

local integrands for two-loop QCD amplitudes

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25th April 2016

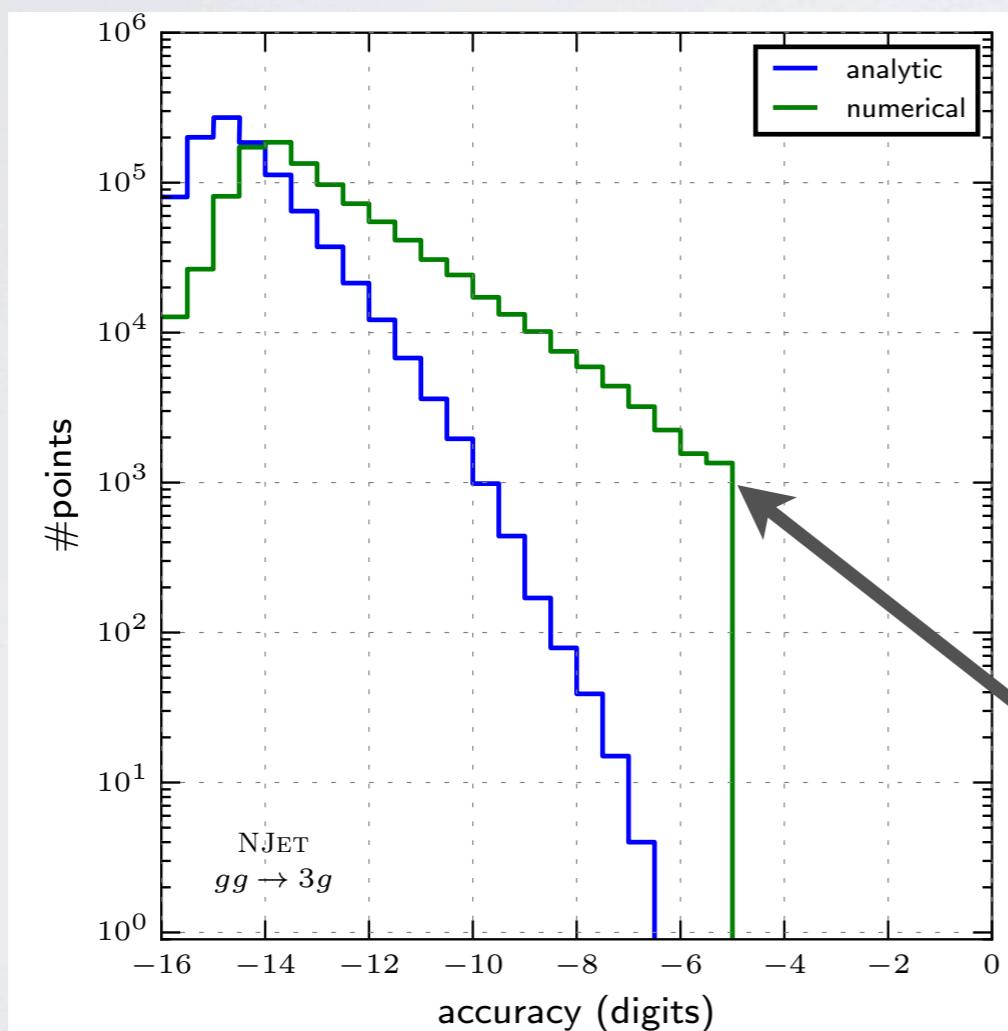
Loops and Legs in Quantum Field Theory, Leipzig, Germany



spurious singularities

big jump between $2 \rightarrow 2$
and $2 \rightarrow 3$ kinematics

numerical D-dimensional
generalised unitarity



applications of loop amplitudes in NNLO
computations more intensive

vs
analytic computation with
finite integrals basis
(e.g. Bern, Dixon, Kosower
[hep-ph/9302280])

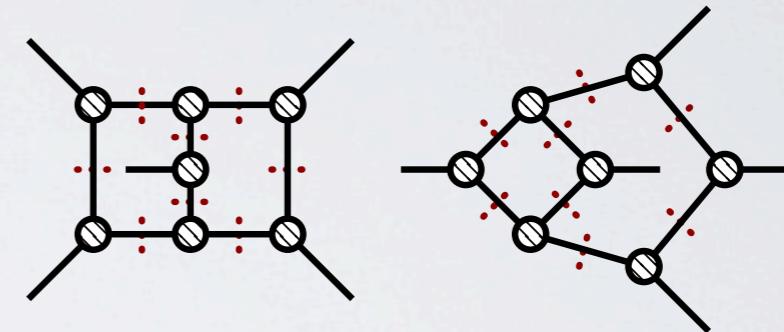
removing spurious poles also
simplifies coefficients \Rightarrow faster
($\sim 100x$ in this case)

need to switch to quadruple
precision evaluation

outline

- d-dimensional generalised unitarity

⇒ multi-loops from trees



- two-loop all-plus amplitudes
- local integrands for d-dimensional amplitudes

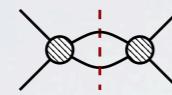
work with Hjalte Frellesvig, Yang Zhang, Gustav Mogull,
Alex Ochirov, Donal O'Connell, Tiziano Peraro

automated one-loop amplitudes

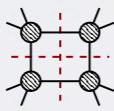
Unitarity: double cuts

[BDDK '94]

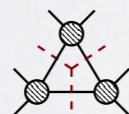
[triple cuts BDK '97]



Generalized unitarity:
quadruple cuts [BCF '04]



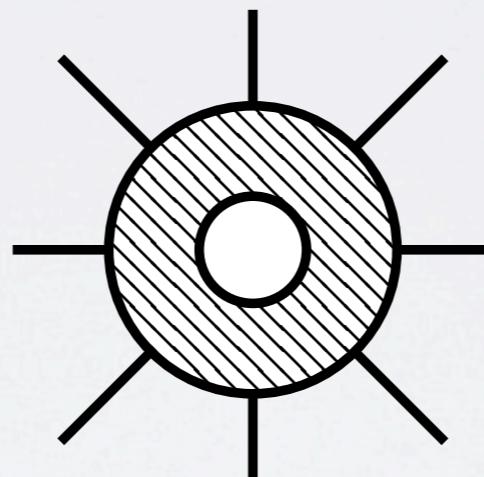
triple cuts [e.g. Forde '07]



$$A = \sum_i (\text{rational})_i (\text{integral})_i$$

find complex contour to isolate
integral coefficient

solving on-shell conditions requires **complex** momenta
⇒ factorise residues into **tree amplitudes**



Integrand reduction [OPP '05]

$$\Delta_3 = \text{Feynman diagram with three external legs and a triple cut} - \text{Feynman diagram with three external legs and a quadruple cut}$$

D-dim. generalized unitarity [GKM '08]

$$A = \int_k \sum_i \frac{\Delta_i(k, p)}{(\text{propagators})_i}$$

multi-scale
kinematic algebra
performed
numerically

explicitly remove poles

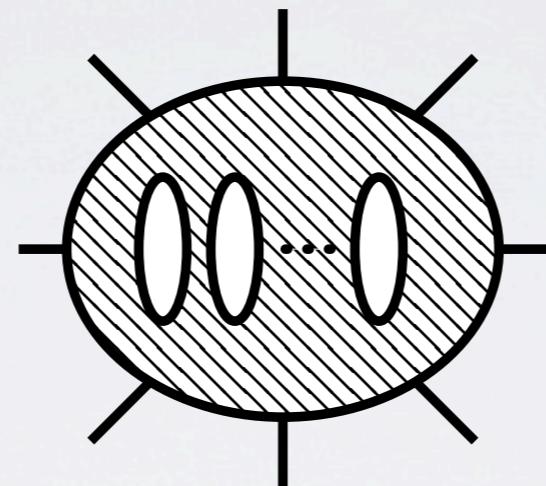
multi-loop amplitudes from trees

Maximal unitarity

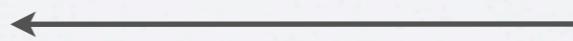
[Kosower, Larsen, Johansson,
Caron-Huot, Zhang, Søgaard]

$$A = \sum_i (\text{rational})_i (\text{integral})_i$$

IBPs must be free of
doubled propagator MI



e.g. IBPs



Integrand reduction via
polynomial division

[Mastrolia, Ossola, SB, Frellesvig,
Zhang, Mirabella, Peraro, Malamos,
Kleiss, Papadopolous, Verheyen,
Feng, Huang]

$$A = \int_k \sum_i \frac{\Delta_i(k, p)}{(\text{propagators})_i}$$

[Gluza, Kosower, Kajda 1009.0472] [Schabinger 1111.4220]
[Ita 1510.05626] [Larsen, Zhang 1511.01071]

a toolbox for multi-loop integrand

momentum twistors
[Hodges (2009)]

six-dimensional
spinor-helicity

generalised unitarity
cuts

$$\mathcal{A} = \sum_i S_i \frac{C(\Delta_i) \Delta_i}{\prod D_\alpha}$$

$$A = \int_k \sum_i \frac{\Delta_i(k, p)}{(\text{propagators})_i}$$

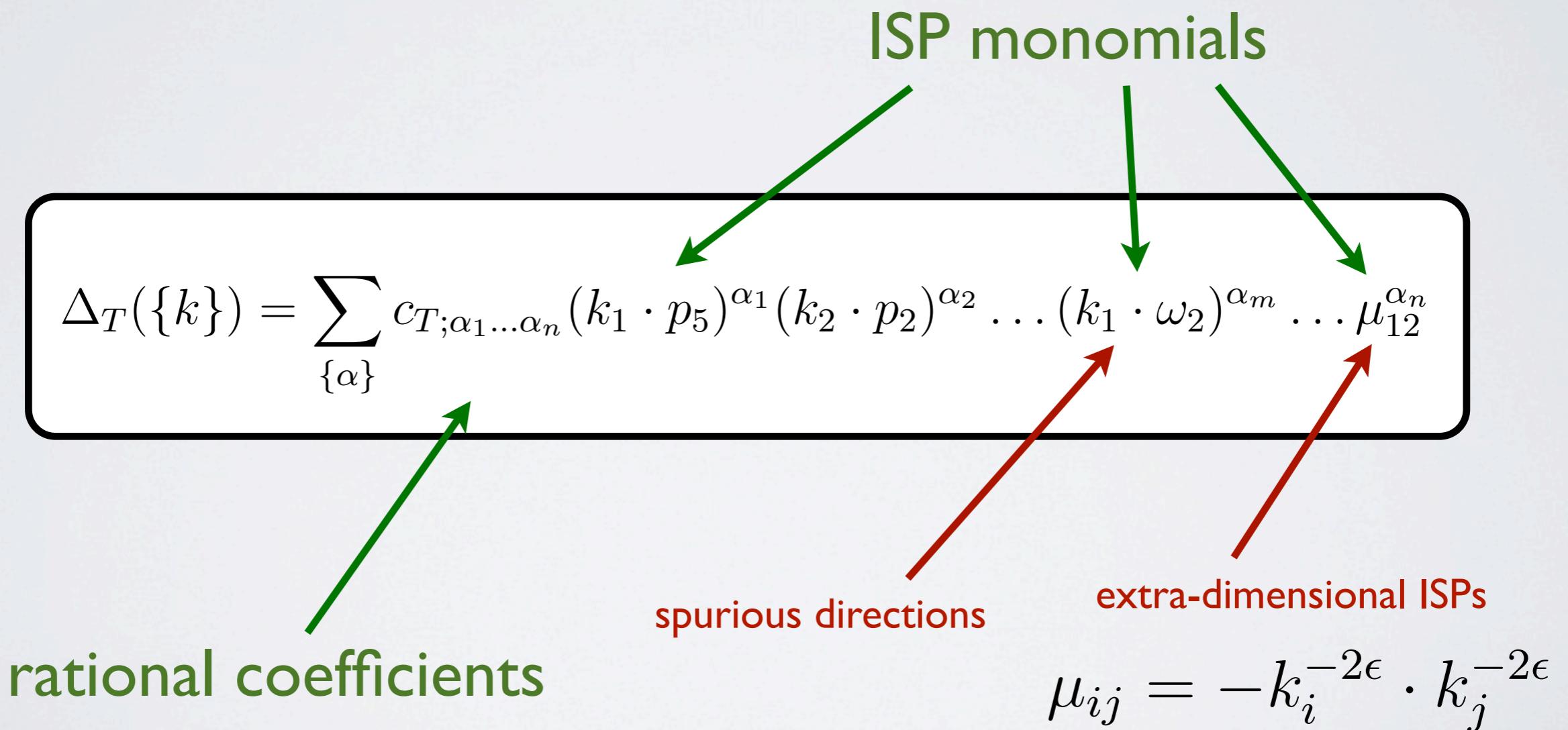
integrand reduction

colour/kinematics
BCJ relations

multi-loop integrand reduction

[Mastrolia, Ossola | 107.6041] [SB, Frellesvig, Zhang | 202.2019]

[Zhang | 205.5707] [Mastrolia, Mirabella, Ossola, Peraro | 205.7087]



integrand reduction

$$\Delta_{c;T} \Big|_{\text{cut}} = \prod_i A_i^{(0)} - \sum_{T'} \frac{\Delta_{c;T'}}{\prod_{l \in T'/T} D_l} \Big|_{\text{cut}}$$

on-shell the numerators can
be written as products of
tree-level amplitudes

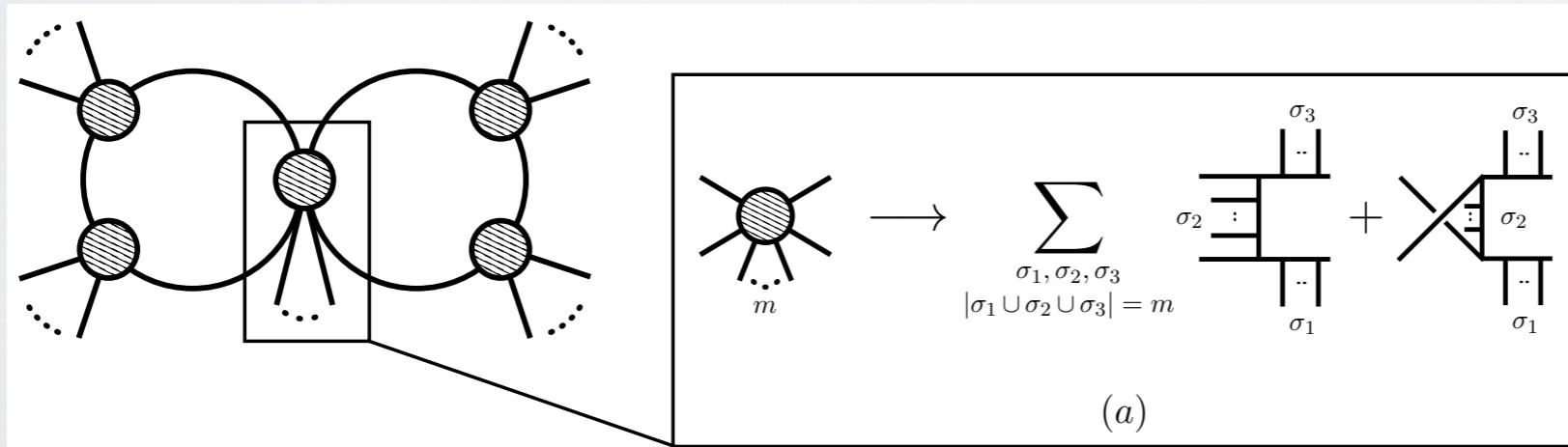
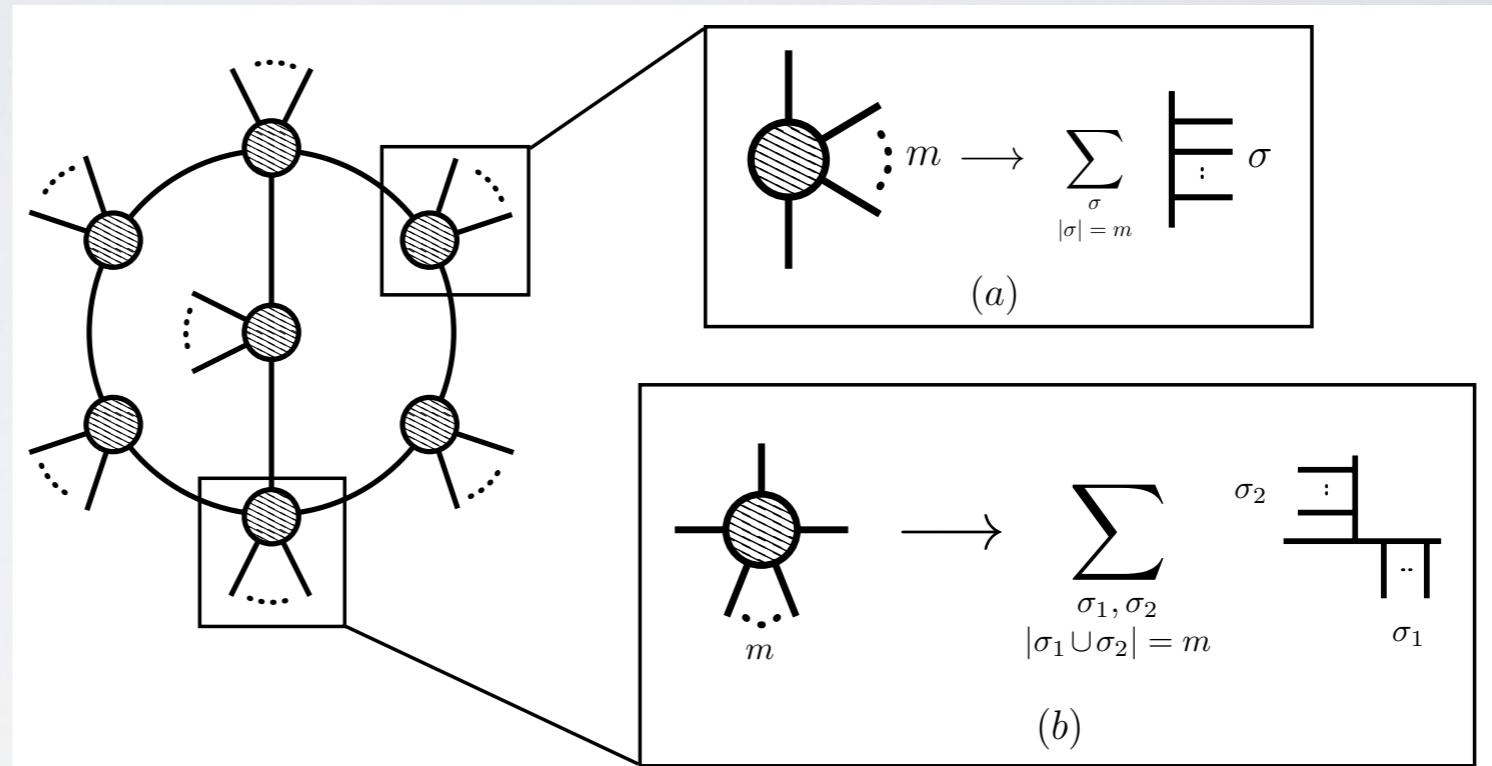
fix basis of **monomials** in
irreducible scalar products
via **polynomial division**
(Gröbner basis)

integrand parameterisations
not unique - freedom in the
choices of ISP monomials

General colour decompositions

[Dixon, Del Duca, Maltoni (1999)]

Inserting the DDM decomposition into colour dressed cuts leads to a compact loop decomposition



general tree-level DDM colour bases including fermions [Johansson, Ochirov arXiv:1507.00332]

applications:
all-plus amplitudes in QCD

one-loop 4pt all-plus

$$+ \begin{array}{c} \text{Diagram: a circle with a smaller circle inside, four external gluon lines labeled '+' at each vertex} \\ + \end{array} = \frac{D_s - 2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \{-st\} \cdot \left\{ I \left(\begin{array}{|c|c|} \hline & & \\ \hline & \text{---} & \\ \hline & & \\ \hline \end{array} \right) [\mu_{11}^2] \right\}$$

$$+ \begin{array}{c} \text{Diagram: a circle with a smaller circle inside, four external gluon lines labeled '+' at each vertex} \\ + \end{array} = \frac{D_s - 2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{-st}{6} + \mathcal{O}(\epsilon)$$

[Bern, Kosower (1991)]

one-loop 5pt all-plus

$$+ \text{Diagram} = \vec{c} \cdot \left\{ I \left(\text{Diagram} \right) [\mu_{11}^2], \right.$$

$$\left. I \left(\text{Diagram} \right) [\mu_{11}^2], I \left(\text{Diagram} \right) [\mu_{11}^2] \right\}$$

$$\text{tr}_\pm(1234) = \frac{1}{2} \text{tr}(1 \pm \gamma_5) \not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4$$

$$\vec{c} = \frac{(D_s - 2)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle \text{tr}_5(1234)} \{ -s_{12}s_{23}s_{34}s_{45}s_{51}, \\ s_{12}s_{23}\text{tr}_-(1345), s_{23}s_{34}\text{tr}_-(2451), s_{34}s_{45}\text{tr}_-(3512), s_{45}s_{51}\text{tr}_-(4123), s_{51}s_{12}\text{tr}_-(5234) \}$$

[Bern, Morgan (1995)]

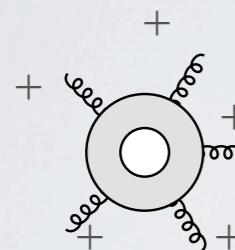
$$+ \text{Diagram} = \frac{(D_s - 2)}{6} \frac{s_{12}s_{23} + s_{23}s_{34} + s_{34}s_{45} + s_{45}s_{51} + s_{51}s_{12} + \text{tr}_5(1234)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} + \mathcal{O}(\epsilon)$$

[Bern, Dixon, Kosower (1993)]

one-loop 5pt all-plus

$$\Delta^{(1)}(1^+, 2^+, 3^+, \dots, n^+) = \frac{D_s - 2}{\langle 12 \rangle^4} \mu_{11}^2 \Delta^{(1), [\mathcal{N}=4]}(1^-, 2^-, 3^+, \dots, n^+)$$

[Bern, Dixon, Dunbar, Kosower (1996)]



$$= \vec{c} \cdot \left\{ I \left(\text{Diagram } 1 \right) [\mu_{11}^3], \right.$$

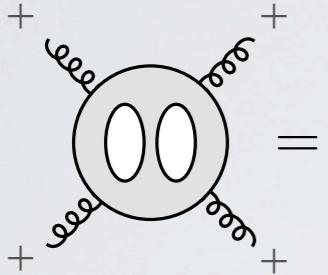
$$I \left(\text{Diagram } 2 \right) [\mu_{11}^2], I \left(\text{Diagram } 3 \right) [\mu_{11}^2], I \left(\text{Diagram } 4 \right) [\mu_{11}^2], I \left(\text{Diagram } 5 \right) [\mu_{11}^2], I \left(\text{Diagram } 6 \right) [\mu_{11}^2] \left. \right\}$$

$$\vec{c} = \frac{(D_s - 2)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \{2\text{tr}_5(1234), s_{12}s_{23}, s_{23}s_{34}, s_{34}s_{45}, s_{45}s_{51}, s_{51}s_{12}\}$$

two-loop 4pt all-plus

(drop Parke-Taylor
pre-factors from now on....)

[Bern, Dixon, Kosower (2000)]



$$= \{s^2t, t^2s, st\} \cdot \{I \left(\text{Diagram A} \right) [F_1], I \left(\text{Diagram B} \right) [F_1], I \left(\text{Diagram C} \right) [F_2 + F_3 \frac{s+(l_1+l_2)^2}{s}] \}$$

**dimension shifting
numerators**

$$F_1 = (D_s - 2) (\mu_{11}\mu_{22} + (\mu_{11} + \mu_{22})^2 + 2(\mu_{11} + \mu_{22})\mu_{12}) + 16(\mu_{12}^2 - \mu_{11}\mu_{22}),$$

$$F_2 = 4(D_s - 2)(\mu_{11} + \mu_{22})\mu_{12},$$

$$F_3 = (D_s - 2)^2 \mu_{11}\mu_{22}.$$

two-loop 5pt all-plus

[SB, Frellesvig, Zhang (2013)]

$$\begin{array}{c}
 \text{Diagram: } \text{A circular loop with two internal vertices labeled '00'. Four external lines are attached to the perimeter, each with a '+' sign.} \\
 + \quad + \quad + \quad + \\
 = \sum_{\text{cyclic}} \left\{ \Delta \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right.} \right) + \Delta \left(\text{Diagram: } \text{A square with a horizontal line across the middle.} \right) + \Delta \left(\text{Diagram: } \text{A square with a vertical line down the middle.} \right) + \Delta \left(\text{Diagram: } \text{A square with a diagonal line from top-right to bottom-left.} \right) \right. \\
 \left. + \Delta \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right, and a vertical line from middle-left to middle-right.} \right) + \Delta \left(\text{Diagram: } \text{A square with a horizontal line across the middle, and a vertical line from middle-left to middle-right.} \right) + \Delta \left(\text{Diagram: } \text{A square with a diagonal line from top-right to bottom-left, and a vertical line from middle-left to middle-right.} \right) + \Delta \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right, and a horizontal line across the middle.} \right) \right\}
 \end{array}$$

$$\Delta \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right.} \right) = \frac{s_{12}s_{23}s_{45}}{\text{tr}_5} \{ s_{34}s_{45}s_{15}, \text{tr}_+(1345) \} \cdot \{ I \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right.} \right) [F_1], I \left(\text{Diagram: } \text{A square with a dashed diagonal line from top-left to bottom-right.} \right) [F_1] \}$$

$$\Delta \left(\text{Diagram: } \text{A square with a horizontal line across the middle.} \right) = \left\{ -\frac{s_{34}s_{45}^2 \text{tr}_+(1235)}{\text{tr}_5} \right\} \cdot \{ I \left(\text{Diagram: } \text{A square with a horizontal line across the middle.} \right) [F_1] \}$$

$$\Delta \left(\text{Diagram: } \text{A square with a vertical line down the middle.} \right) = \left\{ \frac{s_{12}s_{23}s_{34}s_{45}s_{15}}{\text{tr}_5} \right\} \cdot \{ I \left(\text{Diagram: } \text{A square with a vertical line down the middle.} \right) [F_1] \}$$

$$\Delta \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right, and a vertical line from middle-left to middle-right.} \right) = -\frac{s_{12}\text{tr}_+(1345)}{2s_{13}} \{ s_{23}, 1 \} \cdot \{ I \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right, and a vertical line from middle-left to middle-right.} \right) [F_2 + F_3 \frac{s_{45} + (l_1 + l_2)^2}{s_{45}}], I \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right, and a vertical line from middle-left to middle-right.} \right) [(2k_1 \cdot \omega)(F_2 + F_3 \frac{s_{45} + (l_1 + l_2)^2}{s_{45}})] \}$$

$$\Delta \left(\text{Diagram: } \text{A square with a diagonal line from top-right to bottom-left, and a vertical line from middle-left to middle-right.} \right) = \left\{ -\frac{(s_{45} - s_{12})\text{tr}_+(1345)}{2s_{13}} \right\} \cdot \{ I \left(\text{Diagram: } \text{A square with a diagonal line from top-right to bottom-left, and a vertical line from middle-left to middle-right.} \right) [F_2 + F_3 \frac{s_{45} + (l_1 + l_2)^2}{s_{45}}] \}$$

$$\begin{aligned} \Delta \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right, and a horizontal line across the middle.} \right) &= \vec{c} \cdot \{ I \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right, and a horizontal line across the middle.} \right) [F_2], I \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right, and a horizontal line across the middle.} \right) [F_3], I \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right, and a horizontal line across the middle.} \right) [F_3(l_1 + l_2)^2], \\ &\quad I \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right, and a horizontal line across the middle.} \right) [F_3(k_1 \cdot 3)(k_2 \cdot 3)], I \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right, and a horizontal line across the middle.} \right) [F_3(k_1 \cdot 3)], I \left(\text{Diagram: } \text{A square with a diagonal line from top-left to bottom-right, and a horizontal line across the middle.} \right) [F_3(k_2 \cdot 3)], \dots \} \end{aligned}$$

two-loop 5pt all-plus

$$\Delta^{(2)}(1^+, 2^+, 3^+, \dots, n^+) = \frac{D_s - 2}{\langle 12 \rangle^4} F_1 \Delta^{(2), [\mathcal{N}=4]}(1^-, 2^-, 3^+, \dots, n^+) + (\text{1-loop})^2$$

all genuine two-loop topologies related to $\mathcal{N}=4$ MHV

non-planar cuts via BCJ
[SB, Mogull, Ochirov, O'Connell (2015)]

complete BCJ numerator
representation
[Mogull, O'Connell (2015)]

$$\begin{aligned} \mathcal{A}_5^{(2)}(1^+, 2^+, 3^+, 4^+, 5^+) = & \sum_{\sigma \in S_5} I \left[C \left(\text{Diagram } 1 \right) \left(\frac{1}{2} \Delta \left(\text{Diagram } 2 \right) + \Delta \left(\text{Diagram } 3 \right) + \frac{1}{2} \Delta \left(\text{Diagram } 4 \right) \right. \right. \\ & \left. \left. + \frac{1}{2} \Delta \left(\text{Diagram } 5 \right) + \Delta \left(\text{Diagram } 6 \right) + \frac{1}{2} \Delta \left(\text{Diagram } 7 \right) \right) \right. \\ & \left. + C \left(\text{Diagram } 8 \right) \left(\frac{1}{4} \Delta \left(\text{Diagram } 9 \right) + \frac{1}{2} \Delta \left(\text{Diagram } 10 \right) + \frac{1}{2} \Delta \left(\text{Diagram } 11 \right) \right. \right. \\ & \left. \left. - \Delta \left(\text{Diagram } 12 \right) + \frac{1}{4} \Delta \left(\text{Diagram } 13 \right) \right) \right. \\ & \left. + C \left(\text{Diagram } 14 \right) \left(\frac{1}{4} \Delta \left(\text{Diagram } 15 \right) + \frac{1}{2} \Delta \left(\text{Diagram } 16 \right) + \frac{1}{2} \Delta \left(\text{Diagram } 17 \right) \right) \right] \end{aligned}$$

two-loop 5pt all-plus amplitude

Gehrman, Henn, Lo Presti [1511.05409]

planar master integrals using canonical differential equation approach

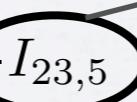
Dunbar, Perkins [1603.07514]

4D unitarity cuts + augmented BCFW

$$A_5^{(2)} = A_5^{(1)} \left[- \sum_{i=1}^5 \frac{1}{\epsilon^2} \left(\frac{\mu^2}{-v_i} \right)^\epsilon \right] + R F_5^{(2)} + \mathcal{O}(\epsilon)$$

$$\begin{aligned} F_5^{(2)} = & \frac{5\pi^2}{12} F_5^{(1)} + \sum_{i=0}^4 \sigma^i \left\{ \frac{v_5 \text{tr} \left[(1 - \gamma_5) \not{p}_4 \not{p}_5 \not{p}_1 \not{p}_2 \right]}{(v_2 + v_3 - v_5)} \right. \\ & \left. + \frac{1}{6} \frac{\text{tr} \left[(1 - \gamma_5) \not{p}_4 \not{p}_5 \not{p}_1 \not{p}_2 \right]^2}{v_1 v_4} + \frac{10}{3} v_1 v_2 + \frac{2}{3} v_1 v_3 \right\} . \quad (8) \end{aligned}$$

simple
function of Li_2



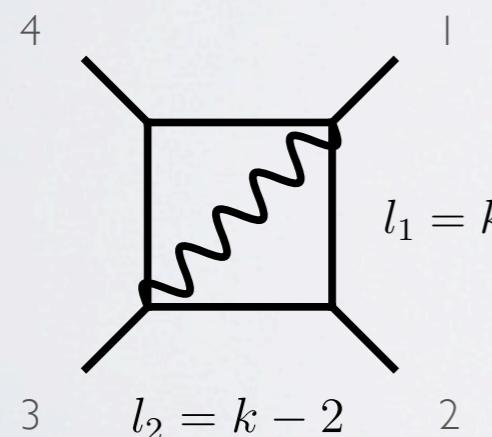
local integrands

[Arkani-Hamed, Bourjaily, Cachazo, Trnka 1012.6032]

[Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka 1008.2958]

manage infra-red
divergences at the
integrand level

simple integrals with
unit leading
singularities



$$= I_4^{4-2\epsilon} [\text{tr} (1l_1l_23)] = I_4^{6-2\epsilon}[1]$$

one-loop integrand bases

$$\begin{aligned} \text{:} \circlearrowleft \text{:} &= \sum c_4 I \left(\text{square loop} \right) + \sum c_3 I \left(\text{triangle loop} \right) + \sum c_2 I \left(\text{circle loop} \right) + \text{ rational} \\ &= \text{:} \circlearrowleft \text{:} \sum -P^2 I \left(\text{triangle loop} \right) \\ &\quad + \sum c_4 I \left(\text{square loop with pole} \right) + \sum c_3^{3m} I \left(\text{triangle loop with pole} \right) + \sum c_2 I \left(\text{circle loop} \right) + \text{ rational} \end{aligned}$$

universal
infrared poles

still complicated functions at
high multiplicity

two-loop 5pt all-plus

[SB, Mogull, Peraro (in progress)]

$$\text{Diagram} = \sum_{\text{cyclic}} \Delta \left(\text{Diagram}_1 \right) + \Delta \left(\text{Diagram}_2 \right) + \Delta \left(\text{Diagram}_3 \right) + \Delta \left(\text{Diagram}_4 \right)$$

$$\Delta \left(\text{Diagram}_1 \right) = \{s_{45}\} \cdot \{I \left(\text{Diagram}_1 \right) [F_1]\}$$

$$\Delta \left(\text{Diagram}_2 \right) = \{s_{12}s_{45}s_{15}\} \cdot \{I \left(\text{Diagram}_2 \right) [F_1]\}$$

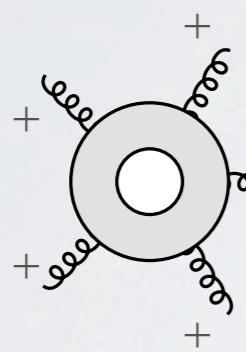
$$\Delta \left(\text{Diagram}_3 \right) = \{1, 1\} \cdot \{I \left(\text{Diagram}_3 \right) [F_2 + F_3], I \left(\text{Diagram}_3 \right) [F_3 \frac{(l_1+l_2)^2}{s_{45}}]\}$$

$$\Delta \left(\text{Diagram}_4 \right) = \{\text{tr}_+(1245), \frac{s_{12}s_{45}s_{15} + (s_{12}+s_{45})\text{tr}_+(1245)}{s_{12}s_{45}}, -1\} \cdot \{$$

$$\left(\text{Diagram}_4 \right) [F_2 + F_3], \left(\text{Diagram}_4 \right) [F_3(l_1 + l_2)^2], \left(\text{Diagram}_4 \right) [F_3 \text{tr}_+(123l_1l_2345)]\}$$

infrared poles

$$A^{(2)}(1^+, 2^+, 3^+, \dots, n^+) = A^{(1)}(1^+, 2^+, 3^+, \dots, n^+) \sum_{i=1}^n \left(\frac{\mu_R^2}{-s_{i,i+1}} \right)^\epsilon + \mathcal{O}(\epsilon^0)$$



$$\text{local one-loop all-plus: } F_I \times \mathcal{N} = 4 \text{ using Arkani-Hamed et al. 1008.2958}$$
$$= \frac{1}{5} \sum_{\text{cyclic}} \left\{ \frac{\text{tr}_+(1345)}{s_{13}}, s_{23}s_{34}, s_{12}s_{15} \right\} \cdot \left\{ I \left(\begin{array}{c} \text{wavy} \\ \text{---} \\ \text{---} \end{array} \right) [\mu_{11}^2], I \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{wavy} \end{array} \right) [\mu_{11}^2], I \left(\begin{array}{c} \text{---} \\ \text{wavy} \\ \text{---} \end{array} \right) [\mu_{11}^2], \right\}$$

integrals in the soft limit

$$F_1 = (D_s - 2)(2\mu_{11}\mu_{22} + \mu_{11}^2 + \mu_{22}^2 + \mu_{12}(\mu_{11} + \mu_{22})) + 16(\mu_{12}^2 - \mu_{11}\mu_{22}) \quad \lim_{k_1 \rightarrow 0} F_1 = (D_s - 2)\mu_{22}^2$$

$$I^{4-2\epsilon} \left(\begin{array}{c} 3 \\ k_2 \\ \hline 2 \\ \hline 4 & k_1 \\ \hline 1 \end{array} \right) [F_1] \xrightarrow{k_1 \rightarrow 0} (D_s - 2) I^{4-2\epsilon} \left(\begin{array}{c} 3 & 2 \\ \hline 4 & 1 \end{array} \right) [\mu_{22}^2] I^{4-2\epsilon} \left(\begin{array}{c} 2 \\ \hline 1 \end{array} \right)$$

$$I^{4-2\epsilon} \left(\begin{array}{c} 3 \\ k_2 \\ \hline 2 \\ \hline 4 & k_1 \\ \hline 1 \end{array} \right) [F_1] \xrightarrow{k_2 \rightarrow 0} (D_s - 2) I^{4-2\epsilon} \left(\begin{array}{c} 3 \\ \hline 4 \end{array} \right) I^{4-2\epsilon} \left(\begin{array}{c} 3 & 2 \\ \hline 4 & 1 \end{array} \right) [\mu_{11}^2]$$

$$I^{4-2\epsilon} \left(\begin{array}{c} 3 & 2 \\ \hline 4 & 1 \end{array} \right) [\mu_{11}^2] = -\frac{1}{6} + \mathcal{O}(\epsilon)$$

$$I^{4-2\epsilon} \left(\begin{array}{c} 2 \\ \hline 1 \end{array} \right) = \frac{c_\Gamma}{\epsilon^2} (-s_{12})^{-1-\epsilon} = -\frac{1}{(4\pi)^2 s_{12} \epsilon^2} + \mathcal{O}(\epsilon^{-1})$$

$$\Rightarrow I^{4-2\epsilon} \left(\begin{array}{c} 3 \\ k_2 \\ \hline 2 \\ \hline 4 & k_1 \\ \hline 1 \end{array} \right) [F_1] = \frac{D_s - 2}{(4\pi)^2 3 \epsilon^2 s_{12}} + \mathcal{O}(\epsilon^{-1}) \quad \text{in this case actually holds to } 1/\epsilon$$

integrals in the soft limit

simple identification of IR poles with local integrands

$$I \left(\begin{array}{c} \text{square loop} \\ | \\ \text{square loop} \end{array} \right) [F_1] = \frac{c_\Gamma}{\epsilon^2} (-s_{45})^{-1-\epsilon} I \left(\begin{array}{c} \text{square loop} \\ | \\ \text{square loop} \end{array} \right) [\mu_{11}^2] + \mathcal{O}(\epsilon^{-1})$$

$$\begin{aligned} I \left(\begin{array}{c} \text{square loop} \\ | \\ \text{square loop} \end{array} \right) [F_1] &= \frac{c_\Gamma}{\epsilon^2} (-s_{45})^{-1-\epsilon} I \left(\begin{array}{c} \text{square loop} \\ | \\ \text{square loop} \end{array} \right) [\mu_{11}^2] + \frac{c_\Gamma}{\epsilon^2 s_{23}} (-s_{12})^{-1-\epsilon} I \left(\begin{array}{c} \text{square loop} \\ | \\ \text{square loop} \end{array} \right) [\mu_{11}^2] \\ &\quad + \frac{c_\Gamma}{\epsilon^2 s_{12}} (-s_{23})^{-1-\epsilon} I \left(\begin{array}{c} \text{square loop} \\ | \\ \text{square loop} \end{array} \right) [\mu_{11}^2] + \mathcal{O}(\epsilon^{-1}) \end{aligned}$$

$$I \left(\begin{array}{c} \text{square loop} \\ | \\ \text{square loop} \end{array} \right) [F_1] = \frac{c_\Gamma}{\epsilon^2} (-s_{45})^{-1-\epsilon} I \left(\begin{array}{c} \text{square loop} \\ | \\ \text{square loop} \end{array} \right) [\mu_{11}^2] + \mathcal{O}(\epsilon^0)$$

$$I \left(\begin{array}{c} \text{square loop} \\ | \\ \text{square loop} \end{array} \right) [F_1] = \frac{c_\Gamma}{\epsilon^2} (-s_{12})^{-1-\epsilon} I \left(\begin{array}{c} \text{square loop} \\ | \\ \text{square loop} \end{array} \right) [\mu_{11}^2] + \frac{c_\Gamma}{\epsilon^2} (-s_{45})^{-1-\epsilon} I \left(\begin{array}{c} \text{square loop} \\ | \\ \text{square loop} \end{array} \right) [\mu_{11}^2] + \mathcal{O}(\epsilon^0)$$

$$I \left(\begin{array}{c} \text{square loop} \\ | \\ \text{square loop} \end{array} \right) [F_1] = I \left(\begin{array}{c} \text{triangle} \\ \diagup \diagdown \\ \text{triangle} \end{array} \right) [1] \times I \left(\begin{array}{c} \text{square loop} \\ | \\ \text{square loop} \end{array} \right) [\mu_{11}^2] + \mathcal{O}(\epsilon^0)$$

two-loop 5pt all-plus

$$\text{Diagram with two internal lines} = \sum_{\text{cyclic}} \Delta \left(\text{Diagram with one internal line} \right) + \Delta \left(\text{Diagram with two internal lines} \right) + \Delta \left(\text{Diagram with three internal lines} \right) + \Delta \left(\text{Diagram with four internal lines} \right)$$

$$\text{Diagram with two internal lines} = \left(\sum_{i=1}^5 s_{i,i+1} I \left(\text{Diagram with one internal line} \right) [1] \right) + \text{finite}$$

$$\text{Diagram with two internal lines} = \frac{1}{5} \sum_{\text{cyclic}} \left\{ \frac{\text{tr}_+(1345)}{s_{13}}, s_{23}s_{34}, s_{12}s_{15} \right\} \cdot \left\{ \begin{array}{l} I \left(\text{Diagram with one internal line} \right) [\mu_{11}^2], I \left(\text{Diagram with two internal lines} \right) [\mu_{11}^2], I \left(\text{Diagram with three internal lines} \right) [\mu_{11}^2], \end{array} \right\}$$

two-loop 6pt all-plus

[SB, Mogull, Peraro (in progress)]

$$\text{Diagram with 6 external legs labeled } + = \sum_{\text{cyclic}} \Delta \left(\text{Diagram 1} \right) + \Delta \left(\text{Diagram 2} \right) + \Delta \left(\text{Diagram 3} \right) + \Delta \left(\text{Diagram 4} \right) + \Delta \left(\text{Diagram 5} \right) + \Delta \left(\text{Diagram 6} \right) + \Delta \left(\text{Diagram 7} \right) + \Delta \left(\text{Diagram 8} \right) + \Delta \left(\text{Diagram 9} \right) + \Delta \left(\text{Diagram 10} \right) + \Delta \left(\text{Diagram 11} \right)$$

follows the expected $\mathcal{N} = 4 \times F_1$ structure



[Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich (2008)]

[Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka (2010)]

two-loop 6pt all-plus

$$\text{Diagram with two loops and six external legs} = \sum_{\text{cyclic}} \Delta \left(\text{Diagram 1} \right) + \Delta \left(\text{Diagram 2} \right) + \Delta \left(\text{Diagram 3} \right) + \Delta \left(\text{Diagram 4} \right) + \Delta \left(\text{Diagram 5} \right) + \Delta \left(\text{Diagram 6} \right) \\
 + \Delta \left(\text{Diagram 7} \right) + \Delta \left(\text{Diagram 8} \right) + \Delta \left(\text{Diagram 9} \right) + \Delta \left(\text{Diagram 10} \right) + \Delta \left(\text{Diagram 11} \right) + \Delta \left(\text{Diagram 12} \right)$$

$$\text{Diagram with two loops and six external legs} = \left(\sum_{i=1}^6 s_{i,i+1} I \left(\text{Diagram 13} \right) [1] \right) \text{Diagram with one loop and six external legs} + \text{finite}$$

$$\text{Diagram with one loop and six external legs} = \frac{1}{6} \sum_{\text{cyclic}} \left\{ \text{tr}_+(123456), \frac{\text{tr}_+(1456)}{s_{14}}, \frac{\text{tr}_+(13(4+5)6)}{s_{13}}, \frac{\text{tr}_+(134(5+6))}{s_{13}}, s_{56}s_{45}, s_{56}s_{16}, s_{56}s_{345} \right\} \cdot \left\{ \begin{array}{l} I \left(\text{Diagram 14} \right) [\mu_{11}^6], I \left(\text{Diagram 15} \right) [\mu_{11}^2], I \left(\text{Diagram 16} \right) [\mu_{11}^2], I \left(\text{Diagram 17} \right) [\mu_{11}^2], \\ I \left(\text{Diagram 18} \right) [\mu_{11}^2], I \left(\text{Diagram 19} \right) [\mu_{11}^2], I \left(\text{Diagram 20} \right) [\mu_{11}^2], \end{array} \right\}$$

outlook

- Local integrands are an efficient way to manage IR divergences
- Simple expressions for 5 and 6 gluon all-plus amplitudes in Yang-Mills
- General basis of local integrands still unknown - connection with polynomial division algorithm?
- Local integrand representations for non-planar sector?

Backup

momentum twistors

[Hodges (2009)]

an all multiplicity parameterisation (not unique)

$$Z_{iA} = \left(\frac{\Sigma_i}{s_{12}}, 1 - \delta_{1i}, \frac{\langle 123i \rangle \langle 34 \rangle [23]}{\langle 1234 \rangle \langle 1i \rangle [12]}, \frac{-\langle 13 \rangle \langle 124i \rangle + \langle 14 \rangle \langle 123i \rangle}{\langle 1234 \rangle \langle 1i \rangle} \right)$$

$$\Sigma_i = \begin{cases} s_{12} & i = 1 \\ \frac{\langle 13 \rangle \langle 2i \rangle}{\langle 23 \rangle \langle 1i \rangle} & i \neq 1 \end{cases}$$

build spinors products
etc. from rational
phase-space points

```
In[21]:= NN = 6;  
GetMTrepALT[NN];  
mtZ // MatrixForm // SymbolForm
```

Out[23]//DisplayForm=

$$\begin{pmatrix} 1 & 0 & \frac{1}{s_{12}} & \frac{\sigma_4}{s_{12}} & \frac{\sigma_5}{s_{12}} & \frac{\sigma_6}{s_{12}} \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & x_{1234} & x_{1235} & 1 \\ 0 & 0 & 1 & 1 & \frac{x_{1235}-x_{1245}}{x_{1234}} & \frac{1-x_{1246}}{x_{1234}} \end{pmatrix}$$

momentum twistors

example: BCFW using rational kinematics

```
In[143]:= NN = 7;
GetMTrepALT[NN];
mtZ // MatrixForm // SymbolForm
mtZ = mtZ /. lpS[1,2]→1; mtW = mtW /. lpS[1,2]→1;
{p1,p2,p3,p4,p5,p6,p7} = Map[ToSPN4[MT[#]]&, Range@NN];
Map[Simplify,BCFW4[AMP[{p1,1,gluon},{p2,1,gluon},{p3,1,gluon},{p4,2,gluon},{p5,2,gluon},{p6,2,gluon},{p7,2,gluon}],1,2]] // SymbolForm
```

```
Jut[145]//DisplayForm=

$$\begin{pmatrix} 1 & 0 & \frac{1}{s_{12}} & \frac{\sigma_4}{s_{12}} & \frac{\sigma_5}{s_{12}} & \frac{\sigma_6}{s_{12}} & \frac{\sigma_7}{s_{12}} \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & x_{1234} & x_{1235} & x_{1236} & 1 \\ 0 & 0 & 1 & 1 & \frac{x_{1235}-x_{1245}}{x_{1234}} & \frac{x_{1236}-x_{1246}}{x_{1234}} & \frac{1-x_{1247}}{x_{1234}} \end{pmatrix}$$

```

```
Jut[148]//DisplayForm=

$$-\frac{i (x_{1234} - x_{1235} + x_{1245})^3}{(-1 + \sigma_4) (\sigma_4 - \sigma_5) (\sigma_5 - \sigma_6) (\sigma_6 - \sigma_7) x_{1234} x_{1235} x_{1245} (\sigma_5 x_{1234} - \sigma_4 x_{1235} + x_{1245})} -$$
  

$$-\frac{i (x_{1234} (x_{1235} - x_{1236}) - x_{1236} x_{1245} + x_{1235} x_{1246})^3}{(-1 + \sigma_4) (\sigma_4 - \sigma_5) (\sigma_5 - \sigma_6) (\sigma_6 - \sigma_7) x_{1234} x_{1235} x_{1236} (-x_{1236} x_{1245} + x_{1235} x_{1246}) (-x_{1236} (\sigma_5 x_{1234} + x_{1245}) + x_{1235} (\sigma_6 x_{1234} + x_{1246}))} +$$
  

$$-\frac{i (x_{1234} (-1 + x_{1236}) - x_{1246} + x_{1236} x_{1247})^3}{(-1 + \sigma_4) (\sigma_4 - \sigma_5) (\sigma_5 - \sigma_6) (\sigma_6 - \sigma_7) x_{1234} x_{1236} (-x_{1246} + x_{1236} x_{1247}) (\sigma_6 x_{1234} + x_{1246} - x_{1236} (\sigma_7 x_{1234} + x_{1247}))}$$

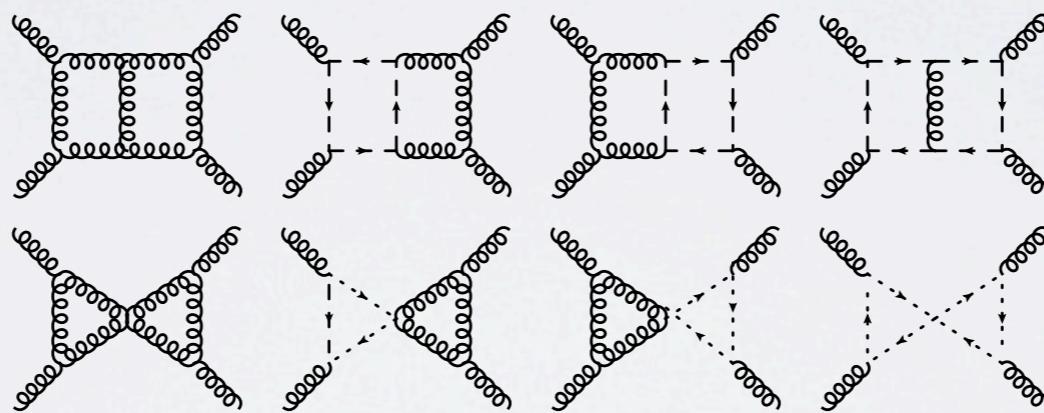
```

[c.f. bcfw mathematica package Bourjaily (2010)]

six-dimensional trees

six dimensional spinor helicity [Cheung, O'Connell 0902.0981]
(one-loop applications [Bern et al. 1010.0494] [Davies 1108.0398])

dimensional reduction to $4 - 2\epsilon$ using additional scalar amplitudes



6D is a convenient way to manage all helicities simultaneously

$$A(1_{a\dot{a}}, 2_{b\dot{b}}, 3_{d\dot{d}}, 4_{d\dot{d}}) = \frac{i}{st} \langle 1_a 2_b 3_c 4_d \rangle [1_{\dot{a}} 2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}}]$$

non-planar from planar

e.g. [Bern, Carrasco, Dixon, Johansson, Roiban |201.5366]

$$A_4(1, 2, 3, 4) = \frac{s_{13}}{s_{12}} A_4(1, 3, 2, 4)$$

factorization

$$\Rightarrow A_3(1, 2, -P_{12}) A_3(P_{12}, 3, 4) = \underset{s_{12}=0}{\text{Res}} (A_4(1, 2, 3, 4)) = s_{13} A_4(1, 3, 2, 4) \Big|_{s_{12}=0}$$

$$\Rightarrow \quad \begin{array}{c} \text{Diagram} \\ \text{with red cuts} \end{array} = (k_1 - P_{123})^2 \quad \begin{array}{c} \text{Diagram} \\ \text{without red cuts} \end{array} \quad \Bigg| \quad (k_1 + k_2 + p_3)^2$$

$$\Rightarrow \boxed{\Delta \left(\text{Diagram} \right) \Big|_{\text{cut}} = \left((k_1 - P_{123})^2 \Delta \left(\text{Diagram} \right) + \Delta \left(\text{Diagram} \right) - \Delta \left(\text{Diagram} \right) \right) \Big|_{\text{cut}}}$$