From elliptic iterated integrals to elliptic multiple zeta values

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based on joint work with Carlos Mafra, Nils Matthes and Oliver Schlotterer
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Loops & Legs in Quantum Field Theory
Leipzig, April 26th, 2016

Introduction

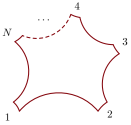
Goal: explore structure and appearance of transcendental numbers of different type in various gauge theories

- *Feynman graphs:* very useful, but challenging
  polylogarithms, iterated integrals/nested sums
  multiple zeta values
- *recently*: elliptic integrals make an appearance
- *idea:* consider more symmetric/constrained theory

This talk:

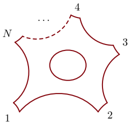
Open string theory as a simple testing ground

Tree-level



multiple polylogarithms
multiple zeta values ζ
Drinfeld associator/KZ-equation
all-order solution

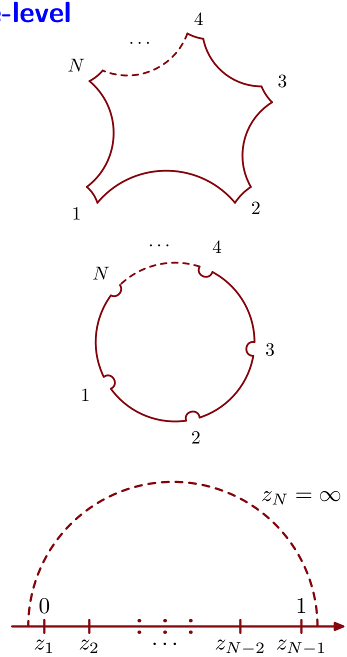
One-loop



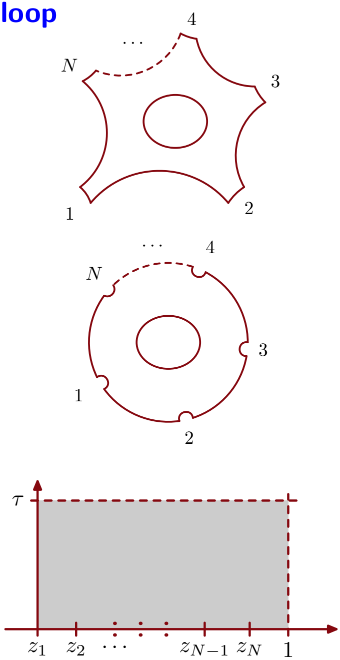
elliptic iterated integrals
elliptic multiple zeta values $\boldsymbol{\omega}$
all-order solution from KZB-equation?

Conformal symmetry

Tree-level



One-loop

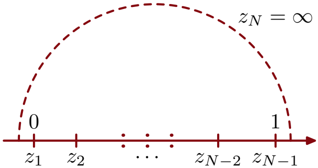


Tree-level

***N*-point tree-level open-string amplitude:** [Veneziano]... [Mafra, Schlotterer Stieberger]

$$\mathbf{A}^{\text{open}}_{\text{string}} = \boldsymbol{F}.\mathbf{A}_{\text{YM}}$$

- dependence on external states in $\mathbf{A}_{\text{YM}}$
- $\boldsymbol{F} = \boldsymbol{F}(s_{ij})$, $s_{ij} = \alpha'(k_i + k_j)^2$
- coefficients are ***multiple zeta values (MZVs)***



$$F^{1,2,\dots,N} = \prod_{i=2}^{N-2} \int_0^{z_{i+1}} \frac{\mathrm{d}z_i}{z_i - a_i} \prod_{i<j}^{N-1} \sum_{n_{ij}=0}^{\infty} (s_{ij})^{n_{ij}} \underbrace{\frac{(\ln|z_{ij}|)^{n_{ij}}}{n_{ij}!}}_{\text{multiple polylogs}}$$

Multiple polylogarithms

$$G(a_1, a_2, \dots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \dots, a_n; t), \quad G(;z) = 1, \quad G(\vec{a}; 0) = G(;0) = 0$$

$$G(\underbrace{0, 0, \dots, 0}_{w}; z) = \frac{1}{w!}(\ln\ z)^w \quad G(\underbrace{1, 1 \dots, 1}_{w}; z) = \frac{1}{w!}\ln^w(1 - z)$$

Multiple zeta values

$$\zeta_{n_1,\dots,n_r} = \sum_{0<k_1<\dots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}} = (-1)^r G(\underbrace{0, 0, \dots, 0, 1}_{n_r}, \dots, \underbrace{0, 0, \dots, 0, 1}_{n_1}; 1) = \zeta_{(w)}$$

Shuffle product:

$$\zeta(a_1,\ldots,a_r)\,\zeta(a_{r+1},\ldots,a_{r+s}) = \zeta(a_1,\ldots,a_r \sqcup\!\sqcup\, a_{r+1},\ldots,a_{r+s}) = \sum_{\sigma\in\Sigma(r,s)} \zeta(a_{\sigma(1)},\ldots,a_{\sigma(r+s)})$$

Stuffle relations:

$$\zeta_n\,\zeta_m = \zeta_{m,n}+\zeta_{n,m}+\zeta_{m+n}$$

5-point-example:

$$F^{(23)} = 1-\zeta_2(s_{12}s_{23}+s_{12}s_{24}+s_{12}s_{34}+s_{13}s_{34}+s_{23}s_{34}) + \zeta_3(s_{12}^2 s_{23}+s_{12}s_{23}^2+s_{12}^2 s_{24}+2s_{12}s_{23}s_{24}+s_{12}s_{24}^2+\cdots)+\cdots + \zeta_{3,5}(\ldots)+\cdots$$

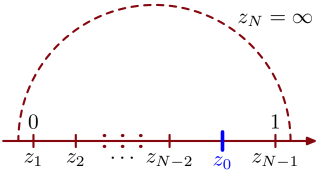
Drinfeld associator

[Broedel, Schlotterer, Stieberger, Terasoma]

Knizhnik-Zamolodchikov equation [Knizhnik, Zamolodchikov]

$$\frac{\mathrm{d}\hat{\mathbf{F}}(z_0)}{\mathrm{d}z_0} = \left(\frac{e_0}{z_0} - \frac{e_1}{z_0 - 1}\right) \hat{\mathbf{F}}(z_0)\,.$$

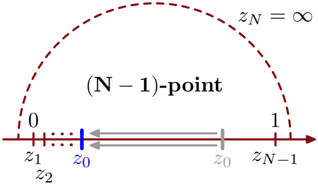
- $z_0 \in \mathbb{C}\backslash\{0,1\}$, Lie-algebra generators e_0, e_1

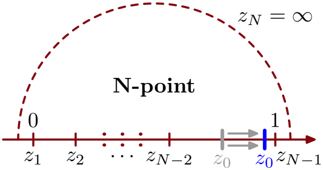


Regularized boundary values

$$C_0 \equiv \lim_{z_0 \to 0} z_0^{-e_0} \hat{\mathbf{F}}(z_0)$$

$$C_1 \equiv \lim_{z_0 \to 1} (1 - z_0)^{e_1} \hat{\mathbf{F}}(z_0)$$



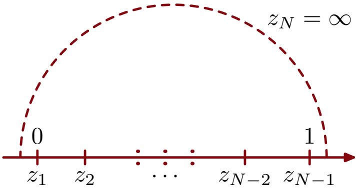


are related by the *Drinfeld associator* Φ: [Drinfeld][Le, Murakami][Furusho][Drummond, Ragoucy]

$$C_1 = \Phi(e_0, e_1)\, C_0, \qquad \Phi(e_0, e_1) = \sum_{w \in \{0,1\}} \tilde{w}[e_0, e_1] \zeta_{(w)}$$

Collect tree-level results

Tree-level



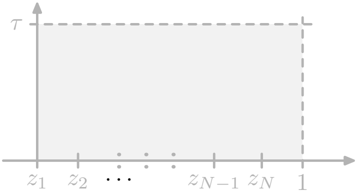
$$\prod_{i=2}^{N-2} \int_0^{z_{i+1}} \mathrm{d}z_i$$

multiple polylogarithms

$$G(a_1, a_2, \ldots, a_n; z) = \int_0^z \mathrm{d}t \, \frac{1}{t - a_1} \, G(a_2, \ldots, a_n; t)$$

partial fraction
multiple zeta values ζ
Drinfeld method - no integrals

One-loop



$$\int_0^1 \mathrm{d}z_N \prod_{i=1}^{N-1} \int_0^{z_{i+1}} \mathrm{d}z_i \, \delta(z_1)$$

elliptic iterated integrals

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_r \\ a_1 & a_2 & \ldots & a_r \end{smallmatrix}; z\right) = \int_0^z \mathrm{d}t \, f^{(n_1)}(t - a_1) \, \Gamma\left(\begin{smallmatrix} n_2 & \ldots & n_r \\ a_2 & \ldots & a_r \end{smallmatrix}; t\right)$$

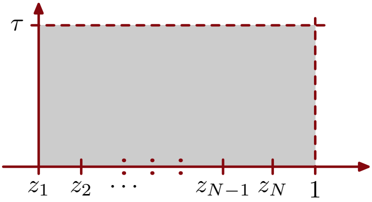
Fay-identities
elliptic multiple zeta values ω
elliptic (KZB) associator?

One-loop

***N*-point one-loop open-string amplitude:**

[Green, Schwarz] · · · [Broedel, Mafra, Matthes, Schlotterer]

- topologies: all genus-one worldsheets with boundaries.
- here: cylinder with insertions on one boundary only: $\mathrm{Im}(z) = 0$
- one imaginary parameter: τ



$$A^{\text{1-loop}}_{\text{string}}(1,2,3,4) = s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4)\int_0^\infty \mathrm{d}\tau\, I_{\text{4pt}}(1,2,3,4)(\tau)$$

$$I_{\text{4pt}}(1,2,3,4)(\tau) = \int_0^1 \mathrm{d}z_N \prod_{i=1}^{N-1}\int_0^{z_{i+1}} \mathrm{d}z_i\,\delta(z_1)\prod_{i<j}\sum_{n_{ij}=0}^{\infty}\frac{1}{n_{ij}!}(s_{ij})^{n_{ij}}\underbrace{(\ln\chi_{ij}(\tau))^{n_{ij}}}_{???}$$

Compare to tree-level:

$$\prod_{i=2}^{N-2}\int_0^{z_{i+1}}\frac{\mathrm{d}z_i}{z_i - a_i}\prod_{i<j}^{N-1}\sum_{n_{ij}=0}^{\infty}\frac{1}{n_{ij}!}(s_{ij})^{n_{ij}}\underbrace{(\ln|z_{ij}|)^{n_{ij}}}_{\text{multiple polylogs}}$$

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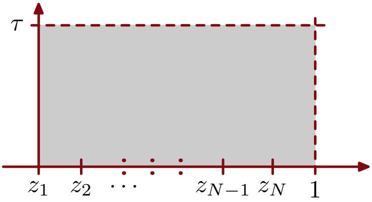

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One-loop

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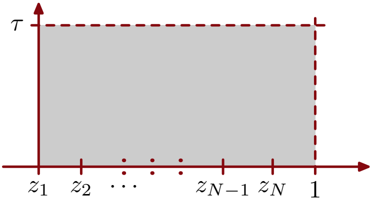

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$$\ln|z_{ij}| = \int_{z_j+1}^{z_i}\frac{\mathrm{d}t}{t - z_j} \qquad \Leftrightarrow \qquad \ln\chi_{ij}(\tau) = \int_{z_j}^{z_i}\mathrm{d}w\, f^{(1)}(w - z_j,\tau)$$

Natural weights for differentials on an elliptic curve:

[Enriquez][Brown Levin]

$$f^{(n)}(z,\tau) = f^{(n)}(z+1,\tau) \quad \text{and} \quad f^{(n)}(z,\tau) = f^{(n)}(z+\tau,\tau)\ .$$

Explicitly: (simplification in our situation because $\mathrm{Im}(z) = 0$)

$$f^{(0)}(z,\tau) \equiv 1 \qquad f^{(1)}(z,\tau) \equiv \partial \ln \theta_1(z,\tau) + 2\pi i \frac{\mathrm{Im} z}{\mathrm{Im}\tau}$$

$$f^{(2)}(z,\tau) \equiv \frac{1}{2}\Big[\Big(\partial \ln \theta_1(z,\tau) + 2\pi i \frac{\mathrm{Im} z}{\mathrm{Im}\tau}\Big)^2 + \partial^2 \ln \theta_1(z,\tau) - \frac{1}{3}\frac{\theta_1'''(0,\tau)}{\theta_1'(0,\tau)}\Big]$$

Parity: $f^{(n)}(-z,\tau) = (-1)^n f^{(n)}(z,\tau)$

Relation to Eisenstein–Kronecker-series:

[Kronecker][Brown Levin]

$$F(z,\alpha,\tau) \equiv \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\theta_1(\alpha,\tau)}\ ,$$

$$\alpha\Omega(z,\alpha,\tau) \equiv \alpha \exp\left(2\pi i \alpha \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\right) F(z,\alpha,\tau) \ = \sum_{n=0}^{\infty} f^{(n)}(z,\tau)\alpha^n$$

Elliptic iterated integrals (suppress τ-dependence from here...)

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix}; z\right) \equiv \int_0^z \mathrm{d}w\, f^{(n_1)}(w-a_1)\, \Gamma\left(\begin{smallmatrix} n_2 & \dots & n_r \\ a_2 & \dots & a_r \end{smallmatrix}; w\right)$$

$\Rightarrow$ can rewrite any integral $\int_{1234}\dots$ into an *elliptic iterated integral*.

Products of differential weights (tree-level) $\Rightarrow$ partial fraction:

$$\int_0^z \mathrm{d}w\, \frac{1}{w-a_1}\frac{1}{w-a_2} = \int_0^z \mathrm{d}w \frac{1}{(w-a_1)(a_1-a_2)} + \frac{1}{(w-a_2)(a_2-a_1)}$$

Products of differential weights (one-loop) $\Rightarrow$ Fay identities

$$\begin{aligned}\int_0^z \mathrm{d}w\, f^1(w-a_1)f^1(w) = \int_0^z & f^{(1)}(w-a_1)f^{(1)}(a_1) - f^{(1)}(w)f^{(1)}(a_1) \\ & + f^{(2)}(w) + f^{(2)}(a_1) + f^{(2)}(w-a_1)\end{aligned}$$

The Fay identity is a form of the trisecant equation for Eisenstein–Kronecker series:

$$\begin{aligned}F(z_1,\alpha_1)F(z_2,\alpha_2) = {} & F(z_1,\alpha_1+\alpha_2)F(z_2-z_1,\alpha_2) \\ & + F(z_2,\alpha_1+\alpha_2)F(z_1-z_2,\alpha_1)\end{aligned}$$

Elliptic multiple zeta values (eMZV's)

$$\omega(n_1, n_2, \ldots, n_r, \tau) \equiv \int\limits_{0 \le z_i \le z_{i+1} \le 1} f^{(n_1)}(z_1, \tau)\mathrm{d}z_1\, f^{(n_2)}(z_2, \tau)\mathrm{d}z_2 \ldots f^{(n_r)}(z_r, \tau)\mathrm{d}z_r$$

$$= \Gamma(n_r, \ldots, n_2, n_1; 1) = \Gamma\left(\begin{smallmatrix} n_r & n_{r-1} & \ldots & n_1 \\ 0 & 0 & \ldots & 0 \end{smallmatrix}; 1\right)$$

Shuffle relation:

$$\omega(n_1, n_2, \ldots, n_r)\, \omega(k_1, k_2, \ldots, k_s) = \omega\big((n_1, n_2, \ldots, n_r) \sqcup\!\sqcup (k_1, k_2, \ldots, k_s)\big)$$

Reflection identity:

$$\omega(n_1, n_2, \ldots, n_{r-1}, n_r) = (-1)^{n_1+n_2+\ldots+n_r} \omega(n_r, n_{r-1}, \ldots, n_2, n_1)$$

Numerous other relations, e.g.

$$0 = \omega(2,3) - \omega(0,5),$$
$$0 = \omega(2,5) - \omega(3,4) - 2\,\omega(0,7)$$
$$0 = \omega(0,0,5) + \omega(0,1,4) + \omega(2,0,3)$$
$$0 = 10\,\omega(0,0,0,5) + 4\,\omega(0,0,3,2) + 2\,\omega(0,2,0,3) - \omega(2)\omega(0,3) - \omega(0,5)$$

Four-point result

$$
\begin{aligned}
I_{\mathsf{4pt}}(1,2,3,4)(\tau) = {} & \omega(0,0,0) \;-\; 2\,\omega(0,1,0,0)\,(s_{12}+s_{23}) \\
& +\; 2\,\omega(0,1,1,0,0)\,\left(s_{12}^2+s_{23}^2\right) \;-\; 2\,\omega(0,1,0,1,0)\,s_{12}s_{23} \\
& +\; \beta_5\,(s_{12}^3+2s_{12}^2s_{23}+2s_{12}s_{23}^2+s_{23}^3) \\
& +\; \beta_{2,3}\,s_{12}s_{23}(s_{12}+s_{23}) \;+\; \mathcal{O}(\alpha'^4)
\end{aligned}
$$

with

$$
\beta_5 = \frac{4}{3}\left[\,\omega(0,0,1,0,0,2)+\omega(0,1,1,0,1,0)-\omega(2,0,1,0,0,0)-\zeta_2\;\omega(0,1,0,0)\right]
$$

$$
\begin{aligned}
\beta_{2,3} = {} & \frac{1}{3}\;\omega(0,0,1,0,2,0)-\frac{3}{2}\;\omega(0,1,0,0,0,2)-\frac{1}{2}\;\omega(0,1,1,1,0,0) \\
& -2\;\omega(2,0,1,0,0,0)-\frac{4}{3}\;\omega(0,0,1,0,0,2)-\frac{10}{3}\,\zeta_2\;\omega(0,1,0,0)
\end{aligned}
$$

How to find a "basis" for eMZVs?

multiple zeta values

- make use of the Hopf algebra comodule structure of MZVs (coproduct)
  $\rightsquigarrow$ map ζ's to words built from non-commutative letters
  $\rightsquigarrow$ *only* shuffle relations, no stuffle etc. [Brown]

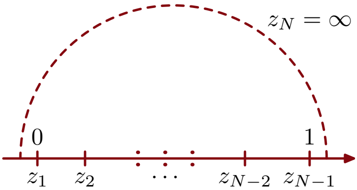
a

elliptic multiple zeta values

- coproduct is replaced by differential equation in τ [Enriquez][Hain][Brown]
  $\rightsquigarrow$ eMZVs can be represented by *iterated Eisenstein integrals*
  $\rightsquigarrow$ labeled by words composed from non-commutative letters
  $\rightsquigarrow$ *only* shuffle relations left
- rigid combinations in eMZVs only? Counting wrong?
  explanation: special derivation algebra
  linked to cusp forms on the elliptic curve [Pollack][Hain][Broadhurst Kreimer]
- number of "basis" eMZVs + canonical choice known [Hain][Brown][Broedel, Matthes Schlotterer]
- using our formalism, one can find new relations in the derivation algebra

Comparison

Tree-level



$$\prod_{i=2}^{N-2} \int_0^{z_{i+1}} \mathrm{d}z_i$$

multiple polylogarithms

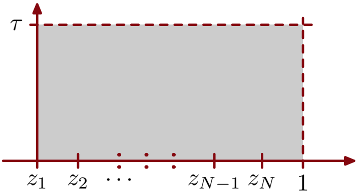
$$G(a_1,a_2,\ldots,a_n;z) = \int_0^z \mathrm{d}t \, \frac{1}{t-a_1} \, G(a_2,\ldots,a_n;t)$$

partial fraction

multiple zeta values ζ

Drinfeld method - no integrals

One-loop



$$\int_0^1 \mathrm{d}z_N \prod_{i=1}^{N-1} \int_0^{z_{i+1}} \mathrm{d}z_i \, \delta(z_1)$$

elliptic iterated integrals

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_r \\ a_1 & a_2 & \ldots & a_r \end{smallmatrix}; z\right) = \int_0^z \mathrm{d}t \, f^{(n_1)}(t-a_1) \, \Gamma\left(\begin{smallmatrix} n_2 & \ldots & n_r \\ a_2 & \ldots & a_r \end{smallmatrix}; t\right)$$

Fay-identities

elliptic multiple zeta values ω

elliptic associator? [Knizhnik, Bernard Zamolodchikov]

Summary

- eMZVs are the natural generalization of MZVs to the elliptic domain
- one-loop amplitudes in open string theory: perfect laboratory
- no divergent eMZVs: open-string result finite
- full amplitude only after τ-integration and consideration of other topologies
- eMZVs might not be the only ingredient: Euler sums?

Give it a try . . .

- numerous relations for eMZVs: `https://tools.aei.mpg.de/emzv`

Future...

- closed/recursive form of the integrand for the one-loop open-string amplitude in terms of iterated Eisenstein integrals (analogous to Drinfeld-method)?
- relation to the elliptic functions from [Adams, Bogner Weinzierl] ?

Thanks!