On the Evaluation and Reduction of Generalized Polylogarithms

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Introduction

On the reduction of generalized polylogarithms to Li_n and $\mathrm{Li}_{2,2}$ and on the evaluation thereof

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ABSTRACT: We give expressions for all generalized polylogarithms up to weight four in terms of the functions log, Li_n, and $\mathrm{Li}_{2,2}$, valid for arbitrary complex variables. Furthermore we provide algorithms for manipulation and numerical evaluation of Li_n and $\mathrm{Li}_{2,2}$, and add codes in Mathematica and C++ implementing the results. With these results we calculate a number of previously unknown integrals, which we add in App. C.

KEYWORDS: Generalized polylogarithms, Multiple polylogarithms, Higher orders, Feynman integrals, Computer algebra.

Generalized Polylogarithms (GPLs) are a class of mathematical functions defined recurcively as

$$G(a_1, \ldots, a_n; x) = \int_0^x \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t)$$

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with $G(;x) \equiv 1$.

$$G(\underbrace{0, \ldots, 0}_{n}; x) \equiv G(\bar{0}_n; x) = \frac{\log^n(x)}{n!}$$

$$G(0;x) = \log(x) \qquad G(a;x) = \log\left(1 - \tfrac{x}{a}\right) \qquad G(\bar{0}_{n-1}, a; x) = -\mathrm{Li}_n\left(\tfrac{x}{a}\right)$$

The shuffle rule: $$G(\bar{a};x)G(\bar{b};x) = \sum_{\bar{c} \,\in\, \bar{a} \text{Ш} \bar{b}} G(\bar{c};x)$$

Motivation

$$G(a_1,\ldots,a_n;x) = \int_0^x \frac{\mathrm{d}t}{t-a_1} G(a_2,\ldots,a_n;t) \qquad G(0;x) = \log(x)$$

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For each process, they can (using IBPs) be expressed in terms of a minimal set, denoted master integrals.

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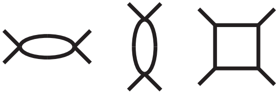
Example:
The one-loop corection to massless $2 \to 2$ scattering
(e.g. QCD contribution to 2-jet production at the LHC)

Three master integrals: two 'bubbles', one 'box'.



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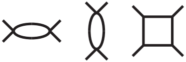

We define the vector $f = \{I_{s-\mathrm{bubble}}, I_{t-\mathrm{bubble}}, I_{\mathrm{box}}\}$ and $x = s/t$.
Then $\frac{df}{dx} = A(x,\epsilon)f$.

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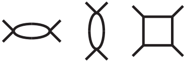

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A smart choice of f gives the 'canonical' form (Henn: [arXiv:1304.1806])

$$df = \epsilon \sum_i d\log(q_i(x)) \tilde{A}_i f$$

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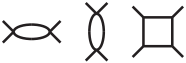

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If we expand $f(\epsilon,x) = \sum \epsilon^i f^{(i)}(x)$, and if $q_1 = x - a_1$, then

$$\frac{df^{(n)}}{dx} = \frac{1}{x-a_1} \tilde{A}_1 f^{(n-1)} + \ldots$$

Reduction

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It was conjectured by Goncharov [arXiv:1110.0458] that all GPLs up to weight 4 (all that is needed at two-loop) can be expressed in terms of

$$\log(x)\ ,\ \ \mathrm{Li}_2(x)\ ,\ \ \mathrm{Li}_3(x)\ ,\ \ \mathrm{Li}_4(x)\ ,\ \ \mathrm{Li}_{2,2}(x,y)$$

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Let's show that...

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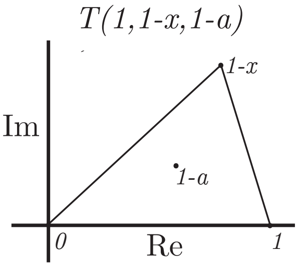

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$$= \log\left(\tfrac{x-a}{1-a}\right)\log(1-x) + \mathrm{Li}_2\left(\tfrac{1-x}{1-a}\right) - \mathrm{Li}_2\left(\tfrac{1}{1-a}\right) + 2\pi i T(1,1-x,1-a)\log(1-a)$$

Goal achieved: $G(a,b;x)$ in terms of $\log$, Li_2.

Reduction

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The reduction consists of two parts:

First map a letter to zero,
then do the remaining integrals.

$$G(A_1,\ldots,A_n;X) \to G(a_1,\ldots,a_{n-1},0,1)$$
$$\to \log, \mathrm{Li}_n, \mathrm{Li}_{2,2} \text{ at weights } \leq 4.$$

The first step can be done at all weights,
The second gets much harder for each weight.

Reduction

$$
\begin{aligned}
G(a,b,c,x) \;=\; & \tfrac{1}{6}\log\left(\tfrac{c-b}{a}\right)^3 - \tfrac{1}{6}\log\left(\tfrac{(a-c)(c-b)}{(a-b)c}\right)^3 - \tfrac{1}{6}\log\left(\tfrac{c-b}{a-x}\right)^3 + \tfrac{1}{6}\log\left(\tfrac{(a-c)(c-b)}{(a-b)(c-x)}\right)^3 - \tfrac{1}{2}\log\left(\tfrac{a}{a-b}\right)\log\left(\tfrac{b}{b-c}\right)^2 + \mathrm{Li}_2\left(\tfrac{x-b}{a-b}\right)\log\left(\tfrac{b-x}{b-c}\right) \\
& -\tfrac{1}{2}\log\left(1-\tfrac{b}{a}\right)\log\left(\tfrac{b}{b-c}\right)^2 + \tfrac{1}{2}\log\left(\tfrac{a-b}{a-x}\right)\log\left(\tfrac{b-x}{b-c}\right)^2 + \tfrac{1}{2}\log\left(\tfrac{a-x}{a-b}\right)\log\left(\tfrac{b-x}{b-c}\right)^2 + \mathrm{Li}_3\left(\tfrac{b}{a}\right) + \mathrm{Li}_3\left(\tfrac{b}{b-a}\right) - \mathrm{Li}_3\left(\tfrac{b(a-c)}{a(b-c)}\right) - \mathrm{Li}_3\left(\tfrac{b(a-c)}{(a-b)c}\right) \\
& -\mathrm{Li}_3\left(-\tfrac{c}{b-c}\right) + \mathrm{Li}_3\left(1-\tfrac{c}{a}\right) - \mathrm{Li}_3\left(\tfrac{a-c}{a-x}\right) - \mathrm{Li}_3\left(\tfrac{b-x}{a-x}\right) + \mathrm{Li}_3\left(\tfrac{(a-c)(b-x)}{(b-c)(a-x)}\right) + \mathrm{Li}_3\left(\tfrac{(a-c)(b-x)}{(a-b)(c-x)}\right) - \mathrm{Li}_3\left(\tfrac{x-b}{a-b}\right) - \mathrm{Li}_3\left(\tfrac{x-c}{a-c}\right) + \mathrm{Li}_3\left(\tfrac{x-c}{b-c}\right) \\
& -\mathrm{Li}_2\left(\tfrac{a-c}{b-c}\right)\log\left(\tfrac{a}{a-c}\right) + \mathrm{Li}_2\left(\tfrac{-c}{b-c}\right)\log\left(\tfrac{a}{a-c}\right) - \mathrm{Li}_2\left(\tfrac{b}{a}\right)\log\left(\tfrac{b}{b-c}\right) - \mathrm{Li}_2\left(\tfrac{b}{b-a}\right)\log\left(\tfrac{b}{b-c}\right) + \mathrm{Li}_2\left(\tfrac{b(a-c)}{a(b-c)}\right)\log\left(\tfrac{b}{b-c}\right) + \mathrm{Li}_2\left(\tfrac{b(a-c)}{(a-b)c}\right)\log\left(\tfrac{b}{b-c}\right) \\
& +\tfrac{1}{2}\log\left(\tfrac{b}{b-c}\right)^2\log\left(\tfrac{(b-a)c}{a(b-c)}\right) + \tfrac{1}{2}\log\left(\tfrac{b}{b-c}\right)^2\log\left(\tfrac{a(c-b)}{(a-b)c}\right) + \mathrm{Li}_2\left(\tfrac{a-c}{b-c}\right)\log\left(\tfrac{a-x}{a-c}\right) - \mathrm{Li}_2\left(-\tfrac{c}{b-c}\right)\log\left(\tfrac{a-x}{a-c}\right) + \mathrm{Li}_2\left(\tfrac{b-x}{a-x}\right)\log\left(\tfrac{b-x}{b-c}\right) \\
& -\mathrm{Li}_2\left(\tfrac{(a-c)(b-x)}{(b-c)(a-x)}\right)\log\left(\tfrac{b-x}{b-c}\right) - \mathrm{Li}_2\left(\tfrac{(a-c)(b-x)}{(a-b)(c-x)}\right)\log\left(\tfrac{b-x}{b-c}\right) - \tfrac{1}{2}\log\left(\tfrac{b-x}{b-c}\right)^2\log\left(-\tfrac{(b-c)(a-x)}{(a-b)(c-x)}\right) - \tfrac{1}{2}\log\left(\tfrac{b-x}{b-c}\right)^2\log\left(\tfrac{(b-a)(c-x)}{(b-c)(a-x)}\right) + \mathrm{Li}_3\left(\tfrac{c}{c-a}\right) \\
& -\mathrm{Li}_2\left(\tfrac{c-b}{a-b}\right)\log\left(1-\tfrac{x}{c}\right) + \mathrm{Li}_2\left(\tfrac{x-b}{a-b}\right)\log\left(1-\tfrac{x}{c}\right) + \log\left(\tfrac{a-x}{a-b}\right)\log\left(\tfrac{b-x}{b-c}\right)\log\left(1-\tfrac{x}{c}\right) - \pi^2\Big(-\tfrac{1}{6}\log\left(\tfrac{c-b}{a}\right) + \tfrac{1}{6}\log\left(\tfrac{(a-c)(c-b)}{(a-b)c}\right) + \tfrac{1}{6}\log\left(\tfrac{c-b}{a-x}\right) \\
& -\tfrac{1}{6}\log\left(\tfrac{(a-c)(c-b)}{(a-b)(c-x)}\right) + 4\log\left(1-\tfrac{b}{c}\right)\mathsf{T}\left(1,1-\tfrac{x}{c},1-\tfrac{b}{c}\right)\mathsf{T}\left(\mathsf{P}\left(\tfrac{b}{c},1-\tfrac{x}{c}\right),1-\tfrac{x}{c},1-\tfrac{a}{c}\right)\Big) + i\pi\Big(\mathsf{T}\left(1,\tfrac{b}{b-c},\tfrac{a}{a-c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{c}{a-c}\right)\right)\log\left(\tfrac{a}{a-c}\right)^2 \\
& +\log\left(\tfrac{b-a}{b-c}\right)^2\mathsf{T}\left(1,\tfrac{b}{b-c},\tfrac{b-a}{b-c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{a-c}{b-c}\right)\right) - \log\left(\tfrac{b-a}{b-c}\right)^2\mathsf{T}\left(1,\tfrac{b-x}{b-c},\tfrac{b-a}{b-c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{a-c}{b-c}\right)\right) - 2\mathrm{Li}_2\left(\tfrac{a-c}{b-c}\right)\mathsf{T}\left(1,1-\tfrac{x}{c},1-\tfrac{a}{c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{a}{c}\right)\right) \\
& +2\mathrm{Li}_2\left(-\tfrac{c}{b-c}\right)\mathsf{T}\left(1,1-\tfrac{x}{c},1-\tfrac{a}{c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{a}{c}\right)\right) - 2\log\left(1-\tfrac{a}{c}\right)\log\left(\tfrac{b-a}{b-c}\right)\mathsf{T}\left(1,1-\tfrac{x}{c},1-\tfrac{a}{c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{a}{c}\right)\right) \\
& -\mathcal{H}_1\left(1-\tfrac{a}{c},1-\tfrac{b}{c}\right)\log\left(\tfrac{(b-a)c}{(a-c)(c-b)}\right)^2\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{b}{c}\right)\right) + 2\log\left(1-\tfrac{b}{c}\right)\log\left(\tfrac{a-b}{a-c}\right)\mathsf{T}\left(1,1-\tfrac{x}{c},1-\tfrac{b}{c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{b}{c}\right)\right) \\
& -2\log\left(1-\tfrac{b}{c}\right)\log\left(\tfrac{a-x}{a-c}\right)\mathsf{T}\left(1,1-\tfrac{x}{c},1-\tfrac{b}{c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{b}{c}\right)\right) + \mathcal{H}_1\left(\tfrac{b-c}{a-c},\tfrac{c-b}{c-x}\right)\log\left(\tfrac{a-x}{b-c}\right)^2\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{c-b}{c-x}\right)\right) \\
& -\mathcal{H}_1\left(\tfrac{c-a}{c-x},\tfrac{c-b}{c-x}\right)\log\left(-\tfrac{(a-b)(c-x)}{(a-c)(c-b)}\right)^2\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{c-b}{c-x}\right)\right) + 2\log\left(\tfrac{b-a}{b-c}\right)\log\left(1-\tfrac{x}{c}\right)\mathsf{T}\left(1,\tfrac{b-x}{b-c},\tfrac{b-a}{b-c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{c-b}{c-x}\right)\right) \\
& +\log\left(\tfrac{a-x}{a-c}\right)^2\mathsf{T}\left(1,\tfrac{b-x}{b-c},\tfrac{a-x}{a-c}\right)\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{x-c}{a-c}\right)\right) + \mathcal{H}_1\left(\tfrac{b-c}{a-c},1-\tfrac{b}{c}\right)\log\left(\tfrac{a}{b-c}\right)^2\mathsf{sgn}\left(\mathsf{Im}\left(\tfrac{b}{c}\right)\right)\Big)\,.
\end{aligned}
$$

Evaluation

We need to evaluate $\mathrm{Li}_n(x)$ and $\mathrm{Li}_{2,2}(x)$

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$\mathrm{Li}_n(x)$ is the classical polylogarithm (or Euler polylogarithm)

$$\mathrm{Li}_n(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^n}$$

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Outside we may use the inversion relation $\mathrm{Li}_n(x) = \pm \mathrm{Li}_n(\frac{1}{x}) + \text{easier}$.

$$\mathrm{Li}_3(x) = \mathrm{Li}_3\left(\tfrac{1}{x}\right) - \tfrac{1}{6}\log^3(-x) - \tfrac{\pi^2}{6}\log(-x)$$

(There are better ways to evaluate $\mathrm{Li}_n(x)$,
e.g. a 'Bernoulli expansion' $\mathrm{Li}_n(e^x) = \sum_\nu \frac{c_\nu}{\nu!} x^\nu$).

Evaluation

$\mathrm{Li}_{2,2}(x,y)$ is given by a similarly looking sum

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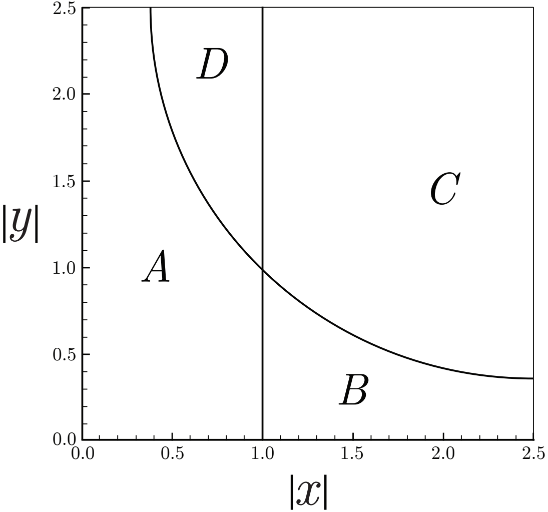
If $|xy| > 1$ we may use the inversion relation

$$\begin{aligned}
\mathrm{Li}_{2,2}(x,y) = {} & \mathrm{Li}_{2,2}\left(\tfrac{1}{x},\tfrac{1}{y}\right) - \mathrm{Li}_4(xy) + 3\Big(\mathrm{Li}_4\left(\tfrac{1}{x}\right) + \mathrm{Li}_4(y)\Big) \\
& + 2\Big(\mathrm{Li}_3\left(\tfrac{1}{x}\right) - \mathrm{Li}_3(y)\Big)\log(-xy) + \mathrm{Li}_2\left(\tfrac{1}{x}\right)\left(\tfrac{\pi^2}{6} + \tfrac{1}{2}\log^2(-xy)\right) \\
& + \tfrac{1}{2}\mathrm{Li}_2(y)\Big(\log^2(-xy) - \log^2(-x)\Big)
\end{aligned}$$

And then if $|x| > 1$ we may use the 'stuffle relation'

$$\mathrm{Li}_{2,2}(x,y) = -\mathrm{Li}_{2,2}(y,x) - \mathrm{Li}_4(xy) + \mathrm{Li}_2(x)\mathrm{Li}_2(y)$$

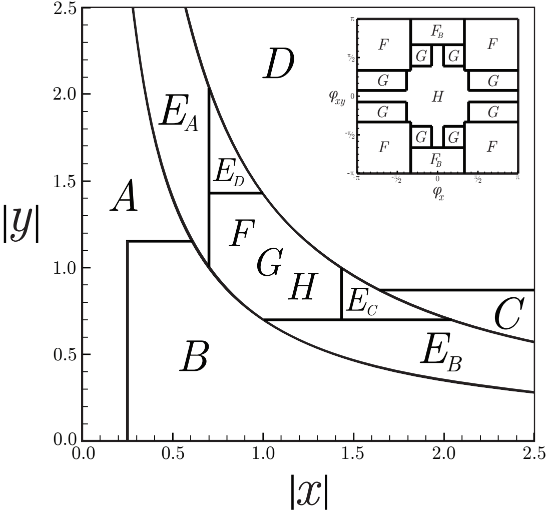


A: Direct summation

B: Stuffle: $x \leftrightarrow y$

C: Inversion: $z \leftrightarrow z^{-1}$

D: Stuffle and inversion



A: Direct summation

B: Stuffle: $x \leftrightarrow y$

C: Inversion: $z \leftrightarrow z^{-1}$

D: Stuffle and inversion

E: Diagonal sum

F: Hölder relation

G: $\log(1-z)$ expansion

H: $\log(z)$ expansion

The added code

With the paper [ArXiv:1601.02649] we have added code implemeting the results.

The reductions are implemented in Mathematica as the replacement rule gtolrules.

```
In[1]:= G(0,a,0,b,x)/.gtolrules
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complex<double> x = 0.2+0.3*I, y = 13.0-8.9*I;
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Please try our code, and send us any critique/bug reports.

Discussion

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The past decade has seen many new developements in the study of GPLs.

Particularly the introduction of 'symbols' and co-products [arXiv:1006.5703, arXiv:1110.0438, arXiv:1123.0454]

$$\mathcal{S}\Big(G(a_n,\ldots,a_1;a_{n+1})\Big) = \sum_{i=1}^{n} \Bigg(\mathcal{S}\Big(G(a_n,\ldots,\hat{a}_i,\ldots,a_1;a_{n+1})\Big) \otimes (a_i - a_{i+1}) - \mathcal{S}\Big(G(a_n,\ldots,\hat{a}_i,\ldots,a_1;a_{n+1})\Big) \otimes (a_i - a_{i-1}) \Bigg)$$

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Symbols capture the algebraic parts of an expression.
We need the full analytical stucture.

Newer developments of the co-products catch most
of the analyticity [arXiv:1401:3546]

To generalize to weight 6 (or n) this will likely be needed.

Discussion

The goal of reducing all GPLs up to weight four to

$$\log(x)\ ,\ \ \mathrm{Li}_2(x)\ ,\ \ \mathrm{Li}_3(x)\ ,\ \ \mathrm{Li}_4(x)\ ,\ \ \mathrm{Li}_{2,2}(x,y)$$

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Thank you for listening...