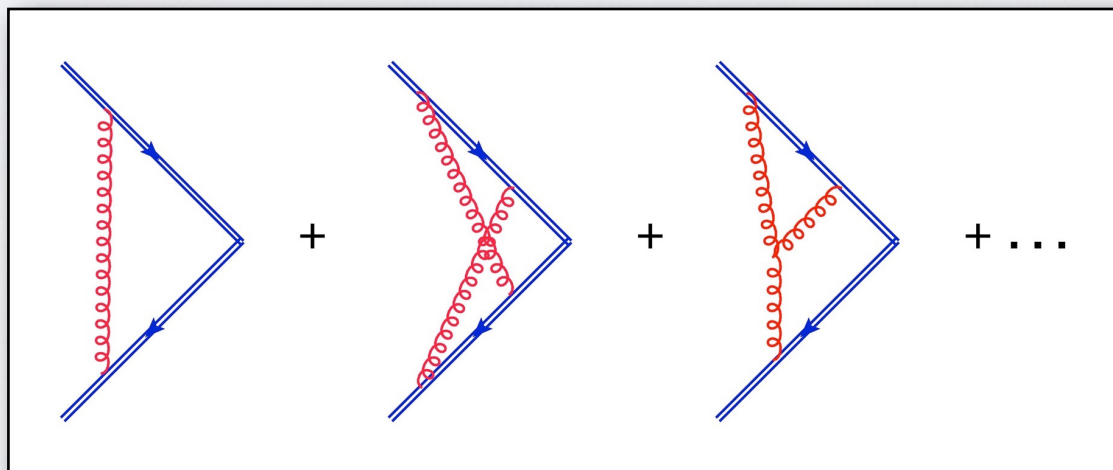
Diagrammatic rules exist to compute directly the logarithm of the correlators.

$$\omega_{2,\text{QED}} =$$



Only **connected** photon **subdiagrams** contribute to the logarithm.

$$\omega_{2,\text{QCD}} =$$




Only gluon **subdiagrams** which are **two-eikonal irreducible** contribute to the logarithm. They have **modified color factors**.

For **eikonal form factors**, these diagrams are called **webs** (Gatheral; Frenkel, Taylor; Sterman).


Beyond the eikonal

The **soft expansion** can be **organized beyond leading power** using either path integral techniques (Laenen, Stavenga, White 08) or diagrammatic techniques (Laenen, LM, Stavenga, White 10). The basic **idea** is **simple**, but the combinatorics **cumbersome**. For **spinors**


$$\frac{\not{p} + \not{k}}{2p \cdot k + k^2} \gamma^\mu u(p) = \left[\frac{p^\mu}{p \cdot k} + \frac{\not{k} \gamma^\mu}{2p \cdot k} - k^2 \frac{p^\mu}{2(p \cdot k)^2} \right] u(p) + \mathcal{O}(k)$$



Eikonal



NE, spin-dependent



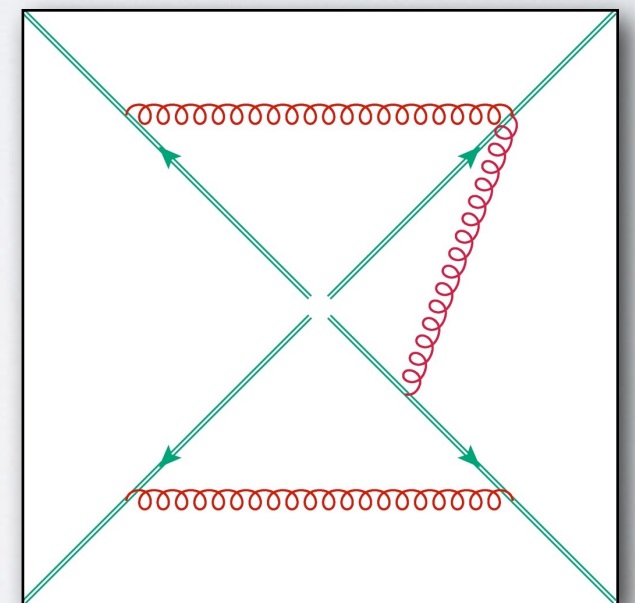
NE, spin-independent

- A class of **factorizable** contributions **exponentiate** via **NE webs**

$$\mathcal{M} = \mathcal{M}_0 \exp \left[\sum_{D_{\text{eik}}} \tilde{C}(D_{\text{eik}}) \mathcal{F}(D_{\text{eik}}) + \sum_{D_{\text{NE}}} \tilde{C}(D_{\text{NE}}) \mathcal{F}(D_{\text{NE}}) \right] .$$

- **Feynman rules** exist for the **NE** exponent, including “**seagull**” vertices.

$$\mathcal{M} = \mathcal{M}_0 \exp [\mathcal{M}_{\text{eik}} + \mathcal{M}_{\text{NE}}] (1 + \mathcal{M}_r) + \mathcal{O}(\text{NNE}) .$$



A next-to-eikonal web

- **Non-factorizable** contributions involve **single gluon emission** from inside the **hard function**, and must be studied using **LBDK's theorem**.

Next-to-eikonal lines

Exponentiating next-to-soft contribution can be organised into next-to-eikonal lines (Laenen, Stavenga, White 08), which generate the appropriate Feynman rules. In momentum space

$$\begin{aligned} \tilde{F}(\beta) = \exp & \left[\int \frac{d^d k}{(2\pi)^d} \tilde{A}_\mu(k) \left(-\frac{\beta^\mu}{\beta \cdot k} + \frac{k^\mu}{2\beta \cdot k} - k^2 \frac{\beta^\mu}{2(\beta \cdot k)^2} - \frac{ik_\nu \Sigma^{\nu\mu}}{p \cdot k} \right) \right. \\ & + \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \tilde{A}_\mu(k) \tilde{A}_\nu(l) \left(\frac{\eta^{\mu\nu}}{2\beta \cdot (k+l)} - \frac{\beta^\nu l^\mu \beta \cdot k + \beta^\mu k^\nu \beta \cdot l}{2(\beta \cdot l)(\beta \cdot k)[\beta \cdot (k+l)]} \right. \\ & \left. \left. + \frac{(k \cdot l)\beta^\mu \beta^\nu}{2(\beta \cdot l)(\beta \cdot k)[\beta \cdot (k+l)]} - \frac{\Sigma^{\mu\nu}}{2p \cdot k} \right) \right]. \end{aligned}$$

- One recognises the eikonal Feynman rule and the NE scalar propagator corrections.
- At NLP level spin dependence enters: Σ is the spin angular momentum operator.
- Seagull vertices, with spin dependence, arise from the cancellation of propagators.
- Complicated momentum-dependent two-gluon correlations arise at next-to-soft level.
- This expression applies to semi-infinite lines, it is easily generalised in coordinate space.

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$$\begin{aligned} \tilde{F}(\beta) = \exp & \left[\int \frac{d^d k}{(2\pi)^d} \tilde{A}_\mu(k) \left(-\frac{\beta^\mu}{\beta \cdot k} + \frac{k^\mu}{2\beta \cdot k} - k^2 \frac{\beta^\mu}{2(\beta \cdot k)^2} - \frac{ik_\nu \Sigma^{\nu\mu}}{p \cdot k} \right) \right. \\ & + \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \tilde{A}_\mu(k) \tilde{A}_\nu(l) \left(\frac{\eta^{\mu\nu}}{2\beta \cdot (k+l)} - \frac{\beta^\nu l^\mu \beta \cdot k + \beta^\mu k^\nu \beta \cdot l}{2(\beta \cdot l)(\beta \cdot k)[\beta \cdot (k+l)]} \right. \\ & \left. \left. + \frac{(k \cdot l)\beta^\mu \beta^\nu}{2(\beta \cdot l)(\beta \cdot k)[\beta \cdot (k+l)]} - \frac{\Sigma^{\mu\nu}}{2p \cdot k} \right) \right]. \end{aligned}$$

Eikonal

Seagulls

NE, spin-independent

NE, spin-dependent

- One recognises the eikonal Feynman rule and the NE scalar propagator corrections.
- At NLP level spin dependence enters: Σ is the spin angular momentum operator.
- Seagull vertices, with spin dependence, arise from the cancellation of propagators.
- Complicated momentum-dependent two-gluon correlations arise at next-to-soft level.
- This expression applies to semi-infinite lines, it is easily generalised in coordinate space.

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Eikonal

Two-gluon correlations

Seagulls

NE, spin-independent

Two-gluon correlations

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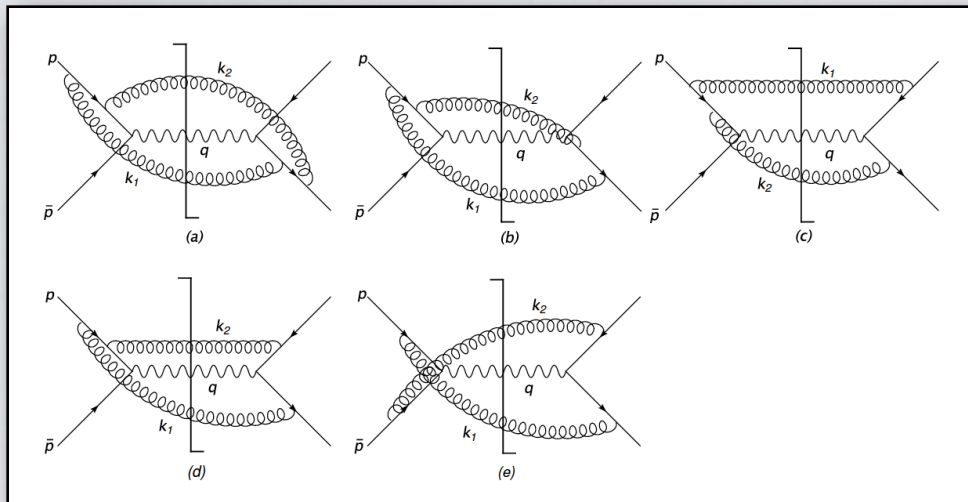
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Double real two-loop Drell-Yan

Multiple real emission contributions to EW annihilation processes involve only factorizable contributions. NE Feynman rules can be tested this level.

Defining the Drell-Yan K-factor as

$$K^{(n)}(z) = \frac{1}{\sigma^{(0)}} \frac{d\sigma^{(n)}(z)}{dz},$$



Real emission Feynman diagrams for the abelian part of the NNLO K-factor.

As a test, we (re)computed the C_F^2 part of K at NNLO from ordinary Feynman diagrams, and then using NE Feynman rules, finding complete agreement. As expected, plus distributions arise from the eikonal approximation.

Next-to-eikonal terms arise from single-gluon corrections: seagull-type contributions vanish for the inclusive cross section.

$$K_{\text{NE}}^{(2)}(z) = \left(\frac{\alpha_s}{4\pi} C_F \right)^2 \left[-\frac{32}{\epsilon^3} \mathcal{D}_0(z) + \frac{128}{\epsilon^2} \mathcal{D}_1(z) - \frac{128}{\epsilon^2} \log(1-z) \right. \\ \left. - \frac{256}{\epsilon} \mathcal{D}_2(z) + \frac{256}{\epsilon} \log^2(1-z) - \frac{320}{\epsilon} \log(1-z) \right. \\ \left. + \frac{1024}{3} \mathcal{D}_3(z) - \frac{1024}{3} \log^3(1-z) + 640 \log^2(1-z) \right],$$

The abelian part of the NNLO K-factor from real emission, omitting constants

BRÜNNHILDE



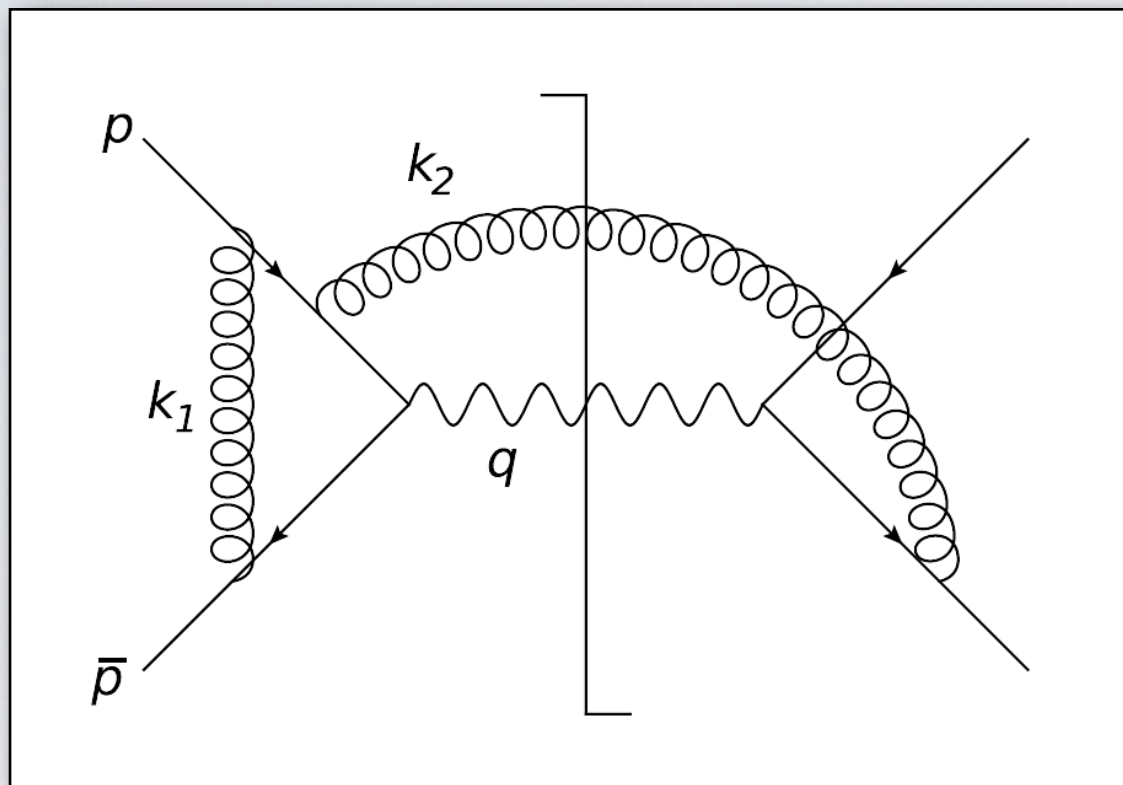
BRÜNNHILDE



Starke Scheite schichtet mir dort ...

A collinear problem

Non-factorizable contributions start at **NNLO**. For **massive** particles they can be traced to the **original LBK** theorem. For **massless** particles a **new contribution** to **NLP** logs **emerges**.

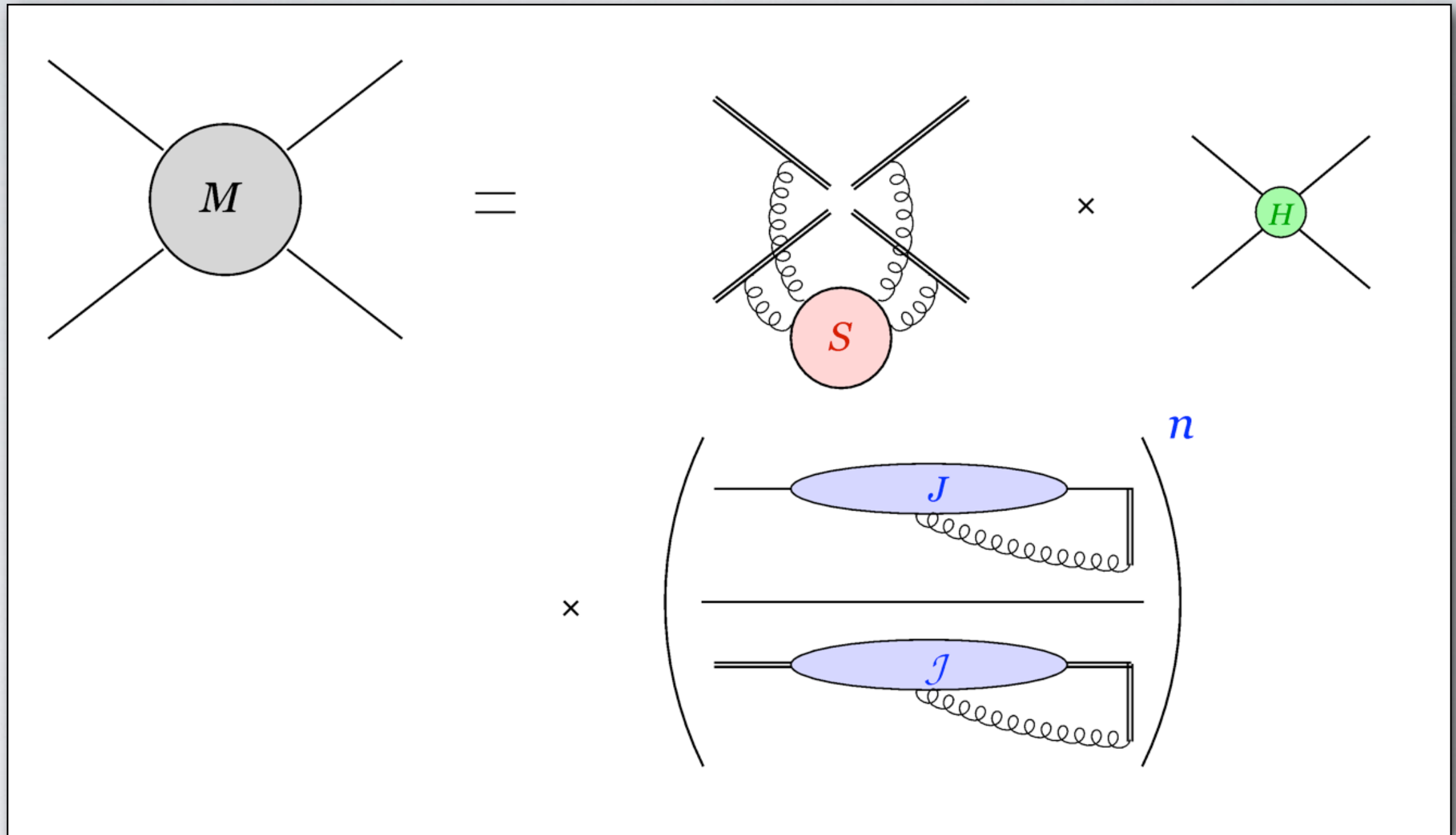


A Feynman diagram containing a collinear enhancement

- Gluon k_2 is **always** (next-to) **soft** for **EW** annihilation **near threshold**.
- When k_1 is (next-to) **soft** all logs are **captured** by **NE** rules.
- Contributions with k_1 **hard** and **collinear** are **missed** by the soft expansion.
- The **collinear pole** interferes with **soft emission** and generates **NLP** logs.
- The problem **first arises** at **NNLO**

- These contributions are **missed** by the **LBK** theorem: it applies to an **expansion** in E_k/m .
- They can be **analyzed** using the **method of regions**: the relevant **factor** is $(p \cdot k_2)^{-\epsilon}/\epsilon$.
- They **cause** the **breakdown** of **next-to-soft theorems** for amplitudes **beyond tree level**.
 \Rightarrow the **soft** expansion and the limit $\epsilon \rightarrow 0$ **do not commute**.
- They **require** an **extension** of **LBK** to $m^2/Q < E_k < m$. It was **provided** by **Del Duca (90)**.

LP factorization: pictorial



A pictorial representation of soft-collinear factorization for fixed-angle scattering amplitudes

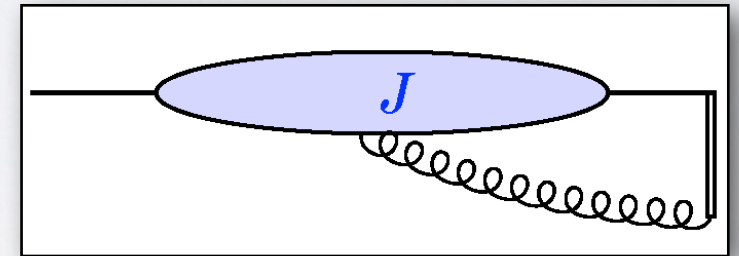
LP factorization: operators

The precise **functional form** of this graphical factorization is

$$\mathcal{M}_L(p_i/\mu, \alpha_s(\mu^2), \epsilon) = \mathcal{S}_{LK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon) H_K\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right) \\ \times \prod_{i=1}^n \left[J_i\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) / \mathcal{J}_i\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right) \right],$$

Here we introduced dimensionless **four-velocities** $\beta_i^\mu = Q p_i^\mu$, $\beta_i^2 = 0$, and **factorization vectors** n_i^μ , $n_i^2 \neq 0$ to define the jets,

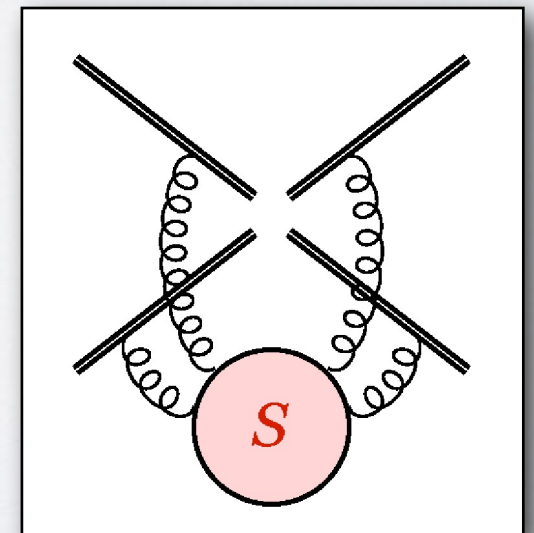
$$J\left(\frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) u(p) = \langle 0 | \Phi_n(\infty, 0) \psi(0) | p \rangle.$$



where Φ_n is the **Wilson line** operator along the direction n^μ .

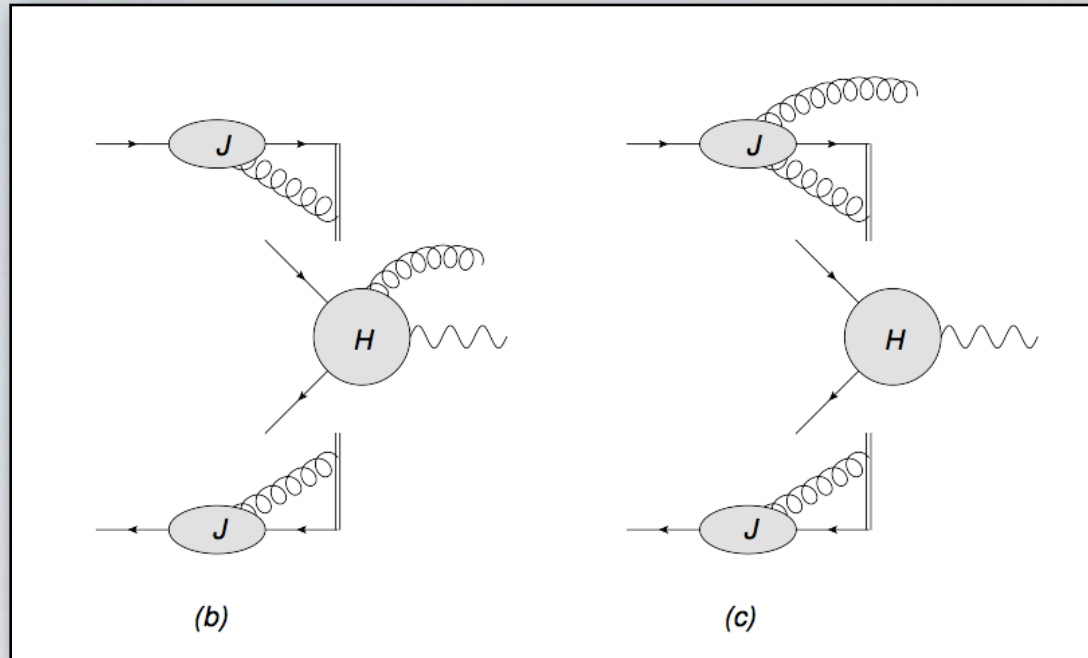
The **soft function S** is a **matrix**, mixing the available color tensors. It is defined by a correlator of **Wilson lines**.

$$(c_L)_{\{a_k\}} \mathcal{S}_{LK}(\beta_i \cdot \beta_j, \epsilon) = \langle 0 | \prod_{k=1}^n [\Phi_{\beta_k}(\infty, 0)]_{a_k}^{b_k} | 0 \rangle (c_K)_{\{b_k\}},$$



where the c_L are a **basis** of **color tensors** for the process at hand.

NLP factorization: a new jet



Soft radiation can arise either from the jets or from the hard function

$$\mathcal{A}_\mu \epsilon^\mu(k) = \mathcal{A}_\mu^J \epsilon^\mu(k) + \mathcal{A}_\mu^H \epsilon^\mu(k),$$

The amplitude for emission from the jets can be precisely defined in terms of a new jet function

$$\mathcal{A}_\mu^J = \sum_{i=1}^2 H(p_i - k; p_j, n_j) J_\mu(p_i, k, n_i) \prod_{j \neq i} J(p_j, n_j) \equiv \sum_{i=1}^2 \mathcal{A}_\mu^{J_i}.$$

Factorized contributions to the radiative amplitude

$$J_\mu(p, n, k, \alpha_s(\mu^2), \epsilon) u(p) = \int d^d y e^{-i(p-k) \cdot y} \langle 0 | \Phi_n(y, \infty) \psi(y) j_\mu(0) | p \rangle,$$

defines the radiative jet.

- At tree level the radiative jet displays the expected dependence on spin.
- Dependence on the gauge vector n^μ starts at loop level: simplifications arise for $n^2 = 0$.

$$\begin{aligned} J^{\nu(0)}(p, n, k) &= \frac{\not{k} \gamma^\nu}{2p \cdot k} - \frac{p^\nu}{p \cdot k} \\ &= -\frac{p^\nu}{p \cdot k} + \frac{k^\nu}{2p \cdot k} - \frac{i k_\alpha \Sigma^{\alpha\mu}}{2p \cdot k}. \end{aligned}$$

Beyond Low's theorem

A slightly modified version of Del Duca's result gives the radiative amplitude in terms of the non-radiative one, its derivatives, and the two “jet” functions.

$$\mathcal{A}^\mu(p_j, k) = \sum_{i=1}^2 \left\{ q_i \left(\frac{(2p_i - k)^\mu}{2p_i \cdot k - k^2} + G_i^{\nu\mu} \frac{\partial}{\partial p_i^\nu} \right) + G_i^{\nu\mu} \left[\frac{J_\nu(p_i, k, n_i)}{J(p_i, n_i)} - q_i \frac{\partial}{\partial p_i^\nu} \left(\ln J(p_i, n_i) \right) \right] \right\} \mathcal{A}(p_i; p_j) .$$

The tensors $G^{\mu\nu}$ project out the eikonal contribution present in the first term.

$$\eta^{\mu\nu} = G^{\mu\nu} + K^{\mu\nu}, \quad K^{\mu\nu}(p; k) = \frac{(2p - k)^\nu}{2p \cdot k - k^2} k^\mu ,$$

The factorized expression for the radiative amplitude can be simplified.

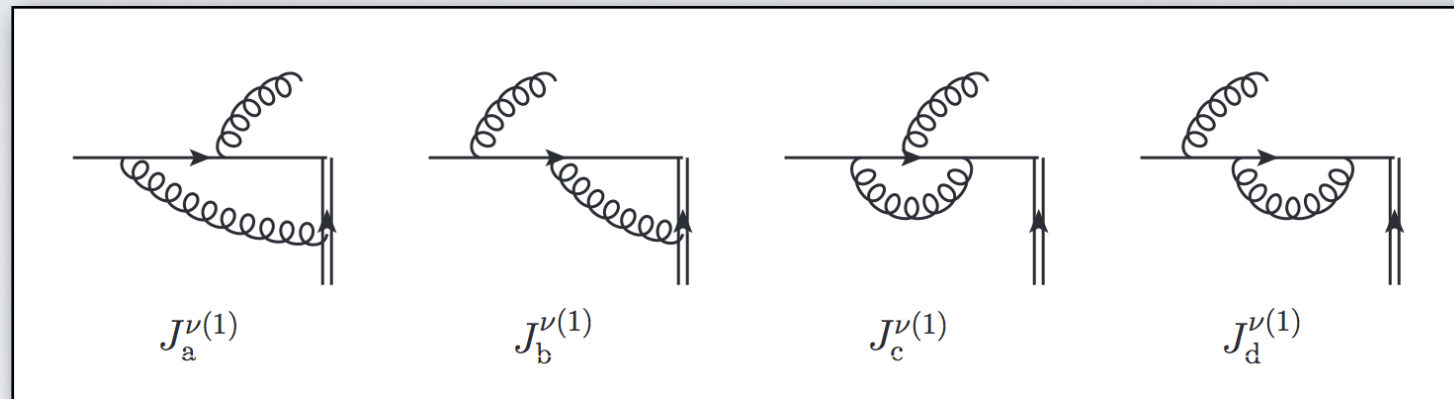
- The jet factor is RG invariant: it can be computed in bare perturbation theory.
- With this choice one can use that $J(p, n) = 1$ for $n^2 = 0$: it is a pure counterterm.
- The choice of reference vectors is then physically motivated (and confirmed by a complete analysis using the method of regions): we take $n_1 = p_2$ and $n_2 = p_1$.

$$\mathcal{A}^\mu(p_j, k) = \sum_{i=1}^2 \left(q_i \frac{(2p_i - k)^\mu}{2p_i \cdot k - k^2} + q_i G_i^{\nu\mu} \frac{\partial}{\partial p_i^\nu} + G_i^{\nu\mu} J_\nu(p_i, k) \right) \mathcal{A}(p_i; p_j) .$$

For general amplitudes, a full subtraction of the residual n dependence should be aimed at.

The one-loop radiative jet

To achieve **NNLO accuracy** at **NLP**, we need the **radiative jet** function at **one loop**.
As a **test**, we compute the C_F^2 contributions. They **simplify** considerably for $n^2 = 0$.



Abelian-like Feynman diagrams for the bare one-loop radiative jet

The result has a **simple structure**, with characteristic **scale** and **spin** dependence

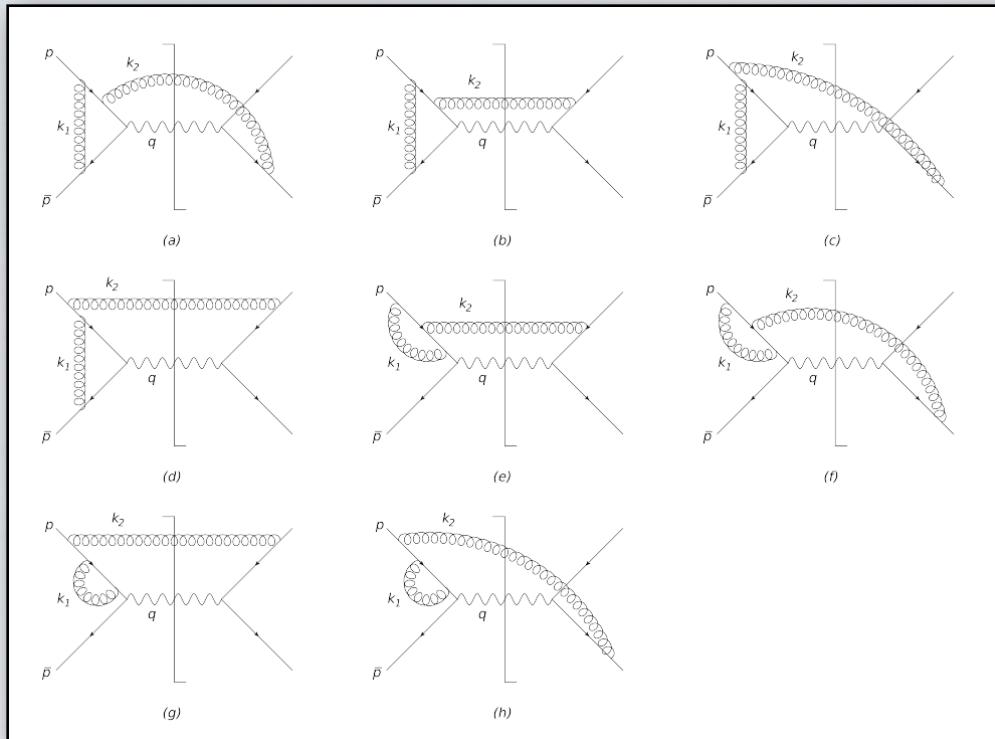
$$J^{\nu(1)}(p, n, k; \epsilon) = (2p \cdot k)^{-\epsilon} \left[\left(\frac{2}{\epsilon} + 4 + 8\epsilon \right) \left(\frac{n \cdot k}{p \cdot k} \frac{p^\nu}{p \cdot n} - \frac{n^\nu}{p \cdot n} \right) - (1 + 2\epsilon) \frac{i k_\alpha \Sigma^{\alpha\nu}}{p \cdot k} \right. \\ \left. + \left(\frac{1}{\epsilon} - \frac{1}{2} - 3\epsilon \right) \frac{k^\nu}{p \cdot k} + (1 + 3\epsilon) \left(\frac{\gamma^\nu \not{n}}{p \cdot n} - \frac{p^\nu \not{k} \not{n}}{p \cdot k p \cdot n} \right) \right] + \mathcal{O}(\epsilon^2, k),$$

As a test, it **obeys** the **Ward identity**

$$k^\mu J_\mu(p, n, k, \alpha_s(\mu^2), \epsilon) = q J(p, n, \alpha_s(\mu^2), \epsilon),$$

Real-virtual two-loop Drell-Yan

Real-virtual corrections to EW annihilation processes involve non-factorizable contributions. NE rules cannot reproduce the perturbative result at NLP, due to collinear interference.



Real-virtual Feynman diagrams for the abelian part of the NNLO K-factor.

As a test of the LBDK factorization, we computed the C_F^2 part of the real-virtual K-factor at NNLO from ordinary Feynman diagrams, and then using the radiative amplitude integrated over phase space. As expected, plus distributions arise from the eikonal approximation, fully determined by the dressed non-radiative amplitude. Derivative terms and the projected radiative jet contribute at NLP.

- All NLP terms are correctly reproduced, including those with no logarithms.
- The radiative jet reproduces exactly the NLP collinear contribution derived by the method of regions.

$$K_{\text{rv}}^{(2)}(z) = \left(\frac{\alpha_s}{4\pi} C_F\right)^2 \left\{ \frac{32}{\epsilon^3} [\mathcal{D}_0(z) - 1] + \frac{16}{\epsilon^2} [-4\mathcal{D}_1(z) + 3\mathcal{D}_0(z) + 4L(z) - 6] \right. \\ \left. + \frac{4}{\epsilon} [16\mathcal{D}_2(z) - 24\mathcal{D}_1(z) + 32\mathcal{D}_0(z) - 16L^2(z) + 52L(z) - 49] \right. \\ \left. - \frac{128}{3}\mathcal{D}_3(z) + 96\mathcal{D}_2(z) - 256\mathcal{D}_1(z) + 256\mathcal{D}_0(z) \right. \\ \left. + \frac{128}{3}L^3(z) - 232L^2(z) + 412L(z) - 408 \right\},$$

The abelian part of the NNLO K-factor from real-virtual diagrams, omitting constants

REGENBOGEN



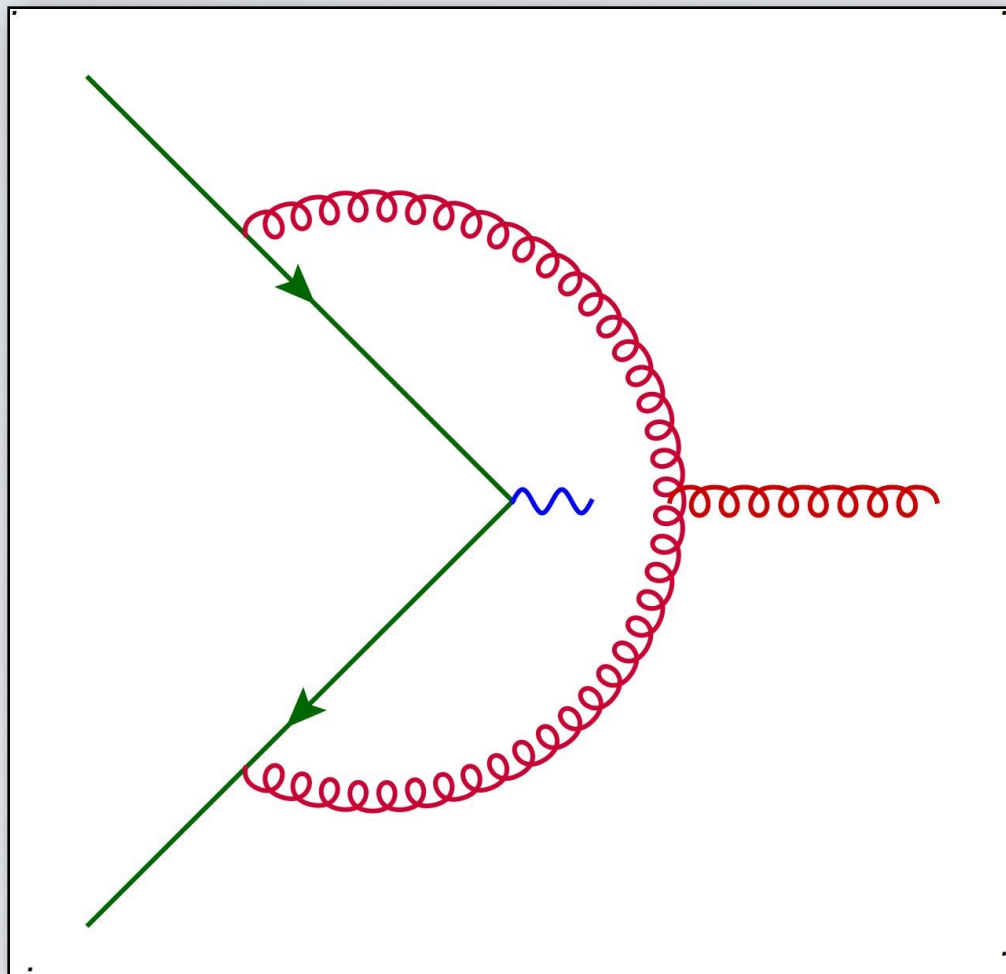
REGENBOGEN



Zur Burg führt die Brücke ...

Non-abelian troubles

The **quasi-abelian** factorization suffers from non-trivial **limitations**. They are exemplified by a **single diagram** contributing to the **real-virtual** part of the **NNLO** cross section.



A Feynman diagram contributing to all singular regions

- This diagram contributes to **all** relevant **momentum regions**.
- Non-abelian **jet contributions** when the virtual gluon is **hard** and **(anti)collinear**.

Scales:

$$(2p \cdot k)^{-\epsilon} \quad (2\bar{p} \cdot k)^{-\epsilon}$$

- (Next-to-)**soft** contributions when the virtual gluon is **soft**.

Scale:

$$\left(\frac{s\mu^2}{2p \cdot k \, 2\bar{p} \cdot k} \right)^\epsilon$$

- The quasi-abelian radiative **jet definition** must be **upgraded** to an **interacting** final state gluon.
- New cases of **double counting** in the factorisation for **(next-to-)soft-collinear** regions.
- Introduce a **radiative** (next-to-)**soft function** and **next-to-eikonal** non-abelian radiative **jets**.

preliminary!

Matrix elements for factorisation

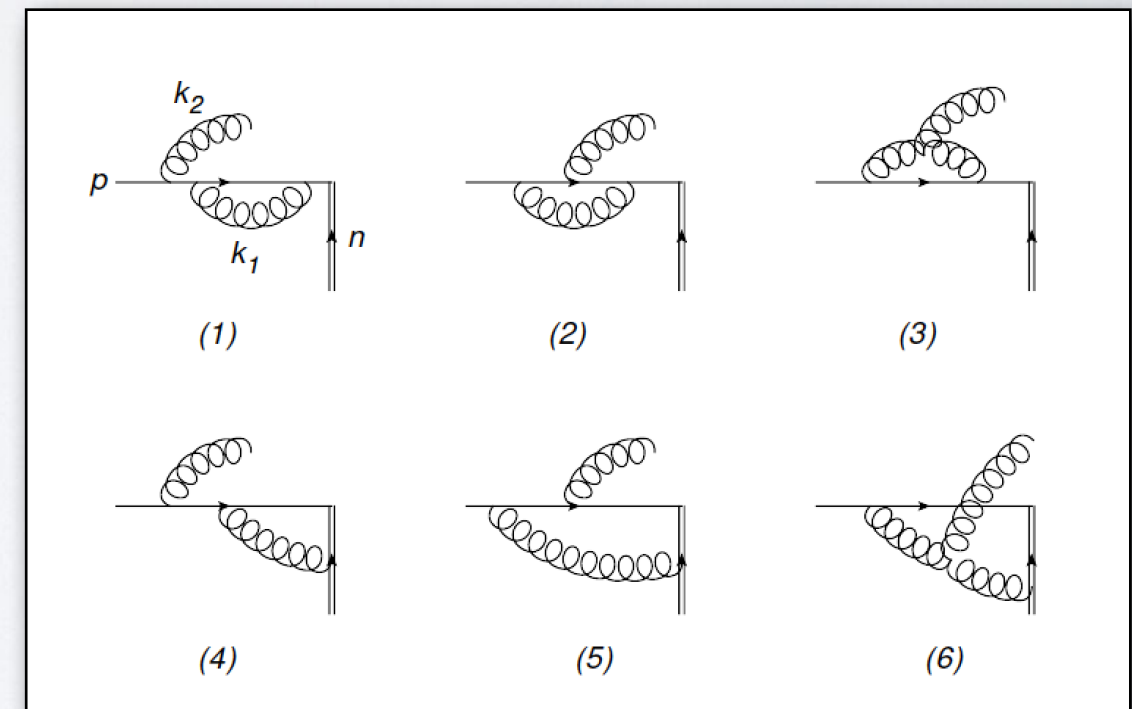
The necessary functions can be defined as matrix elements with single-particle physical states.

$$\epsilon_{(\lambda)}^*(k) \cdot J(p, k, n) u_{(s)}(p) = \langle k, \lambda | \Phi_n(0, \infty) \psi(0) | p, s \rangle ,$$

$$\epsilon_{(\lambda)}^*(k) \cdot \mathcal{J}(p, k, n) = \langle k, \lambda | \Phi_n(0, \infty) \Phi_\beta(\infty, 0) | 0 \rangle ,$$

$$\epsilon_{(\lambda)}^*(k) \cdot W(\beta, \bar{\beta}, k) = \langle k, \lambda | \Phi_\beta(0, \infty) \Phi_{\bar{\beta}}(\infty, 0) | 0 \rangle .$$

- All functions are **transverse** (vanish when $\varepsilon \rightarrow k$).
- All functions can be understood as **single particle contributions** to **cross-section-level** quantities.
- All functions can naturally **upgraded to next-to-soft** level by **replacing** the Wilson lines Φ_β with F_β .
- **Jet** functions also involve **emissions** from the gauge Wilson **line** Φ_n but these **vanish** for $n^2 = 0$.
- **Scale** dependence **matches** the **method of regions**.



Diagrams contributing to the non-abelian radiative jet

preliminary!

Non-abelian factorization

The quasi-abelian **NLP factorisation** formula for the **real-virtual** amplitude **generalises** to

$$\begin{aligned} \mathcal{A}^{\mu a}(p_j, k) \epsilon_\mu(k) = & \epsilon_\mu(k) \sum_{i=1}^2 \left\{ \left(\frac{1}{2} \bar{W}^{\mu a} + \mathbf{T}_i^a G_i^{\nu\mu} \frac{\partial}{\partial p_i^\nu} \right) \mathcal{A}(\{p_i\}) \right. \\ & + \left(J^{\mu a}(p_i, k, n_i) - \tilde{\mathcal{J}}^{\mu a}(\beta_i, k, n_i) - \mathbf{T}_i^a G_i^{\nu\mu} \frac{\partial}{\partial p_i^\nu} \frac{J(p_i, n_i)}{\tilde{\mathcal{J}}(\beta_i, n_i)} \right) \\ & \left. \times \tilde{\mathcal{H}}(p_j, n_j) \tilde{\mathcal{S}}(\beta_j, n_j) \prod_{j \neq i} \frac{J(p_j, n_j)}{\tilde{\mathcal{J}}(\beta_j, n_j)} \right\}, \end{aligned}$$

- The **first** line contains all **(next-to-)soft effects**, corresponding to a **non-abelian Low's** theorem.
- The function **W** has been promoted to **next-to-soft accuracy**, and **divided** by the **non-radiative soft function** to reconstruct the non-radiative amplitude.
- **Jet** functions are **fully subtracted** with **next-to-eikonal jets**, leaving only **hard collinear** terms.
- The choice **n² = 0** leads to a more **transparent expression**, easier to evaluate.

$$\mathcal{A}^{\mu a}(p_j, k) \epsilon_\mu(k) = \epsilon_\mu(k) \sum_{i=1}^2 \left(\frac{1}{2} W^{\mu a} + \mathbf{T}_i^a G_i^{\nu\mu} \frac{\partial}{\partial p_i^\nu} + J^{\mu a}(p_i, k, n_i) - \tilde{\mathcal{J}}^{\mu a}(\beta_i, k, n_i) \right) \mathcal{A}(\{p_i\}).$$

All functions have been **computed at one loop**, and **tested** by reproducing the **full non-abelian** contribution to the **two-loop real-virtual NNLO Drell-Yan** cross section at **NLP accuracy**.

GÖTTERDÄMMERUNG



A Perspective

- 📌 Perturbation theory continues to display new and unexplored structures.
- 📌 Leading power threshold resummation is highly developed and provides some of the most precise predictions in perturbative QCD.
- 📌 Low's theorem is the first of many hints that NLP logs can be understood and organized.
- 📌 Different approaches catch a number of towers of NLP logs in simple processes.
- 📌 The next-to-soft approximation is well understood, using both diagrammatic and path integral approaches, even for multi-parton processes.
- 📌 Hard collinear emissions spoil Low's theorem: a new radiative jet function emerges.
- 📌 Non-abelian contributions introduce significant complications but can be handled.
- 📌 A complete treatment of NLP threshold logs is at hand.
- 📌 Much work to do to organize a true resummation formula, even for EW annihilation: we have a more intricate “factorization”, we need to study and implement evolution equations.
- 📌 In order to achieve complete generality, we will need to include final state jets.

Recht so! Habt Dank! Ein wenig Rast ...