Cutkosky Rules and Outer Space

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work joint with Spencer Bloch

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Motivation

Outer Space

The cubical chain complex

Cutkosky's theorem

Parametric Wonderland

Discriminants and anomalous thresholds

Dispersion

Example

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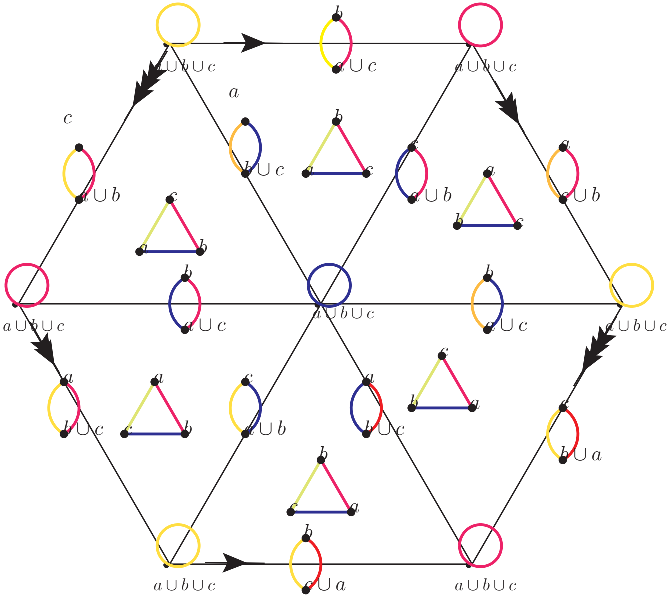
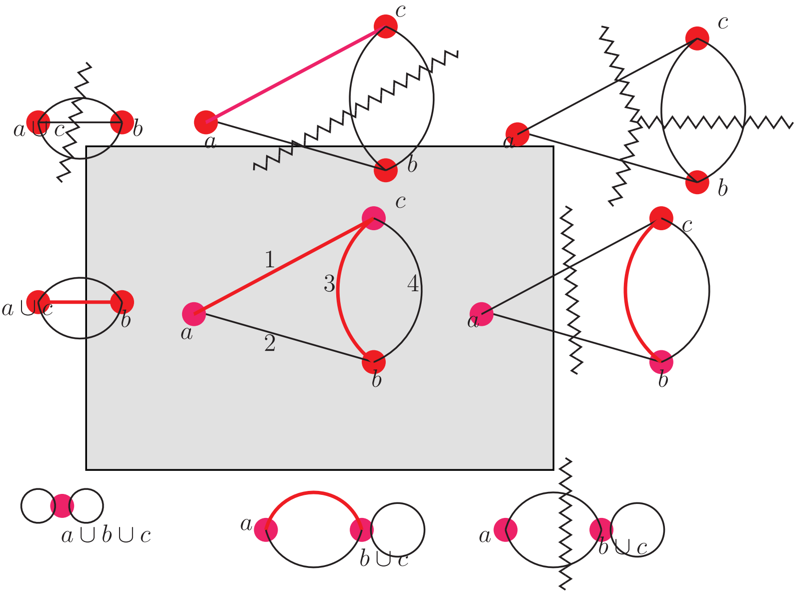
- Some literature:
  Spencer Bloch, DK
  Cutkosky rules and Outer Space, [arXiv:1512.01705 [hep-th].
  F. Pham
  Introduction à l'Etude Topologique des Singularites de Landau, Gautheir-Villars (1967).
  A. Hatcher, K. Vogtmann
  Rational Homology of Aut(F_n), Math. Research Lett. 5 (1998) 759-780.

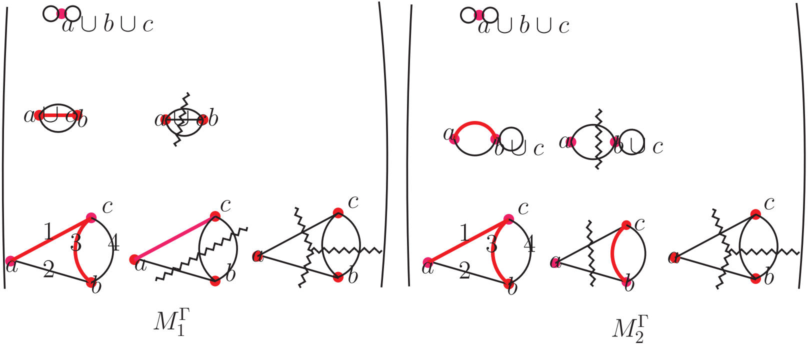
A cell complex for graphs: Outer Space

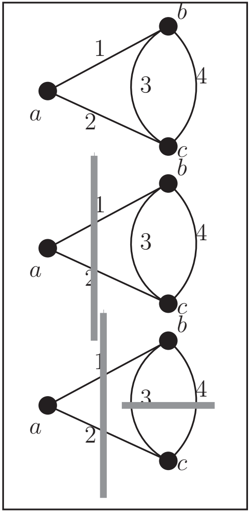


The cubical chain complex



Two matrices, obtained from the two possible orderings of edges in the spanning tree. In total, 5 spanning trees each on two edges → 10 matrices.

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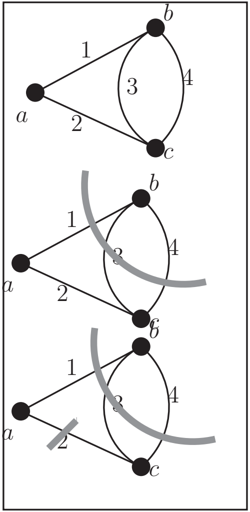
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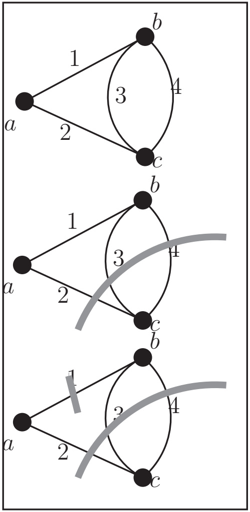
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Cutkosky's theorem

Theorem (Cutkosky)

Assume the quotient graph G'' has a physical singularity at an external momentum point $p'' \in (\bigoplus_{V''} \mathbb{R}^D)^0$, i.e. the intersection $\bigcap_{e \in E''} Q_e$ of the propagator quadrics associated to edges in E'' has such a singularity at a point lying over p''. Let $p \in (\bigoplus_V \mathbb{R}^D)^0$ be an external momentum point for G lying over p''. Then the variation of the amplitude $I(G)$ around p is given by Cutkosky's formula

$$var(I(G)) = (-2\pi i)^{\#E''} \int \frac{\prod_{e \in E''} \delta^+(\ell_e)}{\prod_{e \in E'} \ell_e}. \quad (1)$$

Parametric Wonderland

- $$\Phi(\Gamma) = \underbrace{\Phi(\Gamma/\gamma)\psi(\gamma)}_{k-1} + \underbrace{\Phi(\Gamma-\gamma)\psi(r_k) - M(\gamma)\psi(\Gamma/\gamma)\psi(\gamma)}_{k} - \underbrace{M(\gamma)\psi(\Gamma-\gamma)\psi(r_k)}_{k+1}$$

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- $$\phi^r_{x,y}(\Gamma) = \sum_{T_1 \cup T_2} (Q(T_1)\cdot Q(T_2))^r \prod_{e \notin T_1 \cup T_2} A_e,$$
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- $$\Phi(\Gamma-\gamma)E^\gamma_k - M(\gamma)\psi(\Gamma/\gamma)E^\gamma_{k-1} = \Phi^u(\Gamma-\gamma),$$
  with $u = (\sum_{e \in E_\gamma} m_e)^2$.

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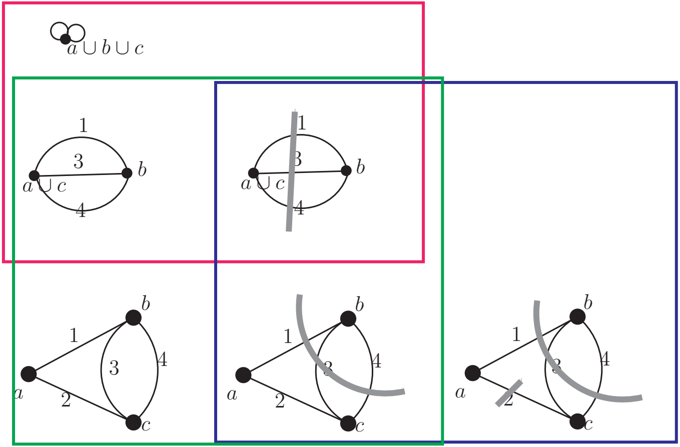
- iii) If for all $T \in T_s^\Gamma$ and for all their forests (Γ, F) we have $s_F > -\infty$, the Feynman integral $\Phi_R(\Gamma)(s)$ is real analytic as a function of s for $s < \min_F\{s_F\}$.

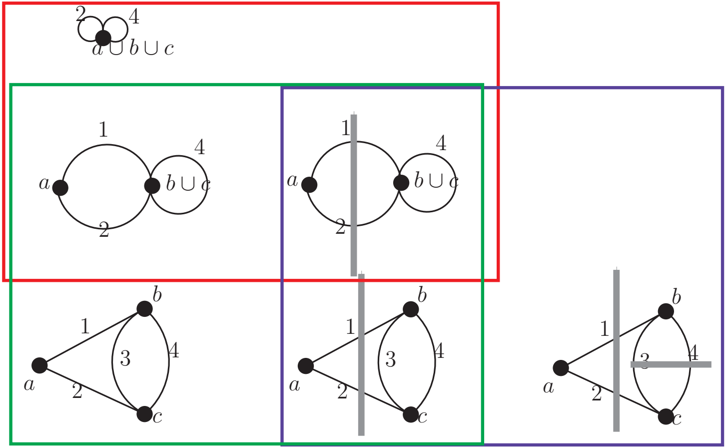
Dispersion

$$
\left(\begin{array}{ccccccc}
1 & & 0 & & 0 & & 0 \\
\uparrow \pi & & \uparrow \pi & & & & \\
\Upsilon^{T_2}_{\Gamma_2} & \rightleftharpoons^{\mathrm{Var}}_{\mathrm{disp}} & \Upsilon^{T_2-e_1}_{\Gamma_2} & & 0 & & 0 \\
\uparrow \pi & & \uparrow \pi & & \uparrow \pi & & \\
\Upsilon^{T_3}_{\Gamma_3} & \rightleftharpoons^{\mathrm{Var}}_{\mathrm{disp}} & \Upsilon^{T_3-e_1}_{\Gamma_3} & \rightleftharpoons^{\mathrm{Var}}_{\mathrm{disp}} & \Upsilon^{T_3-e_1-e_2}_{\Gamma_3} & & 0 \\
\uparrow \pi & & \uparrow \pi & & \uparrow \pi & & \uparrow \pi \\
\Upsilon^{T_4=T}_{\Gamma_4=\Gamma} & \rightleftharpoons^{\mathrm{Var}}_{\mathrm{disp}} & \Upsilon^{T_4-e_1}_{\Gamma_4} & \rightleftharpoons^{\mathrm{Var}}_{\mathrm{disp}} & \Upsilon^{T_4-e_1-e_2}_{\Gamma_4} & \rightleftharpoons^{\mathrm{Var}}_{\mathrm{disp}} & \Upsilon^{T_4-e_1-e_2-e_3}_{\Gamma_4}
\end{array}\right).
$$

An example for a graph with a length 3 spanning tree. Moving up, we shrink edges, moving right, we put more edges on the mass-shell. Dispersion integrals move left.



Along the diagonal, only normal thesholds appear in dispersion



For the subdiagonals, compute your anomalous thresholds.

Example: The triangle

$$\Phi_\Delta = \overbrace{p_a^2 A_1 A_2 - (m_1^2 A_1 + m_2^2 A_1)(A_1 + A_2)}^{=\Phi_{\Gamma/e_3}} + A_3((p_b^2 - m_3^2 - m_1^2)A_1 + (p_c^2 - m_1^2 - m_3^2$$

so

$$\Phi_\Delta = \Phi_{\Delta/e_3} + A_3 \Phi^{m_3^2}_{\Delta - e_3} - A_3^2 m_3^2 \overbrace{\psi_{\Delta - e_1}}^{=1},$$

as announced ($A_3 = t_\gamma$):

$$X = \Phi_{\Delta/e_3},\ Y = \overbrace{(p_b^2 - m_3^2 - m_1^2)}^{=:l_1} A_1 + \overbrace{(p_c^2 - m_1^2 - m_3^2)}^{=:l_2} A_2,\ Z = m_3^2.$$

We have $Y_0 = m_2 l_1 + m_1 l_2$, and need $Y_0 > 0$ for a Landau singularity.

cont'd

Solving $\Phi(\Delta/e_3) = 0$ for a Landau singularity determines the familiar physical threshold in the $s = p_a^2$ channel, leading for the reduced graph to

$$p_Q : s_0 = (m_2 + m_3)^2,\ p_A : A_1 m_1 = A_2 m_2.$$

We let $D = Y^2 + 4XZ$ be the discriminant. For a Landau singularity we need

$$D = 0.$$

We have

$$\Phi_\Delta = -m_3^2 \left(A_3 - \frac{Y + \sqrt{D}}{2m_3^2} \right) \left(A_3 - \frac{Y - \sqrt{D}}{2m_3^2} \right),$$

where Y, D are functions of A_1, A_2 and $m_1^2, m_2^2, m_3^2, s, p_b^2, p_c^2$.

cont'd

We can write

$$0 = D = Y^2 + 4Z(sA_1A_2 - N),$$

with $N = (A_1 m_1^2 + A_2 m_2^2)(A_1 + A_2)$ s-independent. This gives

$$s(A_1, A_2) = \frac{4ZN - (A_1 l_1 + A_2 l_2)^2}{4ZA_1A_2} =: \frac{A_1}{A_2}\rho_1 + \rho_0 + \frac{A_2}{A_1}\rho_2.$$

Define two Kallen functions $\lambda_1 = \lambda(p_b^2, m_1^2, m_3^2)$ and $\lambda_2 = \lambda(p_c^2, m_2^2, m_3^2)$. Both are real and non-zero off their threshold or pseudo-threshold. Then, for

$$r := \lambda_1/\lambda_2 > 0,$$

we find the threshold s_1 at

$$s_1 = \frac{4m_3^2(\sqrt{\lambda_1}m_1^2 + \sqrt{\lambda_2}m_2^2)(\sqrt{\lambda_1} + \sqrt{\lambda_2}) - (\sqrt{\lambda_1}l_2 + \sqrt{\lambda_2}l_1)^2}{4m_3^2\sqrt{\lambda_1}\sqrt{\lambda_2}}.$$

cont'd

On the other hand for $r < 0$ and therefore the coefficients of ρ_1, ρ_2 above of different sign we find a minimum

$$s_1 = -\infty, \tag{2}$$

along either $A_1 = 0$ or $A_2 = 0$. Get dispersion from other channels, looking at other spanning trees, that is.

Things are not simpler than they can be, and not more difficult than they must be.