Computation of Heavy Quarkonium Spectrum in Perturbative QCD

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☆Plan of talk

1. Technical developments
2. Physics predictions
3. Challenge (Analytic evaluation of a_3)
4. Summary

Heavy quarkonium states ($t\bar{t}, b\bar{b}, c\bar{c}, b\bar{c}$)

Unique system: Properties of individual hadrons predictable in pert. QCD

Two theoretical foundations for computing higher-order corr. systematically

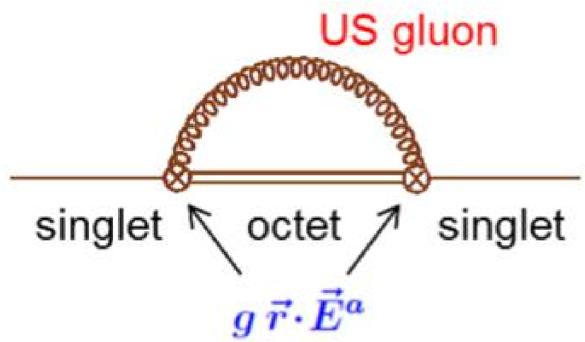
- Threshold expansion — Beneke, Smirnov
- EFT (pNRQCD, vNRQCD) — Pineda, Soto, Brambilla, Vairo; Luke, Manohar, Rothstein

Computation of quarkonium spectrum up to NNNLO

Kiyo, YS: 1408.5590

$$\mathcal{L}_{\mathrm{pNRQCD}} = S^\dagger\left(i\partial_t - \hat{H}_S\right)S + O^{a\dagger}\left(iD_t - \hat{H}_O\right)^{ab}O^b + g\,S^\dagger\vec{r}\cdot\vec{E}^a O^a + \cdots$$

Energy levels given by poles of the full propagator of S in pNRQCD.



$$\frac{1}{E - \hat{H}_S + i\varepsilon}$$

UV div., NNNLO, same as Lamb shift

$$\downarrow$$

$$\delta E_n = \left\langle n\middle|\left(\hat{H}_S\right)_{NLO}\middle|n\right\rangle + \sum\!\!\!\!\!\!\!\int_i \frac{\left|\left\langle n\middle|\left(\hat{H}_S\right)_{NLO}\middle|i\right\rangle\right|^2}{E_n - E_i} + \cdots$$

Pert. theory in Quantum Mech.

Easily converted to infinite sum(s) using
an infinite sum rep. of the Coulomb Green fn.

Breakdown of infinite sum to finite sums (and known transcendental numbers)

e.g. $$A(n,\ell)=\sum_{k=1}^{\infty}\frac{(n-\ell+k-1)!}{(n+\ell+k)!k^3}.$$

$$\frac{(n-\ell+k-1)!}{(n+\ell+k)!}=\prod_{m=-\ell}^{\ell}\frac{1}{n+k+m}=\sum_{m=-\ell}^{\ell}\frac{R(\ell,m)}{n+k+m},\qquad R(\ell,m)=\frac{(-1)^{\ell-m}}{(\ell+m)!(\ell-m)!}$$

$$\sum_{k=1}^{\infty}\frac{1}{(k+i)k^3}=\frac{\zeta(3)}{i}-\frac{\zeta(2)}{i^2}+\frac{S_1(i)}{i^3}$$ where $S_1(i)=\sum_{k=1}^{i}\frac{1}{k}$ denotes the harmonic sum.

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A general algorithm exists, which can evaluate, e.g. Anzai, YS

$$f(i) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{m-k}}{(k+i)^2(k+m+i)(m+2i+1)}$$

- Use shift invariance of the summand

$$i \to i + \Delta i, \quad k \to k + \Delta k,$$
$$m \to m + \Delta m$$

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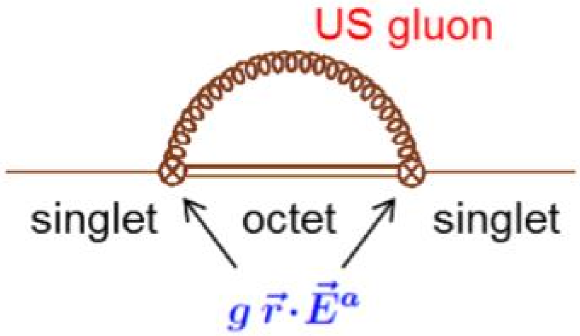
- Use shift invariance of the summand

$$i \to i + \Delta i, \quad k \to k + \Delta k,$$
$$m \to m + \Delta m$$

Reduced to a combination of nested sums:

$$Z(i;\{b_j\};\{\lambda_j\}) = \sum_{i \geq n_1 > n_2 > \cdots > n_N > 0} \frac{\lambda_1^{n_1}\lambda_2^{n_2}\cdots\lambda_N^{n_N}}{n_1^{b_1}n_2^{b_2}\cdots n_N^{b_N}}$$ ($b_j \in \mathbb{N}$, $\lambda_j \in$ roots of unity)

Algebraic derivation of US correction (QCD Bethe log)

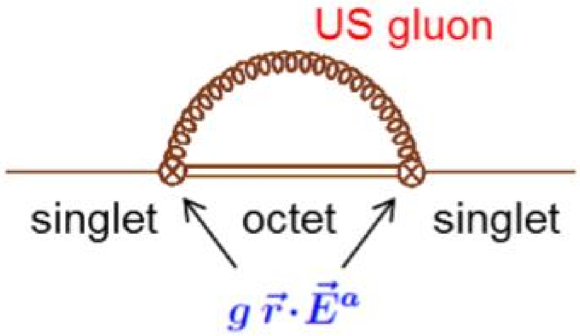


Brambilla, Pineda, Soto, Vairo
Kniehl, Penin

$$E^{\rm us}_{n\ell} = -ig^2\bar{\mu}^{2\epsilon}\frac{T_F}{N_C}\int_0^\infty dt\langle\vec{r}\cdot\vec{E}^a(t,\vec{0})\exp[-i(H_O^{(d)} - E_{n,C}^{(d)})t]\vec{r}\cdot\vec{E}^a(0,\vec{0})\rangle_{n\ell}$$

$$= \frac{1}{2}C_F g^2\bar{\mu}^{2\epsilon}\frac{1-d}{d}C(d)\langle r^i(H_O^{(d)} - E_{n,C}^{(d)})^d r^i\rangle_{n\ell}. \qquad d = D-1 = 3-2\epsilon$$

$$\langle E^{ia}(t,\vec{0})E^{ja}(0,\vec{0})\rangle = -i\delta^{aa}\int\frac{d^Dk}{(2\pi)^D}\frac{e^{ik_0t}}{k^2+i0}(k^ik^j - k_0^2\delta^{ij}) + \mathcal{O}(\alpha_s)$$



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includes $1/\epsilon$

$$= \frac{1}{2} C_F g^2 \bar{\mu}^{2\epsilon} \frac{1-d}{d} C(d) \langle r^i (H_O^{(d)} - E_{n,C}^{(d)})^d r^i \rangle_{n\ell}. \qquad d = D-1 = 3-2\epsilon$$

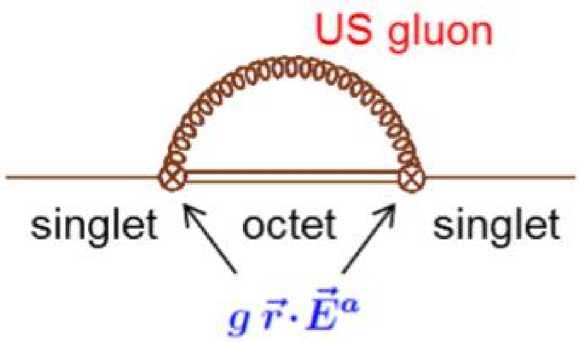
up to $O(\epsilon)$ necessary

$$(H_O^{(d)} - E_{n,C}^{(d)})^{3-2\epsilon} \approx (H_O^{(d)} - E_{n,C}^{(d)})^3 [1 - 2\epsilon \log(H_O^{(d)} - E_{n,C}^{(d)})]$$

$$\bar{\mu}^{2\epsilon} \langle r^i (H_O^{(d)} - E_{n,C}^{(d)})^d r^i \rangle_{n\ell} = \left\langle X - 2\epsilon r^i (H_O^{(3)} - E_{n,C}^{(3)})^3 \log\left(\frac{H_O^{(3)} - E_{n,C}^{(3)}}{\bar{\mu}}\right) r^i \right\rangle_{n\ell} + \mathcal{O}(\epsilon^2)$$

$$X = r^i (H_O^{(d)})^3 r^i - \frac{3}{2}\{H_S^{(d)}, r^i (H_O^{(d)})^2 r^i\} + \frac{3}{2}\{(H_S^{(d)})^2, r^i H_O^{(d)} r^i\} - \frac{1}{2}\{(H_S^{(d)})^3, \vec{r}^2\}$$

Use commutation relation $[r_i, p_j] = i\,\delta_{ij}$ and $\langle n|[H_S, \mathcal{O}]|n\rangle = 0$ to simplify X.



Brambilla, Pineda, Soto, Vairo
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$$E^{\rm us}_{n\ell} = -i g^2 \bar{\mu}^{2\epsilon} \frac{T_F}{N_C} \int_0^\infty dt \langle \vec{r}\cdot\vec{E}^a(t,\vec{0}) \exp[-i(H_O^{(d)} - E_{n,C}^{(d)})t] \vec{r}\cdot\vec{E}^a(0,\vec{0})\rangle_{n\ell}$$

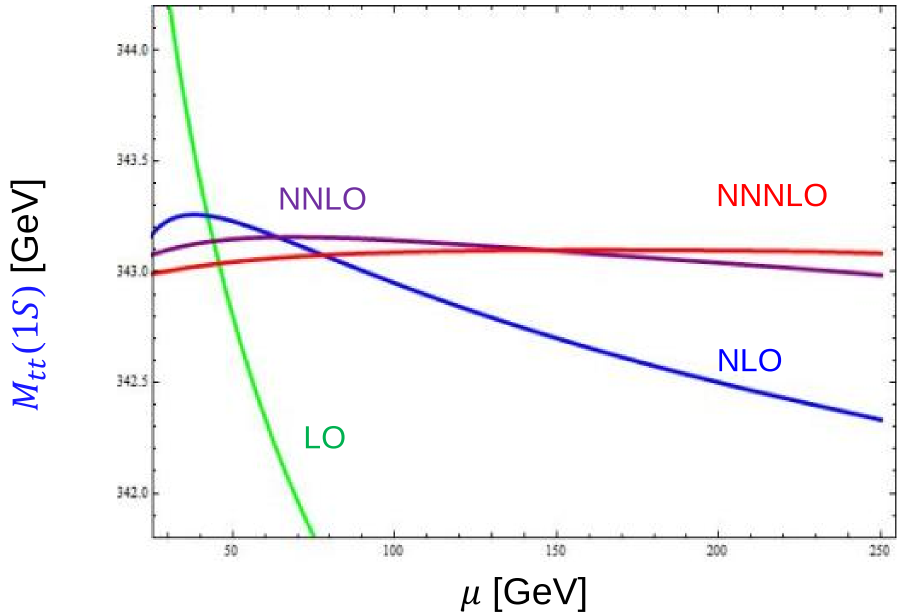
$$= \frac{1}{2} C_F g^2 \bar{\mu}^{2\epsilon} \frac{1-d}{d} C(d) \langle r^i (H_O^{(d)} - E_{n,C}^{(d)})^d r^i \rangle_{n\ell}. \qquad d = D - 1 = 3 - 2\epsilon$$

$$= \langle H^{\rm us} \rangle_{n\ell} - \frac{2C_F\alpha_s}{3\pi} \left\langle r^i (H_O^{(3)} - E_{n,C}^{(3)})^3 \log\left(\frac{H_O^{(3)} - E_{n,C}^{(3)}}{\mu}\right) r^i \right\rangle_{n\ell}.$$

includes $1/\epsilon$

Physics Predictions

- Scale dependence

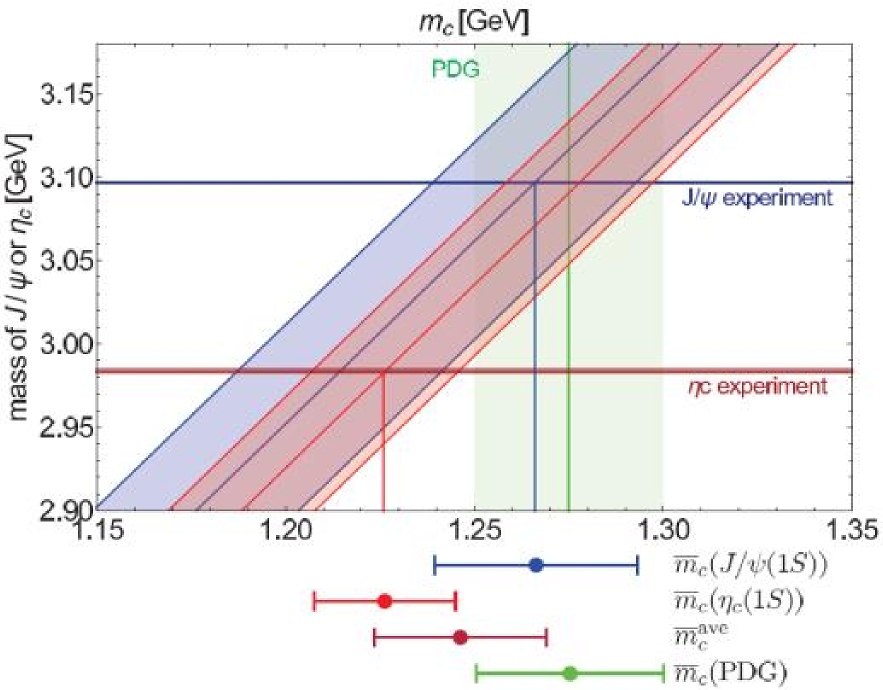


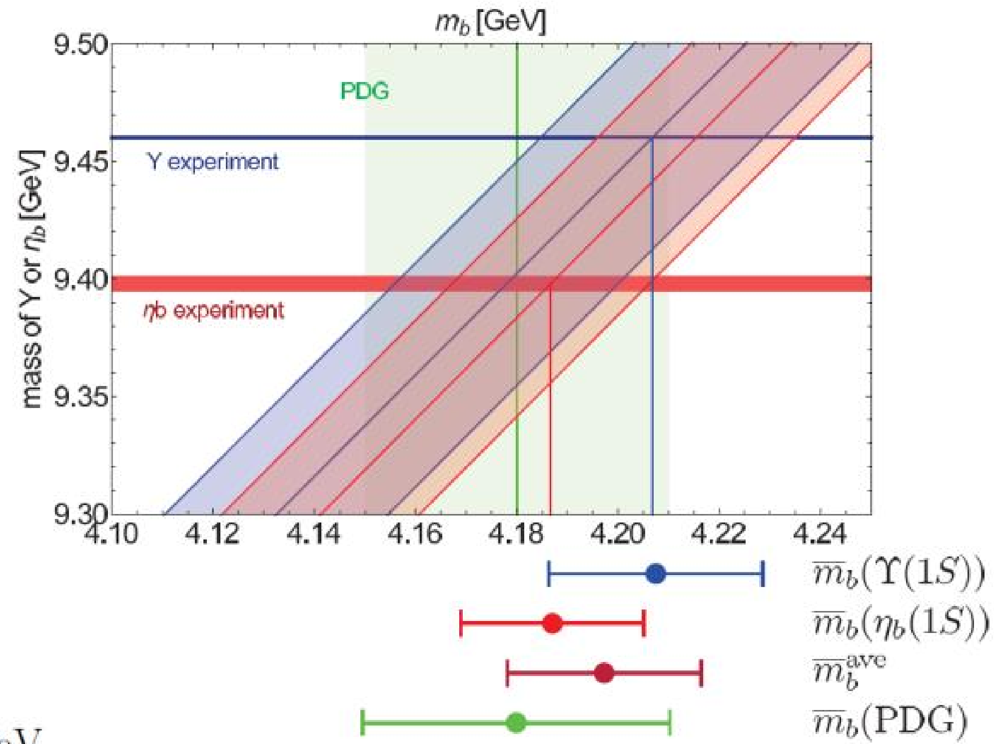
Use $\overline{\text{MS}}$ mass

Marquard, Smirnov, Smirnov, Steinhauser

- $\overline{m}_b, \overline{m}_c$ determination

Kiyo, Mishima, YS





$$\overline{m}_c^{\rm ave} = 1246 \pm 2\ (d_3)\ \pm 4\ (\alpha_s)\ \pm 23\ (\text{h.o.}) \quad \text{MeV}$$

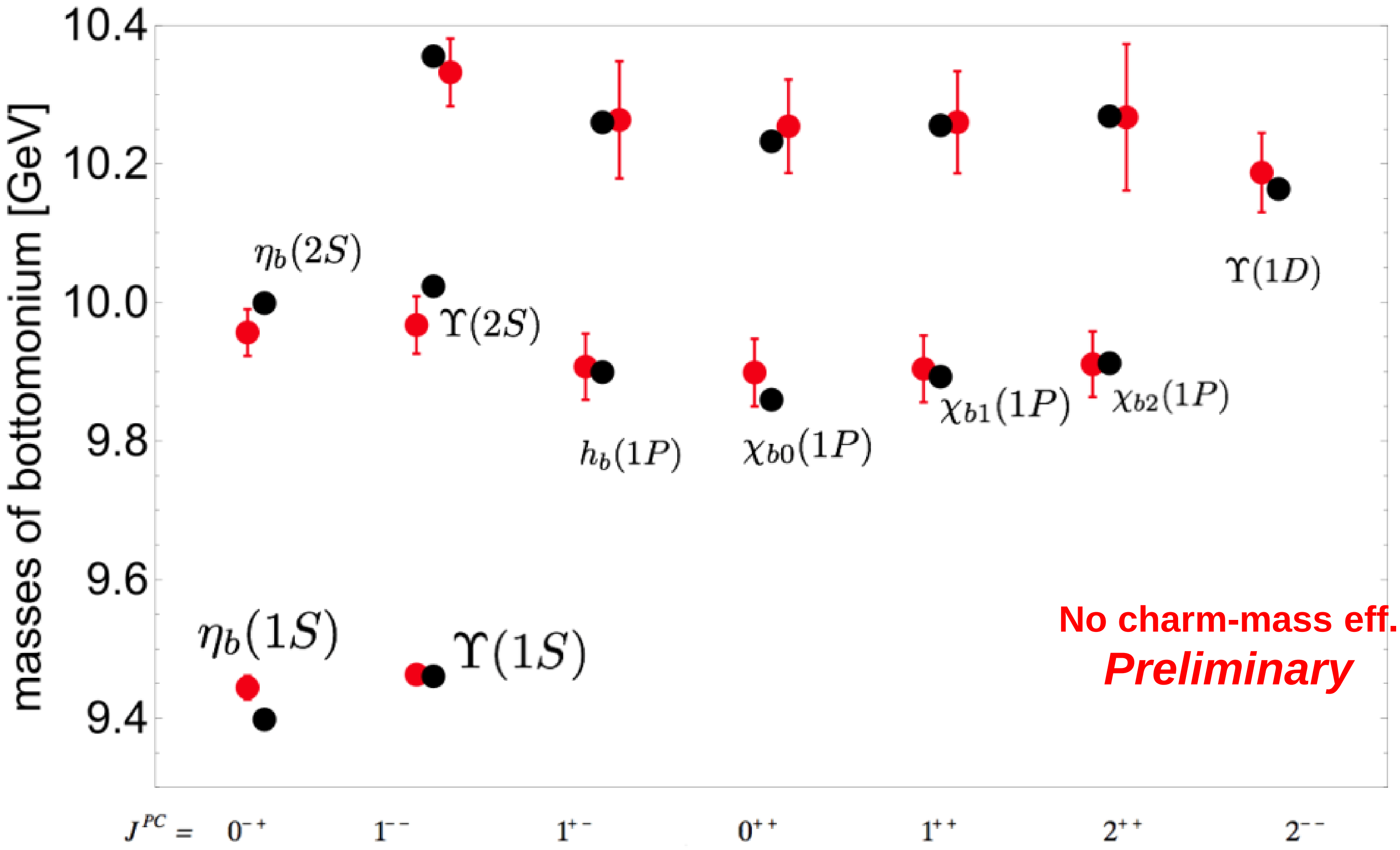
PDG value $\overline{m}_c = 1275 \pm 25$ MeV

$$\overline{m}_b^{\rm ave} = 4197 \pm 2\ (d_3)\ \pm 6\ (\alpha_s)\ \pm 20\ (\text{h.o.}) \pm 5\ (m_c) \quad \text{MeV}$$

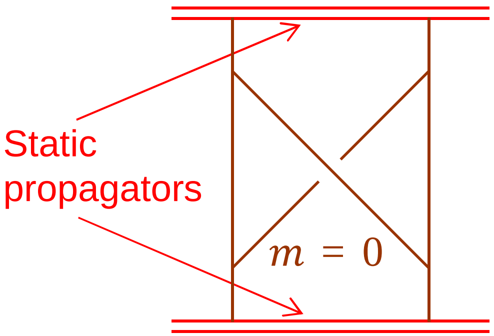
PDG value $\overline{m}_b = 4.18 \pm 0.03$ GeV

- Bottomonium spectroscopy

Based on Kiyo, YS
(Plot made by G. Mishima)



3. Challenge: Analytic evaluation of a_3 (3-loop QCD potential)



Unknown expansion coefficient in ϵ

$$= \sum_k r_k \, Z\left(\infty; \{b_i^{(k)}\}; \{\lambda_i^{(k)}\}\right) + \boxed{C_0}$$

Generalized MZVs

$$Z(\infty; \{b_i\}; \{\lambda_i\}) = \sum_{n_1 > n_2 > \cdots > n_N > 0} \frac{\lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_N^{n_N}}{n_1^{b_1} n_2^{b_2} \cdots n_N^{b_N}} \qquad (b_i \in \mathbb{N},\ \lambda_i \in \mathbb{C})$$

- Not reducible to MZVs (as yet)

$$\int_0^\Lambda dx \, \frac{1}{\sqrt{x}\sqrt{1-4x}} \int_{1/4}^x dy \, \frac{1+2y}{y\sqrt{1-y}\sqrt{1-4y}} \int_1^y dz \, \frac{\operatorname{Arctan} z}{z\sqrt{1-z}}$$

$$= \underbrace{C_2}_{\text{expressed by MZVs}} \log^2 \Lambda + \underbrace{C_1}_{\text{expressed by MZVs}} \log \Lambda + \underbrace{\boxed{C_0}}_{\text{unknown}} + \mathcal{O}\left(\frac{1}{\Lambda} \log^n \Lambda\right) \quad \text{as } \Lambda \to \infty$$

Colleagues are invited to reveal the nature of this extension(?) of MZVs.

Summary

- Non-relativistic bound-state theory for QED/QCD has become fairly mature and amenable to a textbook-level understanding and computations.

- Recently NNNLO corrections to the complete spectrum and threshold production cross section have been computed. Kiyo, YS Beneke, Kiyo, Marquard, Penin, Piclum, Steinhauser

  Applied a new technology for evaluating multiple sums;
  All computations arithmetic (no diagrammatic analysis).

  Challenge: Analytic evaluation of a_3 remains

- Applications

  Bottomonium spectroscopy at NNNLO: reasonable agreement with exp. data. Kiyo, YS

  $\bar{m}_b$ and $\bar{m}_c$ $\overline{\mathrm{MS}}$ mass determination from $J/\psi(1S)$, $\eta_c(1S)$ and $\Upsilon(1S)$, $\eta_b(1S)$.

  Kiyo, Mishima, YS:

Postdictions or predictions?

- Fine and hyperfine splittings of charmonium/bottomonium reproduced.

  Two exceptions around 2003: Recksiegel, Y.S.; Kniehl, Penin, Pineda, Smirnov, Steinhauser

  charmonium hyperfine splitting $\Psi(2S)-\eta_c(2S)$

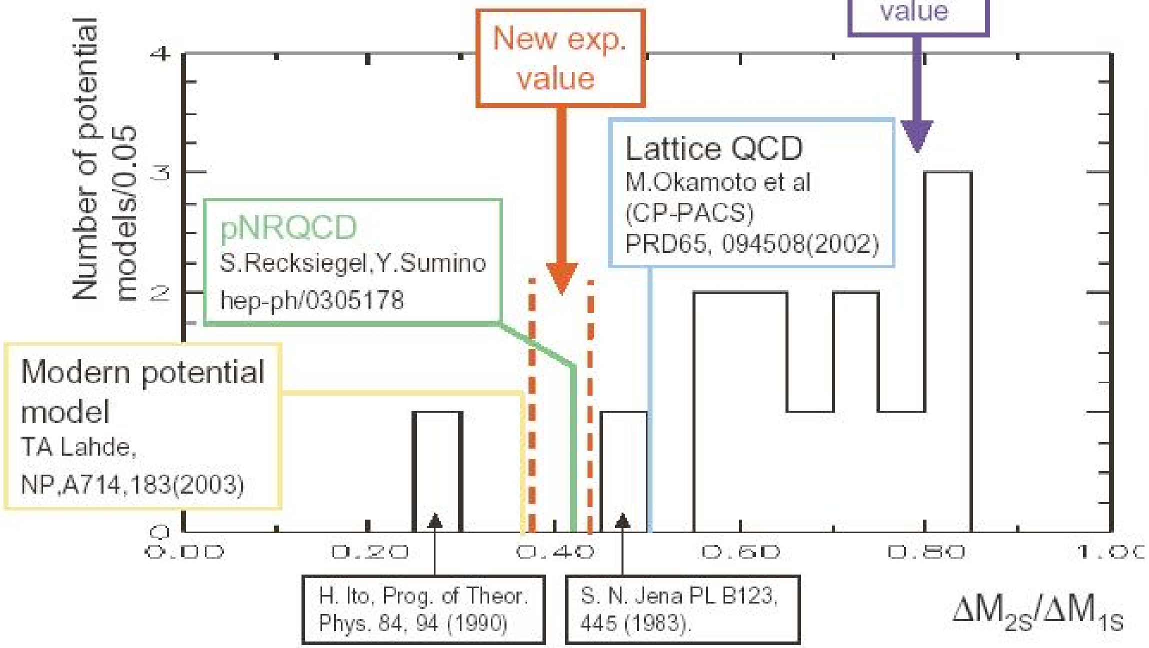
  bottomonium hyperfine splitting $\Upsilon(1S)-\eta_b(1S)$

  Both are solved in favor of pert. QCD predictions.

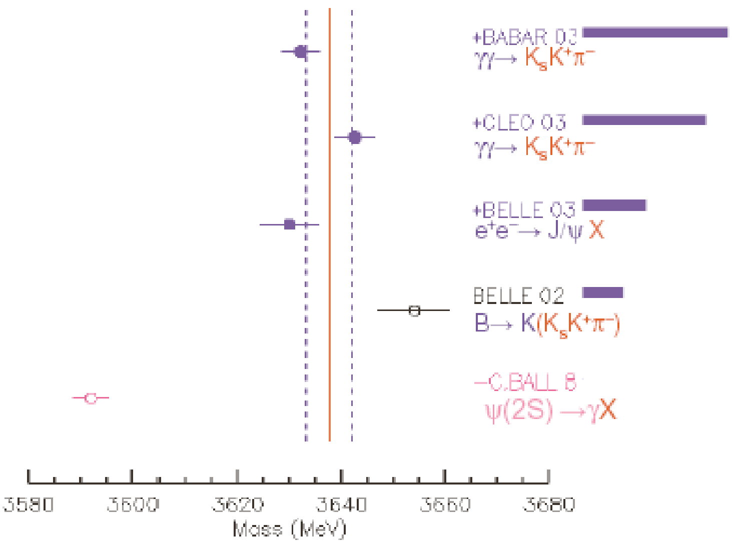
Slides from Skwarnicki's plenary talk at Lepton-Photon 2003

Predictions for hyperfine splitting ratio

- For 20 years theorists were exposed to the experimental value of ΔM_{2S}=M(ψ(2S))-M(η_c(2S)) which was wrong by a factor of 2
- Predictions for $\Delta M_{2S}/\Delta M_{1S}$



3637.7±4.4 MeV



CL=14% scale factor=1.3

New measurements of mass are consistent

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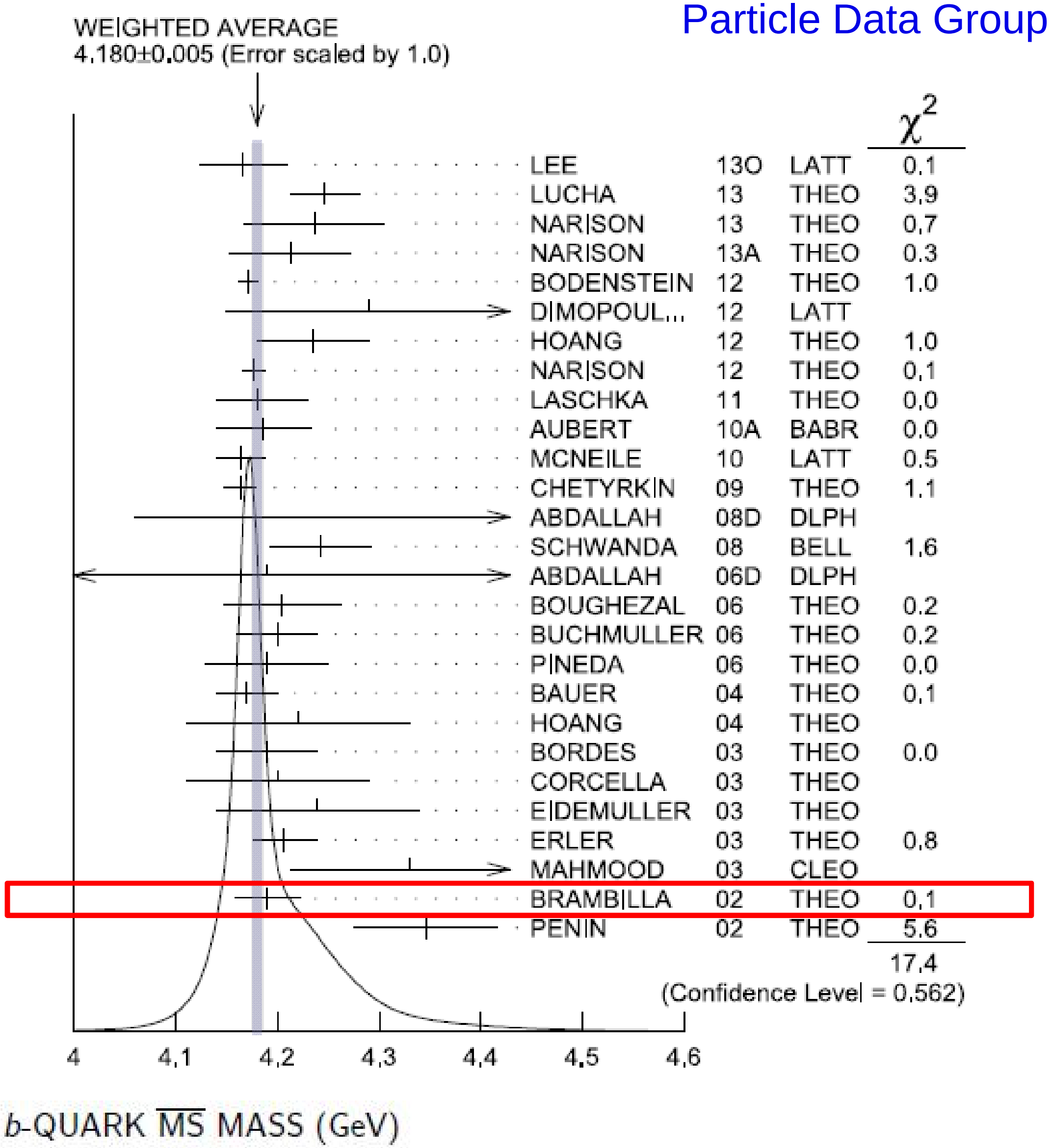
Recksiegel, Y.S.; Kniehl, Penin, Pineda, Smirnov, Steinhauser

Particle Data Group 2014



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$$\sum_{k=1}^{\infty}\frac{1}{(k+i)k^3}=\frac{\zeta(3)}{i}-\frac{\zeta(2)}{i^2}+\frac{S_1(i)}{i^3}$$ where $S_1(i)=\sum_{k=1}^{i}\frac{1}{k}$ denotes the harmonic sum.

$$=\sum_{k=1}^{\infty}\left[\frac{1}{i}\frac{1}{k^3}-\frac{1}{i^2}\frac{1}{k^2}+\frac{1}{i^3}\left(\frac{1}{k}-\frac{1}{k+i}\right)\right]$$

$$=\frac{1}{i}\zeta(3)-\frac{1}{i^2}\zeta(2)+\frac{1}{i^3}\underbrace{\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+i}\right)}_{\frac{1}{k}+\frac{1}{k+1}+\cdots+\frac{1}{i}\equiv S_1(i)}$$

$$f(m) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(1+k+m)^2}. \tag{20}$$

This can be expressed as

$$\begin{aligned} f(m) &= [f(m) + f(m-1)] - [f(m-1) + f(m-2)] \\ &\quad + \cdots + (-1)^m [f(2) + f(1)] - (-1)^m f(1) \\ &= \sum_{j=2}^{m} (-1)^{m-j} [f(j) + f(j-1)] - (-1)^m f(1). \end{aligned} \tag{21}$$

Bulk of the sum in the "difference" of two adjacent terms gets canceled since the shifts $m \to m-1$, $k \to k+1$ leave the denominator in Eq. (20) unchanged

$$f(j) + f(j-1) = \left(\sum_{k=1}^{\infty} - \sum_{k=2}^{\infty} \right) \frac{(-1)^k}{(k+j)^2} = \frac{-1}{(j+1)^2}. \tag{22}$$

Thus,

$$f(m) = (-1)^m \sum_{j=2}^{m} \frac{(-1)^{1-j}}{(j+1)^2} - (-1)^m \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+2)^2}. \tag{23}$$

In the first term, apart from the coefficient $(-1)^m$, dependence on the external index m enters only through the upper bound of the summation, while the second term, apart from $(-1)^m$, is free of m and is essentially an MZV.

$$H_S^{(d)} = \frac{\hat{\vec{p}}^2}{m} + V_S^{(d)}(r), \qquad H_O^{(d)} = \frac{\hat{\vec{p}}^2}{m} + V_O^{(d)}(r),$$

$$V_S^{(d)}(r) = -C_F \frac{\alpha_s}{r} (\bar{\mu} r)^{2\epsilon} A(\epsilon), \qquad V_O^{(d)}(r) = \left(\frac{C_A}{2} - C_F\right) \frac{\alpha_s}{r} (\bar{\mu} r)^{2\epsilon} A(\epsilon),$$

$$A(\epsilon) = \frac{\Gamma(\frac{1}{2} - \epsilon)}{\pi^{\frac{1}{2} - \epsilon}}.$$

Regularization needed to deal with $\delta(\vec{r})$ in commutation relations.

$$V_S^{(d)}(r) \to -C_F \frac{\alpha_s}{r} (\bar{\mu} r)^{2(\epsilon+u)} A(\epsilon),$$

$$V_O^{(d)}(r) \to \left(\frac{C_A}{2} - C_F\right) \frac{\alpha_s}{r} (\bar{\mu} r)^{2(\epsilon+u)} A(\epsilon).$$

$$X = r^i (H_O^{(d)})^3 r^i - \frac{3}{2}\{H_S^{(d)}, r^i (H_O^{(d)})^2 r^i\} + \frac{3}{2}\{(H_S^{(d)})^2, r^i H_O^{(d)} r^i\} - \frac{1}{2}\{(H_S^{(d)})^3, \vec{r}^2\}$$

$$\begin{aligned}
X = &-\frac{4\alpha_s A\, C_A(2u^2 + u(4\epsilon - 1) + \epsilon(2\epsilon - 1))(\bar{\mu} r)^{2(u+\epsilon)}}{m^2 r^3} r^i r^j \hat{p}^j \hat{p}^i \\
&+ \frac{\alpha_s^3 A^3 C_A^3 (\bar{\mu} r)^{6(u+\epsilon)}}{8r} \\
&+ \frac{4\alpha_s A\, u(2u + 2\epsilon - 1)(C_A(u+1)(u+\epsilon) - C_F)(\bar{\mu} r)^{2(u+\epsilon)}}{m^2 r^3} \\
&+ \frac{\alpha_s^2 A^2 C_A [C_A\{2u^2 + 4u(\epsilon + 1) + 2\epsilon^2 + \epsilon + 2\} - 4C_F(2u^2 + 4u\epsilon + u + 2\epsilon^2 + \epsilon - 1)](\bar{\mu} r)^{4(u+\epsilon)}}{2mr^2} \\
&+ \frac{2\alpha_s A C_A\{8u^3 + 4u^2(4\epsilon + 1) + 2u(4\epsilon^2 + 6\epsilon - 3) + 8\epsilon^2 - 6\epsilon + 1\}(\bar{\mu} r)^{2(u+\epsilon)}}{m^2 r^3} i\vec{r}\cdot\hat{\vec{p}} \\
&- \frac{2\alpha_s A C_A(2u + 2\epsilon - 1)(\bar{\mu} r)^{2(u+\epsilon)}}{m^2 r} \hat{\vec{p}}^2 .
\end{aligned}$$

$$\frac{(\bar{\mu} r)^{2(u+\epsilon)}}{r^3} i\vec{r}\cdot\hat{\vec{p}} = \frac{iu}{r^3}(\bar{\mu} r)^{2(u+\epsilon)} + \frac{i}{4(u+\epsilon) - 2}\left(\left[\hat{\vec{p}}^2, \frac{(\bar{\mu} r)^{2(u+\epsilon)}}{r}\right] - \frac{(\bar{\mu} r)^{2(u+\epsilon)}}{r}\hat{\vec{p}}^2\right).$$

$$\left[H_S^{(3)}, \frac{C_F\alpha_s}{4mr}\left(\cancel{\frac{1}{2}} - i\vec{r}\cdot\hat{\vec{p}}\right)\right] = -\frac{C_F\alpha_s}{4m^2}\left\{\frac{1}{r}, \hat{\vec{p}}^2\right\} + \frac{\pi C_F\alpha_s}{m^2}\delta^3(\vec{r}) + \frac{C_F\alpha_s}{2m^2 r^3} r^i r^j \hat{p}^j \hat{p}^i + \frac{C_F^2\alpha_s^2}{4mr^2},$$

☆ Reduction of MZVs with various λ_is

e.g. $\lambda_i \in \{\pm 1, \pm e^{\pm\pi i/4}, \pm\frac{1\pm 2i}{\sqrt{5}}, \pm i\}$

Shuffle relations are ineffective for reduction in such cases,
too many variables, insufficient constraints,…

We reduced MZVs to a smaller set in the following way.

Let $\omega_8 \equiv e^{\pi i/4}$, $Q \equiv \dfrac{1+2i}{\sqrt{5}}$, and

type ω_8: $\lambda_i = \omega_8^p$

type Q: $\lambda_i = Q^p\,\omega_8^{p'}$

$p, p' \in \mathbb{Z}$

- Convert MZVs using the algorithm 1 in Anzai's talk:

$$Z(\infty; a_1, \ldots, a_N; \underbrace{\lambda_1, \ldots, \lambda_N}_{\text{type } Q}) \longrightarrow Z(\infty; a_1, \ldots, a_N; \underbrace{\lambda_1}_{\text{type } Q}, \underbrace{\lambda_2, \ldots, \lambda_N}_{\text{type } \omega_8})$$

then use shuffle relations in integral rep.

- $Q \to Q^{-1}$, $Q \to Q^2$

Physics predictions

At NNLO (~2000)

- Global level structure of bottomonium is reproduced. Brambilla, Y.S., Vairo

- Fine and hyperfine splittings of charmonium/bottomonium reproduced.

  Two exceptions in ~2003: Recksiegel, Y.S.; Kniehl, Penin, Pineda, Smirnov, Steinhauser

  charmonium hyperfine splitting $\Psi(2S)-\eta_c(2S)$

  bottomonium hyperfine splitting $\Upsilon(1S)-\eta_b(1S)$

  Solved in favor of pert. QCD predictions.

At NNNLO

$$Z(\infty; a_1, a_2, \ldots, a_N; \lambda_1, \lambda_2, \ldots, \lambda_N) = \sum_{n_1 > n_2 > \cdots > n_N > 0} \frac{\lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_N^{n_N}}{n_1^{a_1} n_2^{a_2} \cdots n_N^{a_N}} \quad ; \quad a_i \in \mathbb{N}$$

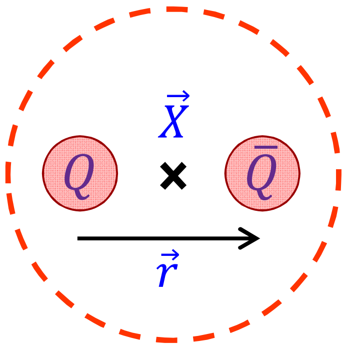
$\lambda_i \in \{1\}$	MZV
$\lambda_i \in \{-1, 1\}$	Sign-alternating Euler sum
$\lambda_i \in$ Roots of unity	e.g. $\lambda_i = e^{i\pi/3}$
$\lambda_i \in \mathbb{C}$	generalized MZV

pNRQCD Lagrangian

$Q\bar{Q}$ composite fields

$$S(\vec{X},\vec{r};t)_{\sigma\bar{\sigma}}\,, \qquad O^a(\vec{X},\vec{r};t)_{\sigma\bar{\sigma}}$$

singlet octet



$$\mathcal{L}_{\text{pNRQCD}} = S^\dagger\left(i\partial_t - \hat{H}_S\right)S + O^{a\dagger}\left(iD_t - \hat{H}_O\right)^{ab}O^b$$
$$+g\, S^\dagger\vec{r}\cdot\vec{E}^a O^a + g\, O^{a\dagger}\vec{r}\cdot\vec{E}^a S + \cdots$$

Color electric field $\vec{E}^a = -\vec{\nabla}A_0^a - \partial_t\vec{A}^a - g f^{abc}A_0^b\vec{A}^c$ at position $\vec{X}$, originating from multipole exp. of $A_\mu\left(\vec{X}\pm\vec{r}/2\right)$ in $\vec{r}$.

$\hat{H}_S, \hat{H}_O$: Quantum mechanical Hamiltonian for singlet and octet states

$$\left(\hat{H}_S\right)_{LO} = \frac{\vec{p}^2}{m} - C_F\frac{\alpha_s}{r}, \qquad \left(\hat{H}_O\right)_{LO} = \frac{\vec{p}^2}{m} + \left(\frac{C_A}{2} - C_F\right)\frac{\alpha_s}{r}$$

$\Rightarrow$ $\beta \sim \frac{p}{m} \sim \frac{1}{mr} \sim \alpha_s$

(in the c.m. frame)

$$\left(\hat{H}_S\right)_{LO} = \frac{\vec{p}^2}{m} - C_F\frac{\alpha_s}{r},$$

$$\left(\hat{H}_S\right)_{NLO} = -C_F\frac{\alpha_s}{r}\cdot\left(\frac{\alpha_s}{4\pi}\right)\cdot\{\beta_0 \log\left(\mu'^2 r^2\right) + a_1\},$$

$$\begin{aligned}\left(\hat{H}_S\right)_{NNLO} = &-\frac{\vec{p}^4}{4m^3} - C_F\frac{\alpha_s}{r}\cdot\left(\frac{\alpha_s}{4\pi}\right)^2\cdot\left\{\beta_0^2\left[\log^2\left(\mu'^2 r^2\right) + \frac{\pi^2}{3}\right] + \left(\beta_1 + 2\beta_0 a_1\right)\log\left(\mu'^2 r^2\right) + a_2\right\}\\ &+ \frac{\pi C_F\alpha_s}{m^2}\delta^3(\vec{r}) + \frac{3C_F\alpha_s}{2m^2r^3}\vec{L}\cdot\vec{S} - \frac{C_F\alpha_s}{2m^2r}\left(\vec{p}^2 + \frac{1}{r^2}r_ir_jp_jp_i\right) - \frac{C_AC_F\alpha_s^2}{2mr^2}\\ &- \frac{C_F\alpha_s}{2m^2}\left\{\frac{\vec{S}^2}{r^3} - 3\frac{\left(\vec{S}\cdot\vec{r}\right)^2}{r^5} - \frac{4\pi}{3}\left(2\vec{S}^2 - 3\right)\delta^3(\vec{r})\right\}\end{aligned}$$

$\left(\hat{H}_S\right)_{NNNLO}$ = known (Wilson coeffs. include IR div.)

Kniehl, Penin, Smirnov, Steinhauser

(a_3: Anzai, Kiyo, YS; Smirnov, Steinhauser)

Physics Predictions

At NNLO (~2000)

- Global level structure of bottomonium is reproduced. Brambilla, Y.S., Vairo

- Fine and hyperfine splittings of charmonium/bottomonium reproduced.

  Two exceptions in ~2003: Recksiegel, Y.S.; Kniehl, Penin, Pineda, Smirnov, Steinhauser

  charmonium hyperfine splitting $\Psi(2S)–\eta_c(2S)$

  bottomonium hyperfine splitting $\Upsilon(1S)–\eta_b(1S)$

  Solved in favor of pert. QCD predictions.

At NNNLO