local integrands for two-loop QCD amplitudes

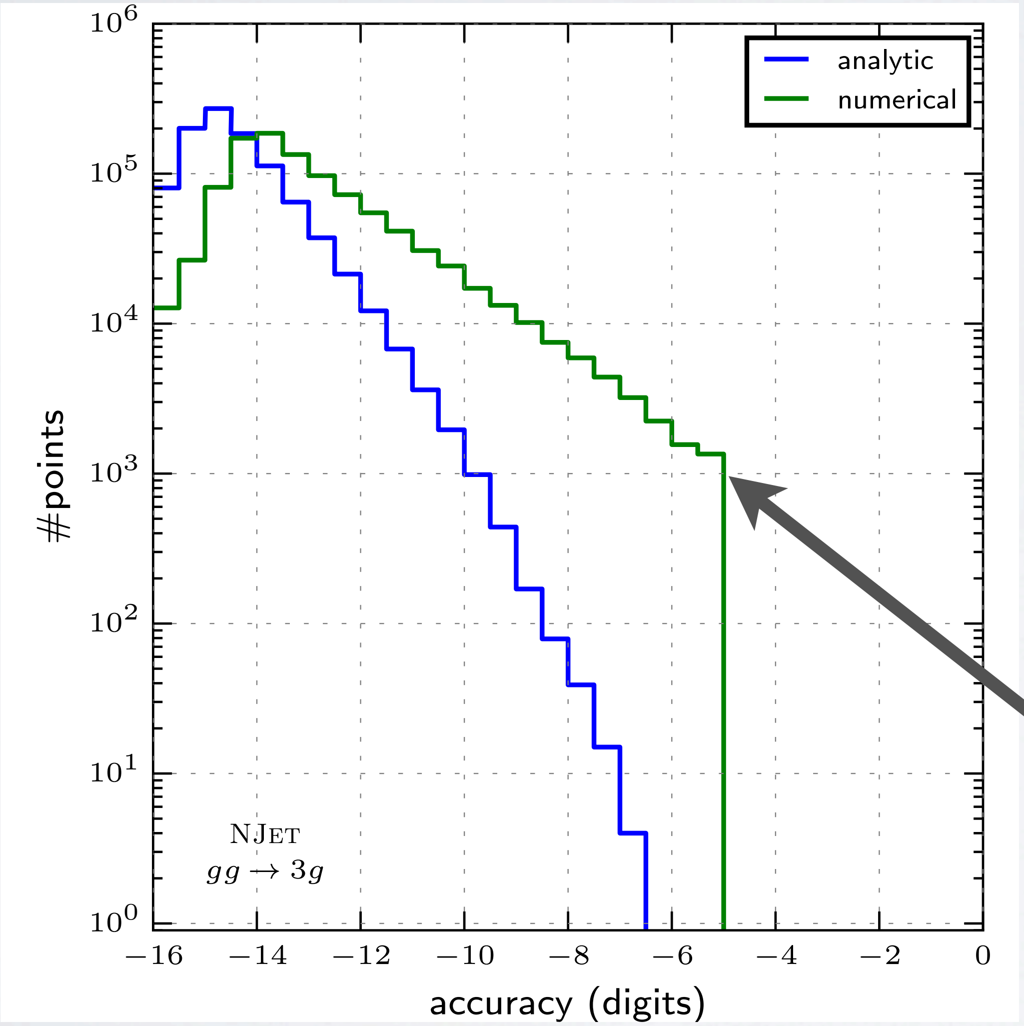
Simon Badger

25th April 2016

Loops and Legs in Quantum Field Theory, Leipzig, Germany

spurious singularities

big jump between 2⟶2 and 2⟶3 kinematics



applications of loop amplitudes in NNLO computations more intensive

numerical D-dimensional generalised unitarity

vs

analytic computation with finite integrals basis (e.g. Bern, Dixon, Kosower [hep-ph/9302280])

removing spurious poles also simplifies coefficients ⇒ faster

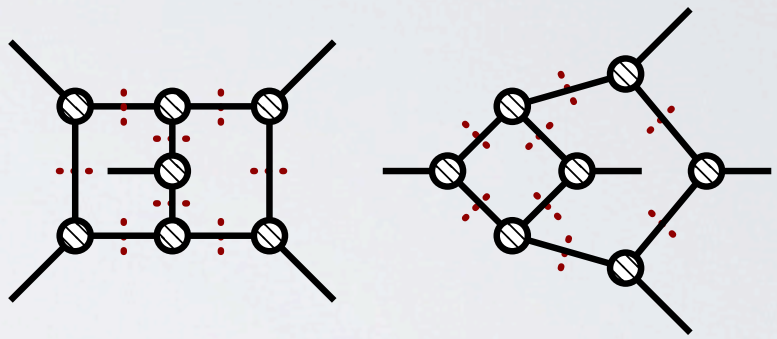
(~100x in this case)

need to switch to quadruple precision evaluation

outline

- d-dimensional generalised unitarity

  ⇒ multi-loops from trees



- two-loop all-plus amplitudes

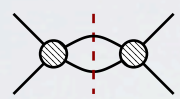
- local integrands for d-dimensional amplitudes

work with Hjalte Frellesvig, Yang Zhang, Gustav Mogull, Alex Ochirov, Donal O'Connell, Tiziano Peraro

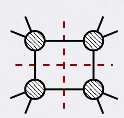
automated one-loop amplitudes

solving on-shell conditions requires **complex** momenta
⇒ factorise residues into **tree amplitudes**

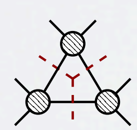
Unitarity: double cuts
[BDDK '94]
[triple cuts BDK '97]



Generalized unitarity:
quadruple cuts [BCF '04]

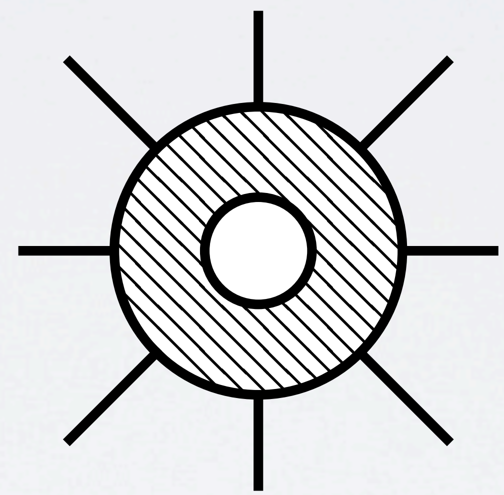


triple cuts [e.g. Forde '07]



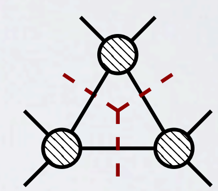
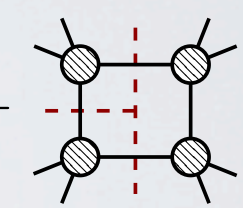
$$A = \sum_i (\text{rational})_i (\text{integral})_i$$

find complex contour to isolate
integral coefficient



multi-scale
kinematic algebra
performed
numerically

Integrand reduction [OPP '05]

$\Delta_3 =$  $-$ 

D-dim. generalized unitarity [GKM '08]

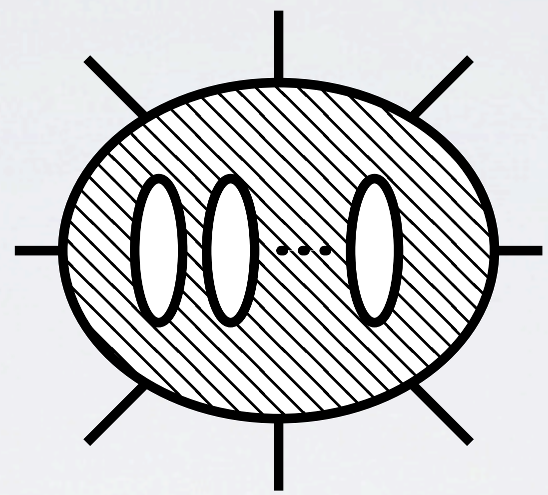
$$A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

explicitly remove poles

multi-loop amplitudes from trees

Maximal unitarity

[Kosower, Larsen, Johansson, Caron-Huot, Zhang, Søgaard]



Integrand reduction via polynomial division

[Mastrolia, Ossola, SB, Frellesvig, Zhang, Mirabella, Peraro, Malamos, Kleiss, Papadopolous, Verheyen, Feng, Huang]

$$A = \sum_i (\text{rational})_i (\text{integral})_i \xleftarrow{\text{e.g. IBPs}} A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

IBPs must be free of doubled propagator MI

[Gluza, Kosower, Kajda 1009.0472] [Schabinger 1111.4220] [Ita 1510.05626] [Larsen, Zhang 1511.01071]

a toolbox for multi-loop integrands

momentum twistors
[Hodges (2009)]

six-dimensional spinor-helicity

$$A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

generalised unitarity cuts

integrand reduction

$$\mathcal{A} = \sum_i S_i \frac{C(\Delta_i)\Delta_i}{\prod D_\alpha}$$

colour/kinematics BCJ relations

multi-loop integrand reduction

[Mastrolia, Ossola 1107.6041] [SB, Frellesvig, Zhang 1202.2019]

[Zhang 1205.5707] [Mastrolia, Mirabella, Ossola, Peraro 1205.7087]

ISP monomials

$$\Delta_T(\{k\}) = \sum_{\{\alpha\}} c_{T;\alpha_1\ldots\alpha_n} (k_1 \cdot p_5)^{\alpha_1} (k_2 \cdot p_2)^{\alpha_2} \ldots (k_1 \cdot \omega_2)^{\alpha_m} \ldots \mu_{12}^{\alpha_n}$$

rational coefficients

spurious directions

extra-dimensional ISPs

$$\mu_{ij} = -k_i^{-2\epsilon} \cdot k_j^{-2\epsilon}$$

integrand reduction

$$\Delta_{c;T}\Big|_{\text{cut}} = \prod_i A_i^{(0)} - \sum_{T'} \frac{\Delta_{c;T'}}{\prod_{l\in T'/T} D_l}\Bigg|_{\text{cut}}$$

on-shell the numerators can be written as products of tree-level amplitudes

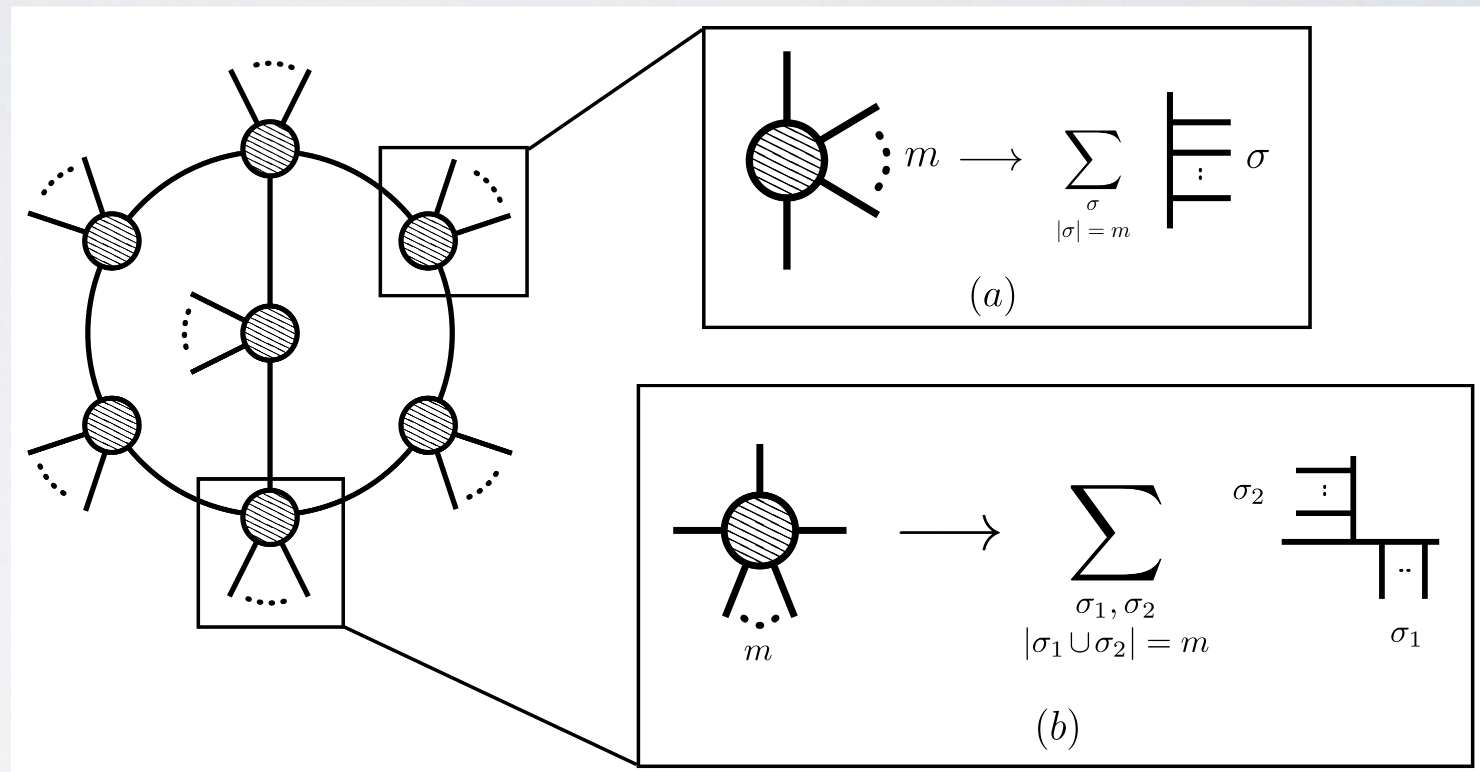
fix basis of **monomials** in **irreducible scalar products** via **polynomial division** (Gröbner basis)

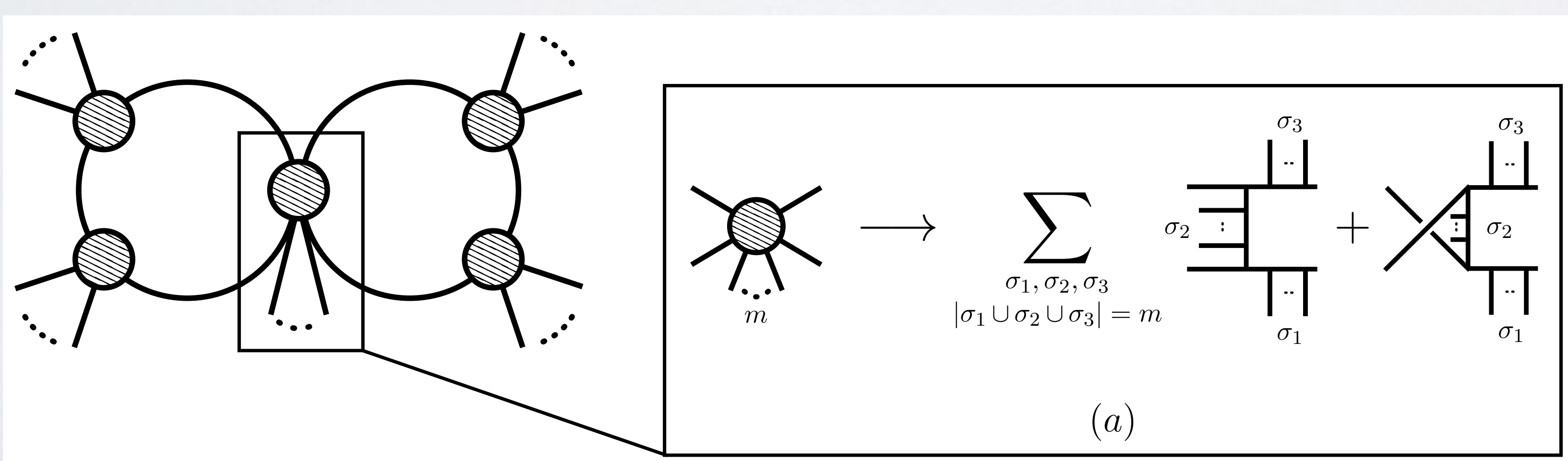
integrand parameterisations not unique - freedom in the choices of ISP monomials

General colour decompositions

[Dixon, Del Duca, Maltoni (1999)]

Inserting the DDM decomposition into colour dressed cuts leads to a compact loop decomposition

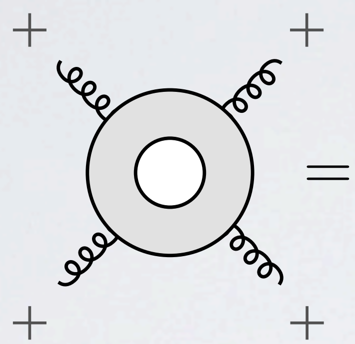




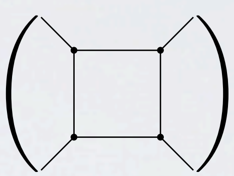
general tree-level DDM colour bases including fermions [Johansson, Ochirov arXiv:1507.00332]

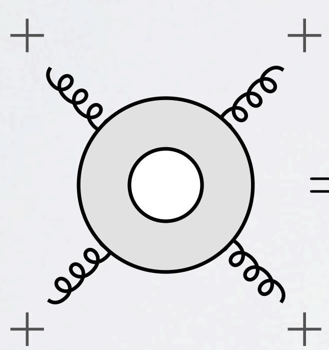
applications:
all-plus amplitudes in QCD

one-loop 4pt all-plus



$$= \frac{D_s - 2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\{-st\}\cdot\left\{ I\left(\,\cdot\, \right)[\mu_{11}^2]\right\}$$



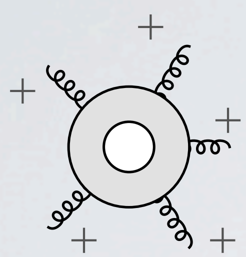
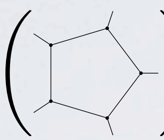


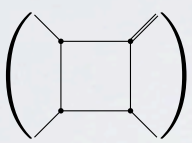
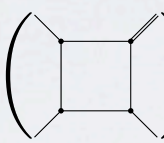
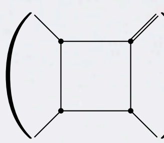
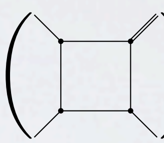
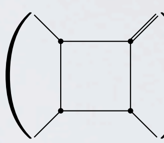
$$= \frac{D_s - 2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{-st}{6} + \mathcal{O}(\epsilon)$$

[Bern, Kosower (1991)]

one-loop 5pt all-plus

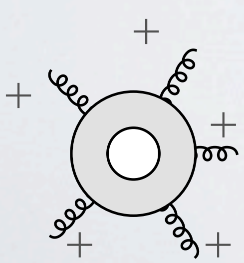
$\mathrm{tr}_{\pm}(1234) = \frac{1}{2}\mathrm{tr}(1 \pm \gamma_5)\not{p}_1\not{p}_2\not{p}_3\not{p}_4)$

 $= \vec{c} \cdot \Big\{ I \Big($  $\Big) [\mu_{11}^2],$

$I \Big($  $\Big) [\mu_{11}^2], I \Big($  $\Big) [\mu_{11}^2], I \Big($  $\Big) [\mu_{11}^2], I \Big($  $\Big) [\mu_{11}^2], I \Big($  $\Big) [\mu_{11}^2] \Big\}$

$$\vec{c} = \frac{(D_s - 2)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5(1234)} \{-s_{12}s_{23}s_{34}s_{45}s_{51},$$
$$s_{12}s_{23}\mathrm{tr}_-(1345), s_{23}s_{34}\mathrm{tr}_-(2451), s_{34}s_{45}\mathrm{tr}_-(3512), s_{45}s_{51}\mathrm{tr}_-(4123), s_{51}s_{12}\mathrm{tr}_-(5234)\}$$

[Bern, Morgan (1995)]

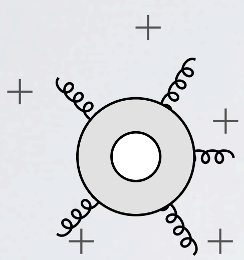
 $= \frac{(D_s - 2)}{6} \frac{s_{12}s_{23} + s_{23}s_{34} + s_{34}s_{45} + s_{45}s_{51} + s_{51}s_{12} + \mathrm{tr}_5(1234)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} + \mathcal{O}(\epsilon)$

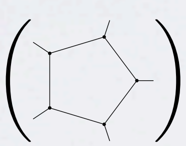
[Bern, Dixon, Kosower (1993)]

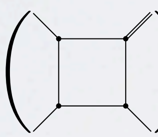
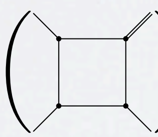
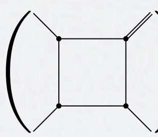
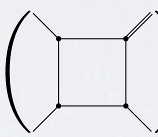
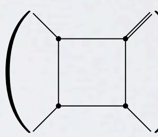
one-loop 5pt all-plus

$$\Delta^{(1)}(1^+,2^+,3^+,\ldots,n^+) = \frac{D_s-2}{\langle 12\rangle^4}\,\mu_{11}^2\,\Delta^{(1),[\mathcal{N}=4]}(1^-,2^-,3^+,\ldots,n^+)$$

[Bern, Dixon, Dunbar, Kosower (1996)]



$= \vec{c}\cdot\Big\{ I\Big($  $\Big)[\mu_{11}^3],$

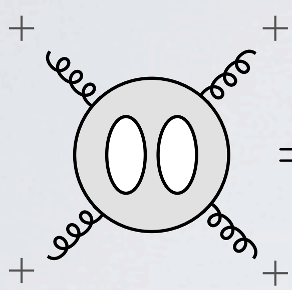
$I\Big($  $\Big)[\mu_{11}^2], I\Big($  $\Big)[\mu_{11}^2], I\Big($  $\Big)[\mu_{11}^2], I\Big($  $\Big)[\mu_{11}^2], I\Big($  $\Big)[\mu_{11}^2]\Big\}$

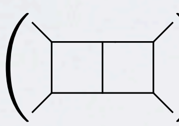
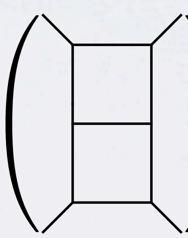
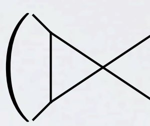
$$\vec{c} = \frac{(D_s-2)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\{2\mathrm{tr}_5(1234), s_{12}s_{23}, s_{23}s_{34}, s_{34}s_{45}, s_{45}s_{51}, s_{51}s_{12}\}$$

two-loop 4pt all-plus

(drop Parke-Taylor pre-factors from now on....)

[Bern, Dixon, Kosower (2000)]



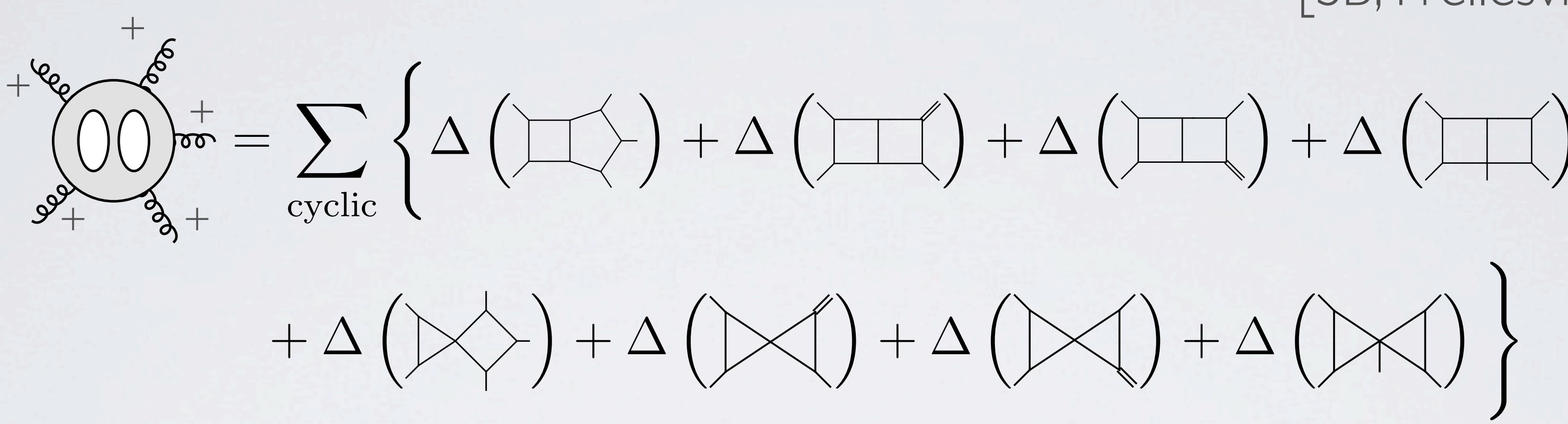
$= \{s^2 t, t^2 s, st\} \cdot \{I\,\big($  $\big)\,[F_1], I\,\big($  $\big)\,[F_1], I\,\big($  $\big)\,[F_2 + F_3 \frac{s+(l_1+l_2)^2}{s}]\}$

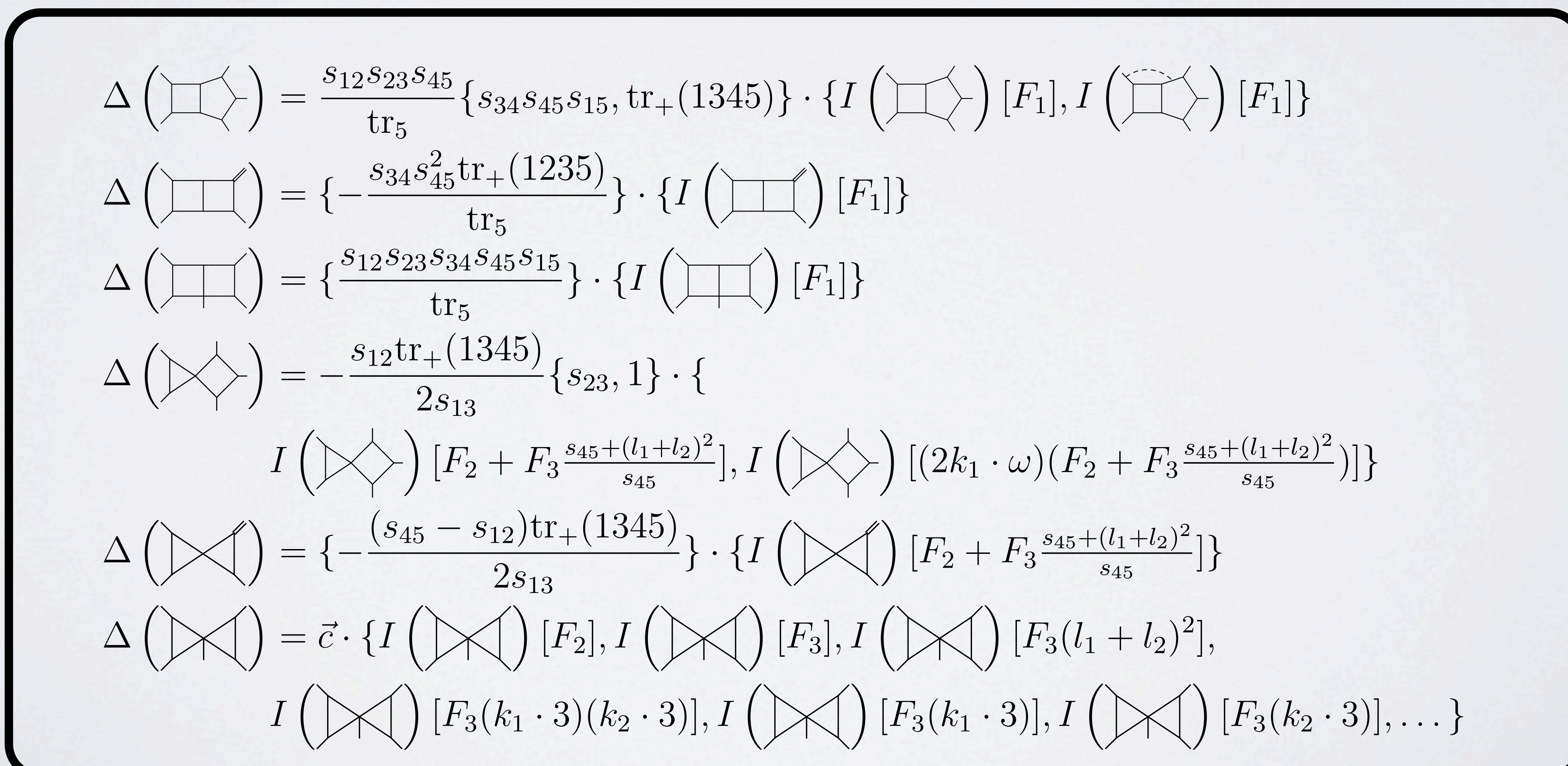
dimension shifting numerators

$$F_1 = (D_s-2)\big(\mu_{11}\mu_{22} + (\mu_{11}+\mu_{22})^2 + 2(\mu_{11}+\mu_{22})\mu_{12}\big) + 16(\mu_{12}^2 - \mu_{11}\mu_{22}),$$
$$F_2 = 4(D_s-2)(\mu_{11}+\mu_{22})\mu_{12},$$
$$F_3 = (D_s-2)^2\mu_{11}\mu_{22}.$$

two-loop 5pt all-plus

[SB, Frellesvig, Zhang (2013)]





two-loop 5pt all-plus

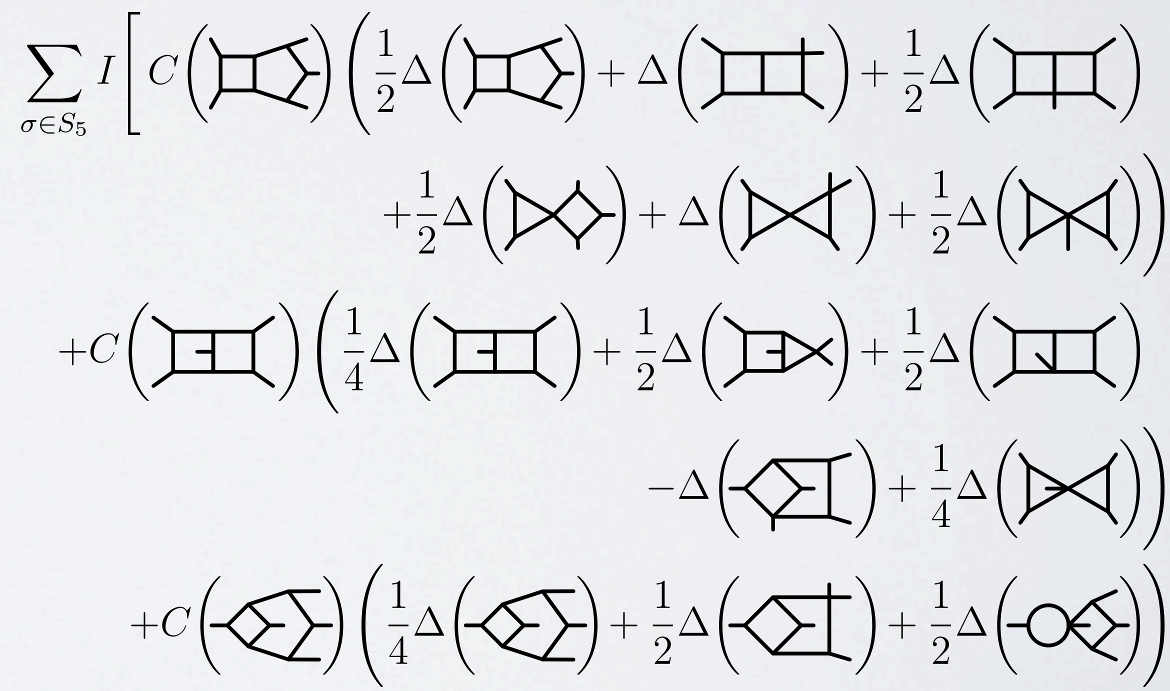
$$\Delta^{(2)}(1^+,2^+,3^+,\ldots,n^+) = \frac{D_s-2}{\langle 12\rangle^4}\,F_1\,\Delta^{(2),[\mathcal{N}=4]}(1^-,2^-,3^+,\ldots,n^+) + (\text{1-loop})^2$$

all genuine two-loop topologies related to $\mathcal{N}$=4 MHV

non-planar cuts via BCJ
[SB, Mogull, Ochirov, O'Connell (2015)]

complete BCJ numerator representation
[Mogull, O'Connell (2015)]

$$\mathcal{A}_5^{(2)}(1^+,2^+,3^+,4^+,5^+) =$$



two-loop 5pt all-plus amplitude

Gehrmann, Henn, Lo Presti [1511.05409]

planar master integrals using canonical differential equation approach

Dunbar, Perkins [1603.07514]

4D unitarity cuts + augmented BCFW

$$A_5^{(2)} = A_5^{(1)} \left[-\sum_{i=1}^{5} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{-v_i} \right)^{\epsilon} \right] + R\, F_5^{(2)} + \mathcal{O}(\epsilon)$$

$$F_5^{(2)} = \frac{5\pi^2}{12} F_5^{(1)} + \sum_{i=0}^{4} \sigma^i \left\{ \frac{v_5 \mathrm{tr}\left[(1-\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2\right]}{(v_2+v_3-v_5)} I_{23,5} + \frac{1}{6} \frac{\mathrm{tr}\left[(1-\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2\right]^2}{v_1 v_4} + \frac{10}{3} v_1 v_2 + \frac{2}{3} v_1 v_3 \right\} . \quad (8)$$

simple function of Li_2

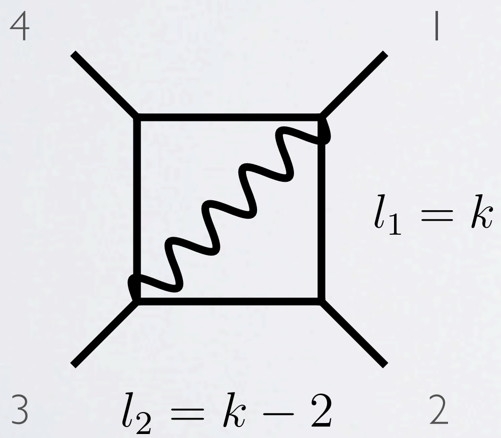
local integrands

[Arkani-Hamed, Bourjailly, Cachazo, Trnka 1012.6032]

[Arkani-Hamed, Bourjailly, Cachazo, Caron-Huot, Trnka 1008.2958]

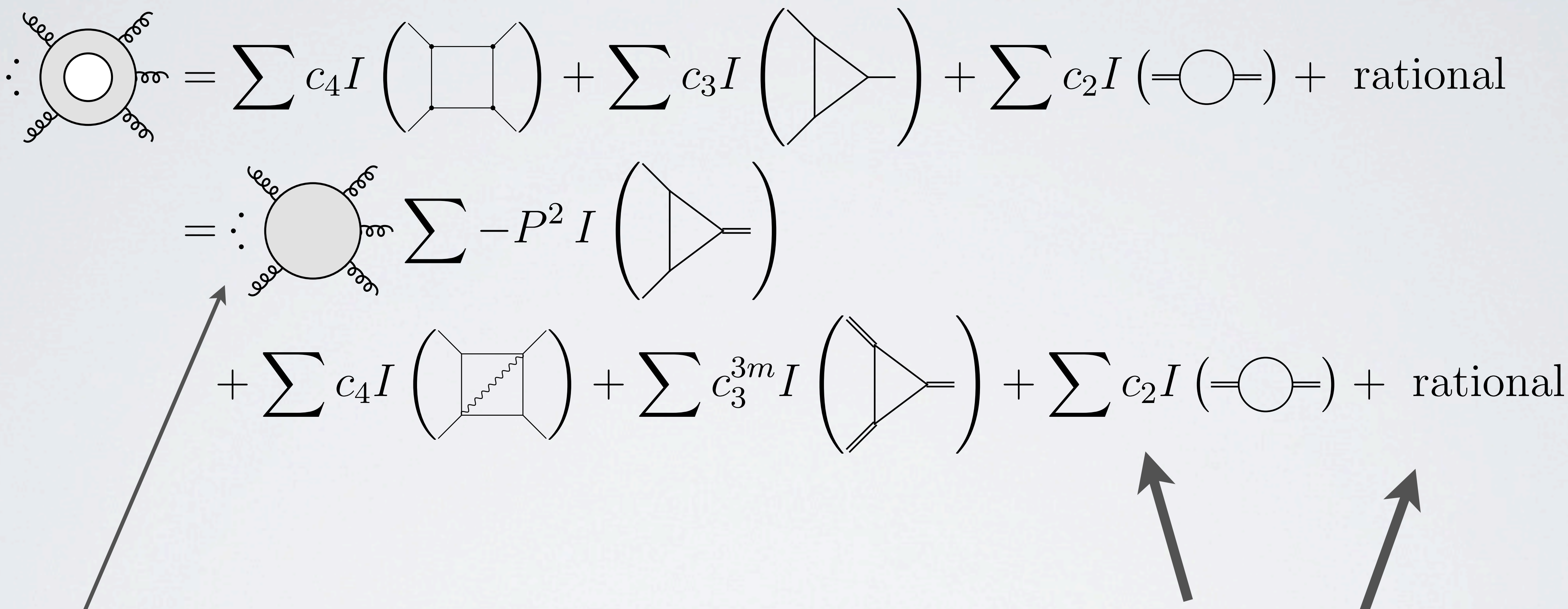
manage infra-red divergences at the integrand level

simple integrals with unit leading singularities



$$= I_4^{4-2\epsilon}\left[\mathrm{tr}\left(1 l_1 l_2 3\right)\right] = I_4^{6-2\epsilon}[1]$$

one-loop integrand bases

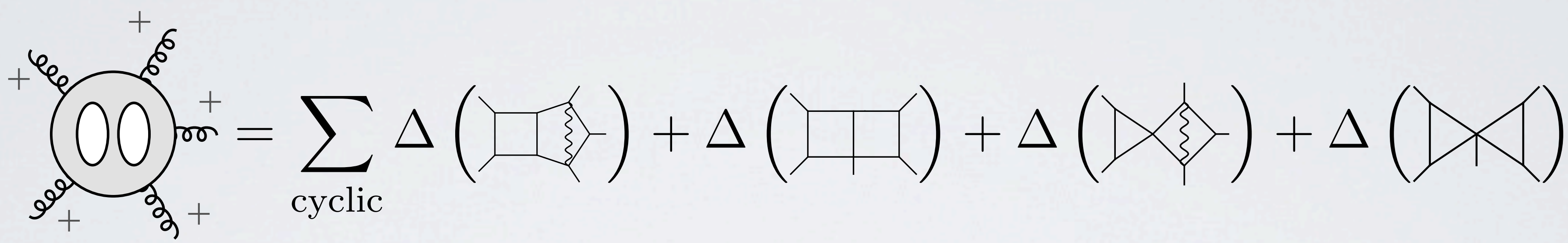


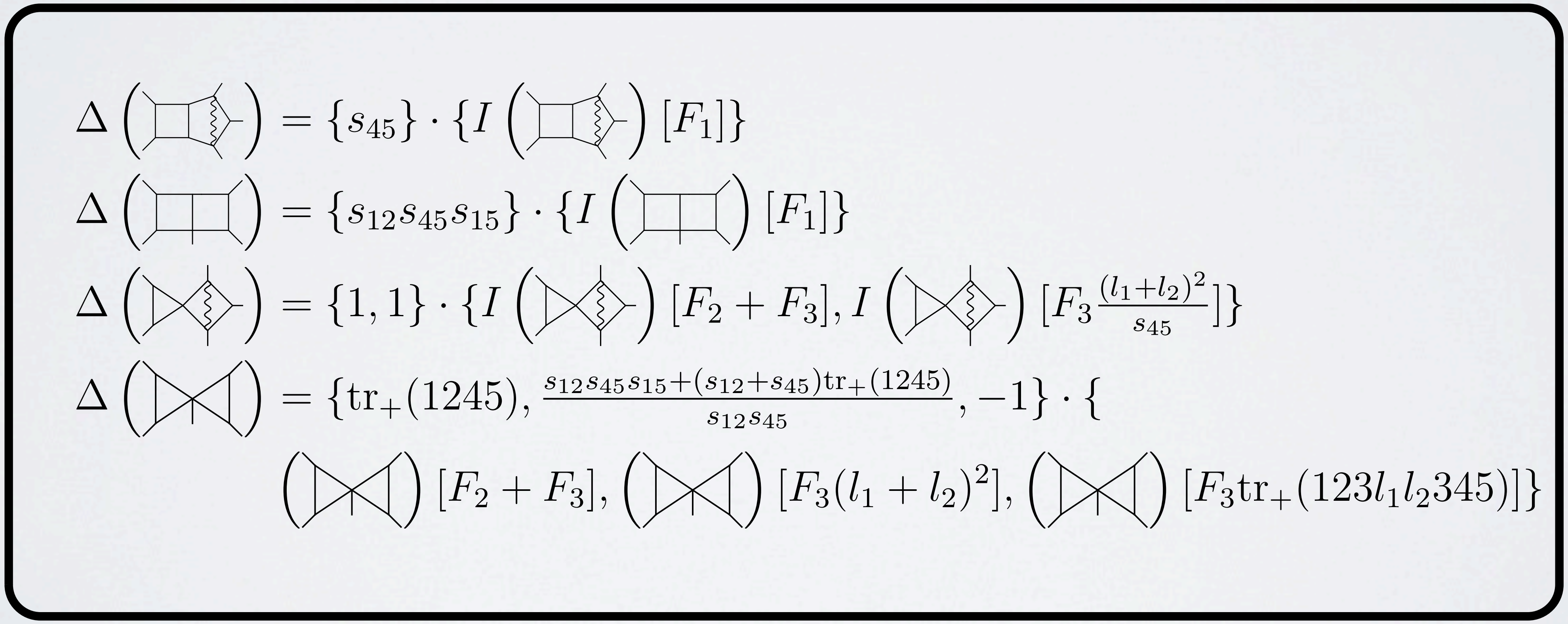
universal infrared poles

still complicated functions at high multiplicity

two-loop 5pt all-plus

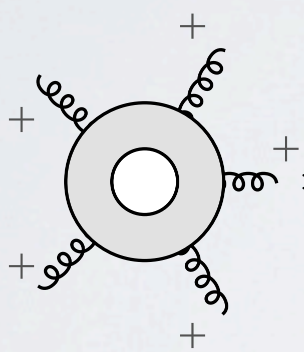
[SB, Mogull, Peraro (in progress)]



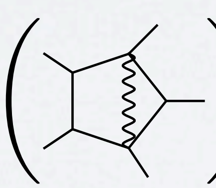
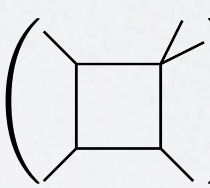
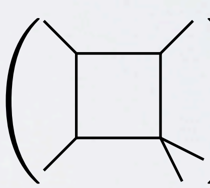


infrared poles

$$A^{(2)}(1^+, 2^+, 3^+, \ldots, n^+) = A^{(1)}(1^+, 2^+, 3^+, \ldots, n^+) \sum_{i=1}^{n} \left(\frac{\mu_R^2}{-s_{i,i+1}} \right)^{\epsilon} + \mathcal{O}(\epsilon^0)$$



$$= \frac{1}{5} \sum_{\text{cyclic}} \left\{ \frac{\text{tr}_+(1345)}{s_{13}}, s_{23}s_{34}, s_{12}s_{15} \right\} \cdot \Bigg\{$$

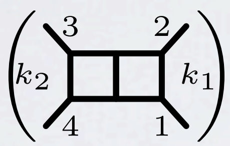
$I\Big($  $\Big)[\mu_{11}^2]$, $I\Big($  $\Big)[\mu_{11}^2]$, $I\Big($  $\Big)[\mu_{11}^2]$, $\Bigg\}$

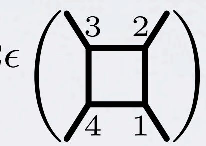
local one-loop all-plus:
$F_1 \times \mathcal{N} = 4$ using
Arkani-Hamed et al.
1008.2958

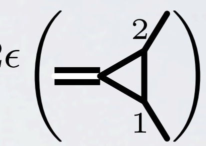
integrals in the soft limit

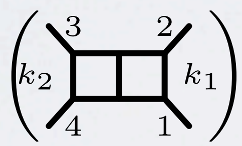
$$F_1 = (D_s - 2)(2\mu_{11}\mu_{22} + \mu_{11}^2 + \mu_{22}^2 + \mu_{12}(\mu_{11} + \mu_{22})) + 16(\mu_{12}^2 - \mu_{11}\mu_{22})$$

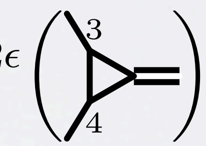
$$\lim_{k_1 \to 0} F_1 = (D_s - 2)\mu_{22}^2$$

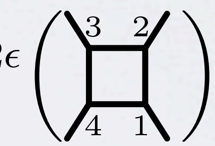
$I^{4-2\epsilon}$ 

$[F_1] \overset{k_1 \to 0}{\to} (D_s - 2) I^{4-2\epsilon}$ 

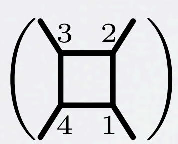
$[\mu_{22}^2]\, I^{4-2\epsilon}$ 

$I^{4-2\epsilon}$ 

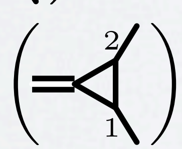
$[F_1] \overset{k_2 \to 0}{\to} (D_s - 2) I^{4-2\epsilon}$ 

$I^{4-2\epsilon}$ 

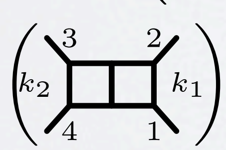
$[\mu_{11}^2]$

$I^{4-2\epsilon}$ 

$[\mu_{11}^2] = -\frac{1}{6} + \mathcal{O}(\epsilon)$

$I^{4-2\epsilon}$ 

$= \frac{c_\Gamma}{\epsilon^2}(-s_{12})^{-1-\epsilon} = -\frac{1}{(4\pi)^2 s_{12}\epsilon^2} + \mathcal{O}(\epsilon^{-1})$

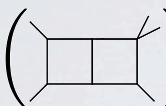
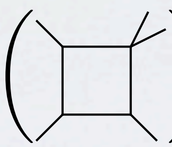
$\Rightarrow I^{4-2\epsilon}$ 

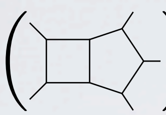
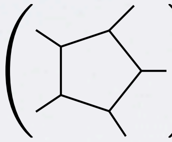
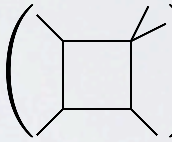
$[F_1] = \frac{D_s - 2}{(4\pi)^2 3\epsilon^2 s_{12}} + \mathcal{O}(\epsilon^{-1})$

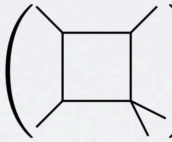
in this case actually holds to 1/ε

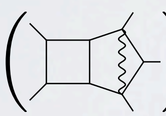
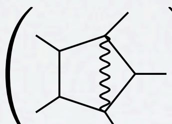
integrals in the soft limit

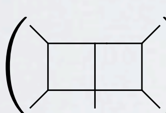
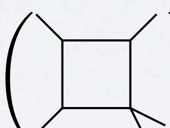
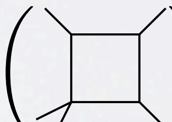
simple identification of IR poles with local integrands

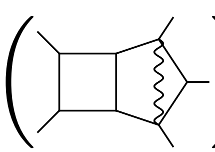
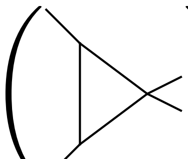
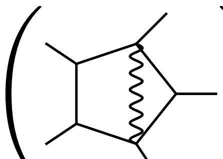
$I\Big($  $\Big)[F_1] = \frac{c_\Gamma}{\epsilon^2}(-s_{45})^{-1-\epsilon} I\Big($  $\Big)[\mu_{11}^2] + \mathcal{O}(\epsilon^{-1})$

$I\Big($  $\Big)[F_1] = \frac{c_\Gamma}{\epsilon^2}(-s_{45})^{-1-\epsilon} I\Big($  $\Big)[\mu_{11}^2] + \frac{c_\Gamma}{\epsilon^2\, s_{23}}(-s_{12})^{-1-\epsilon} I\Big($  $\Big)[\mu_{11}^2]$

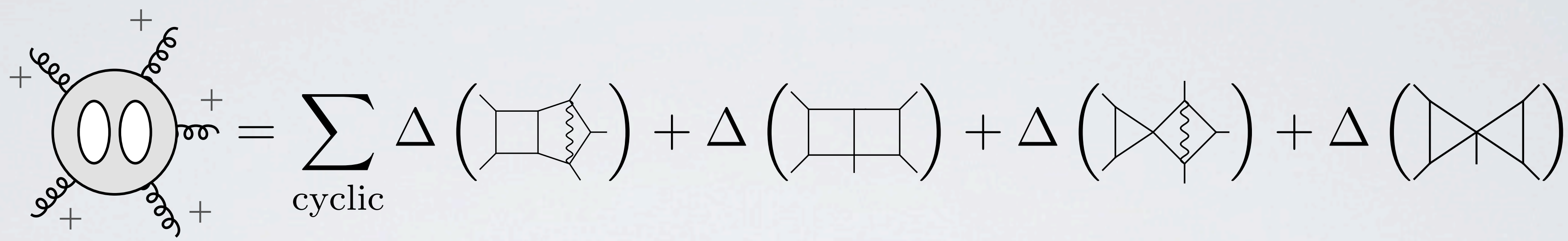
$+ \frac{c_\Gamma}{\epsilon^2\, s_{12}}(-s_{23})^{-1-\epsilon} I\Big($  $\Big)[\mu_{11}^2] + \mathcal{O}(\epsilon^{-1})$

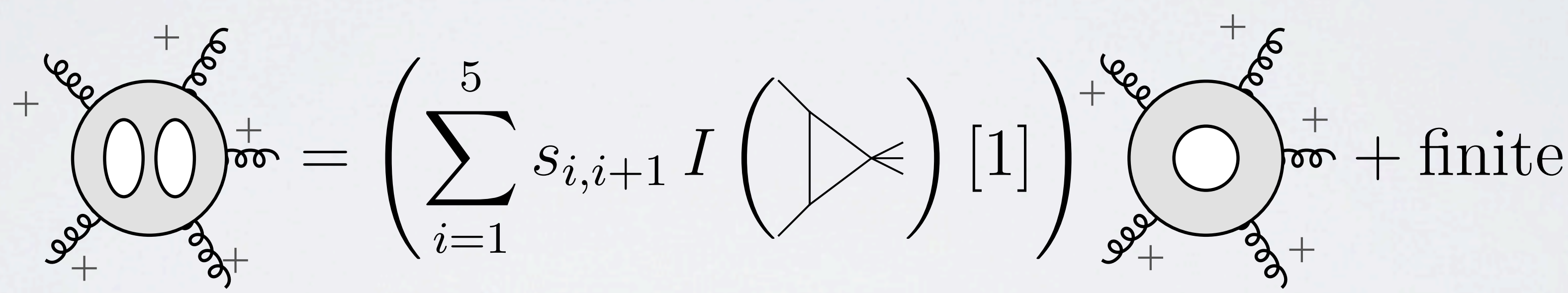
$I\Big($  $\Big)[F_1] = \frac{c_\Gamma}{\epsilon^2}(-s_{45})^{-1-\epsilon} I\Big($  $\Big)[\mu_{11}^2] + \mathcal{O}(\epsilon^{0})$

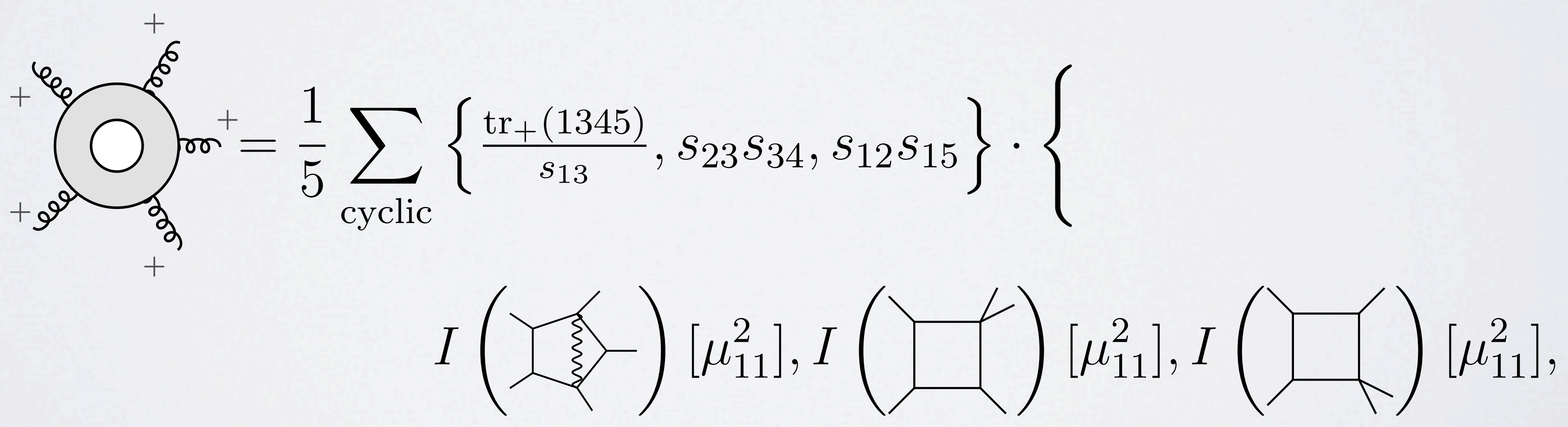
$I\Big($  $\Big)[F_1] = \frac{c_\Gamma}{\epsilon^2}(-s_{12})^{-1-\epsilon} I\Big($  $\Big)[\mu_{11}^2] + \frac{c_\Gamma}{\epsilon^2}(-s_{45})^{-1-\epsilon} I\Big($  $\Big)[\mu_{11}^2] + \mathcal{O}(\epsilon^{0})$

$I\Big($  $\Big)[F_1] = I\Big($  $\Big)[1] \times I\Big($  $\Big)[\mu_{11}^2] + \mathcal{O}(\epsilon^{0})$

two-loop 5pt all-plus

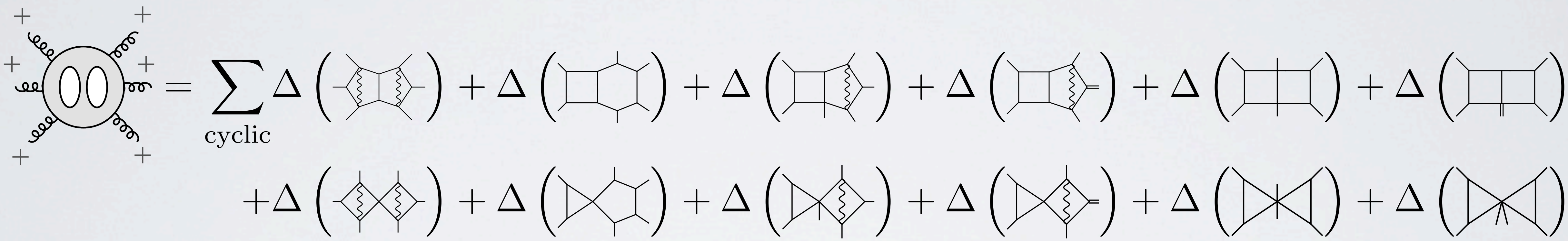






two-loop 6pt all-plus

[SB, Mogull, Peraro (in progress)]

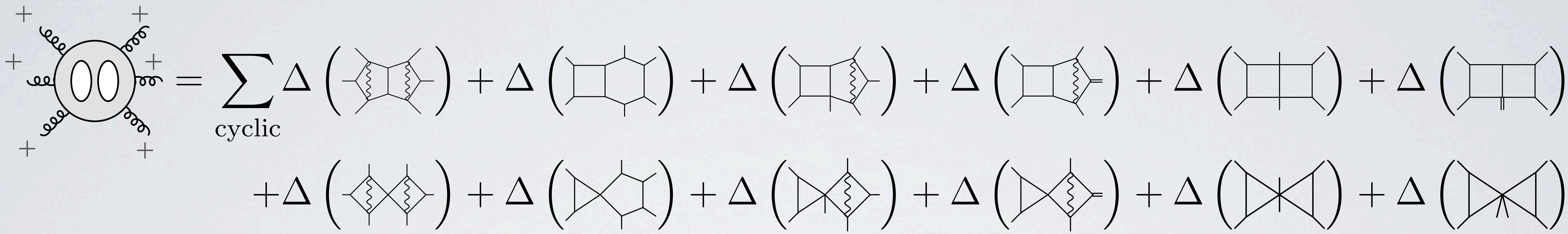


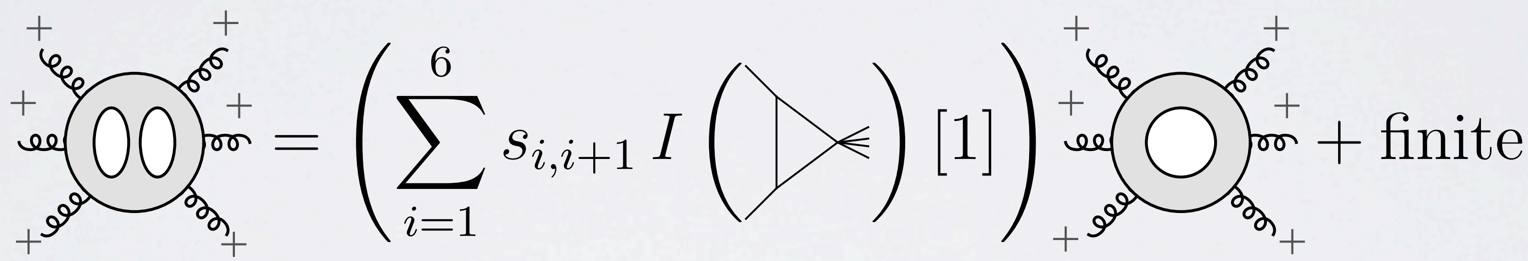
follows the expected $\mathcal{N} = 4 \times F_1$ structure

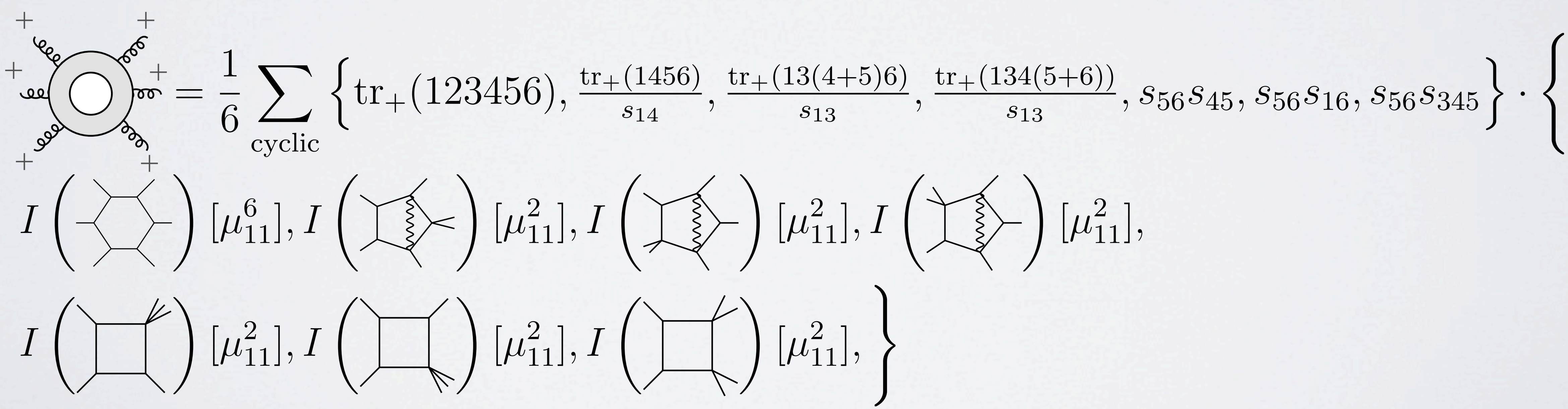
[Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich (2008)]

[Arkani-Hamed, Bourjailly, Cachazo, Caron-Huot, Trnka (2010)]

two-loop 6pt all-plus







outlook

- Local integrands are an efficient way to manage IR divergences
- Simple expressions for 5 and 6 gluon all-plus amplitudes in Yang-Mills
- General basis of local integrands still unknown - connection with polynomial division algorithm?
- Local integrand representations for non-planar sector?

Backup

momentum twistors

[Hodges (2009)]

an all multiplicity parameterisation (not unique)

$$Z_{iA} = \left(\frac{\Sigma_i}{s_{12}},\, 1 - \delta_{1i},\, \frac{\langle 123i \rangle \langle 34 \rangle [23]}{\langle 1234 \rangle \langle 1i \rangle [12]},\, \frac{-\langle 13 \rangle \langle 124i \rangle + \langle 14 \rangle \langle 123i \rangle}{\langle 1234 \rangle \langle 1i \rangle} \right)$$

$$\Sigma_i = \begin{cases} s_{12} & i = 1 \\ \frac{\langle 13 \rangle \langle 2i \rangle}{\langle 23 \rangle \langle 1i \rangle} & i \neq 1 \end{cases}$$

In[21]:=

```
NN = 6;
GetMTrepALT[NN];
mtZ // MatrixForm // SymbolForm
```

Out[23]//DisplayForm=

$$\begin{pmatrix} 1 & 0 & \frac{1}{s_{12}} & \frac{\sigma_4}{s_{12}} & \frac{\sigma_5}{s_{12}} & \frac{\sigma_6}{s_{12}} \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & x_{1234} & x_{1235} & 1 \\ 0 & 0 & 1 & 1 & \frac{x_{1235} - x_{1245}}{x_{1234}} & \frac{1 - x_{1246}}{x_{1234}} \end{pmatrix}$$

build spinors products etc. from rational phase-space points

momentum twistors

example: BCFW using rational kinematics

In[143]:=

```
NN = 7;
GetMTrepALT[NN];
mtZ // MatrixForm // SymbolForm
mtZ = mtZ /. lpS[1,2]→1; mtW = mtW /. lpS[1,2]→1;
{p1,p2,p3,p4,p5,p6,p7} = Map[ToSPN4[MT[#]]&,Range@NN];
Map[Simplify,BCFW4[AMP[{p1,1,gluon},{p2,1,gluon},{p3,1,gluon},{p4,2,gluon},{p5,2,gluon},{p6,2,gluon},{p7,2,gluon}],1,2]] // SymbolForm
```

Out[145]//DisplayForm=

$$\begin{pmatrix} 1 & 0 & \frac{1}{s_{12}} & \frac{\sigma_4}{s_{12}} & \frac{\sigma_5}{s_{12}} & \frac{\sigma_6}{s_{12}} & \frac{\sigma_7}{s_{12}} \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & x_{1234} & x_{1235} & x_{1236} & 1 \\ 0 & 0 & 1 & 1 & \frac{x_{1235}-x_{1245}}{x_{1234}} & \frac{x_{1236}-x_{1246}}{x_{1234}} & \frac{1-x_{1247}}{x_{1234}} \end{pmatrix}$$

Out[148]//DisplayForm=

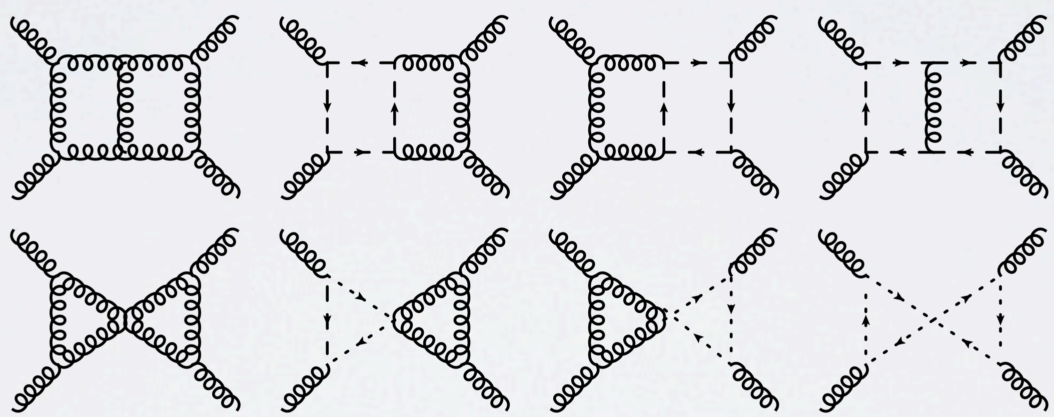
$$-\frac{i\,(x_{1234}-x_{1235}+x_{1245})^3}{(-1+\sigma_4)(\sigma_4-\sigma_5)(\sigma_5-\sigma_6)(\sigma_6-\sigma_7)\,x_{1234}\,x_{1235}\,x_{1245}\,(\sigma_5 x_{1234}-\sigma_4 x_{1235}+x_{1245})} - \frac{i\,(x_{1234}(x_{1235}-x_{1236})-x_{1236}x_{1245}+x_{1235}x_{1246})^3}{(-1+\sigma_4)(\sigma_4-\sigma_5)(\sigma_5-\sigma_6)(\sigma_6-\sigma_7)\,x_{1234}\,x_{1235}\,x_{1236}\,(-x_{1236}x_{1245}+x_{1235}x_{1246})\,(-x_{1236}(\sigma_5 x_{1234}+x_{1245})+x_{1235}(\sigma_6 x_{1234}+x_{1246}))} + \frac{i\,(x_{1234}(-1+x_{1236})-x_{1246}+x_{1236}x_{1247})^3}{(-1+\sigma_4)(\sigma_4-\sigma_5)(\sigma_5-\sigma_6)(\sigma_6-\sigma_7)\,x_{1234}\,x_{1236}\,(-x_{1246}+x_{1236}x_{1247})\,(\sigma_6 x_{1234}+x_{1246}-x_{1236}(\sigma_7 x_{1234}+x_{1247}))}$$

[c.f. bcfw mathematica package Bourjaily (2010)]

six-dimensional trees

six dimensional spinor helicity [Cheung, O'Connell 0902.0981]
(one-loop applications [Bern at al. 1010.0494] [Davies 1108.0398])

dimensional reduction to $4-2\epsilon$ using additional scalar amplitudes



6D is a convenient way to manage all helicities simultaneously

$$A(1_{a\dot{a}}, 2_{b\dot{b}}, 3_{d\dot{d}}, 4_{d\dot{d}}) = \frac{i}{st}\langle 1_a 2_b 3_c 4_d\rangle [1_{\dot{a}} 2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}}]$$

non-planar from planar

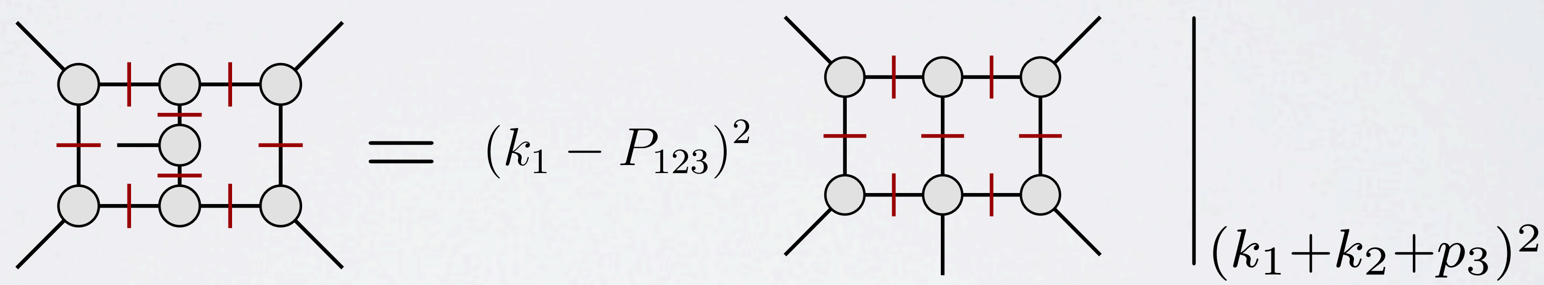
e.g. [Bern, Carrasco, Dixon, Johansson, Roiban 1201.5366]

$$A_4(1,2,3,4) = \frac{s_{13}}{s_{12}} A_4(1,3,2,4)$$

factorization

$$\Rightarrow \quad A_3(1,2,-P_{12})A_3(P_{12},3,4) = \operatorname*{Res}_{s_{12}=0}\left(A_4(1,2,3,4)\right) = s_{13}A_4(1,3,2,4)\big|_{s_{12}=0}$$

$\Rightarrow$



$\Rightarrow$