Three-loop heavy flavour corrections to deep-inelastic scattering

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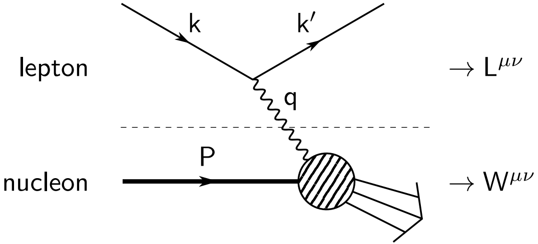
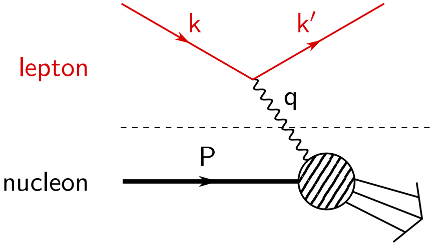
Heavy flavour contributions to deep-inelastic scattering



Kinematic variables: $Q^2 = -q^2$, $x = \frac{Q^2}{2P.q}$

Cross section: $\frac{d\sigma}{dx dQ^2} \propto L_{\mu\nu} W^{\mu\nu}$

Heavy flavour contributions to deep-inelastic scattering



$\rightarrow L^{\mu\nu}$

$\rightarrow W^{\mu\nu}$

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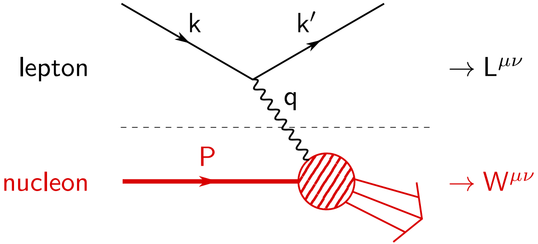

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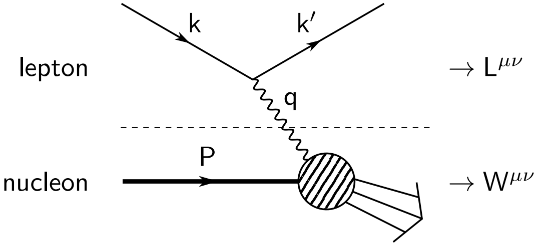

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Structure functions contain light and heavy quark contributions.

Motivation for NNLO heavy flavour corrections

- Precision of DIS world data: $\sim 1\%$ for F_2
  $\rightarrow$ requires $\mathcal{O}(\alpha_s^3)$ description
- Heavy quarks yield essential contributions to structure functions $\sim 20 - 30\%$ in the small x region
- Heavy quark contributions to the scaling violations have different shape than massless contributions

$\Rightarrow$ NNLO heavy quark contributions are important for precise measurement of the strong coupling constant

$$\delta\alpha_s(M_Z) \approx 1\%$$

and heavy quark masses [Alekhin et al. '12 (and updates)]

$$m_c(m_c) = 1.25 \pm 0.02(\text{exp})^{+0.03}_{-0.02}(\text{scale})^{+0.00}_{-0.07}(\text{thy})\text{GeV}$$

$$m_b(m_b) = 3.91 \pm 0.14(\text{exp})^{+0.00}_{-0.11}(\text{thy})\text{GeV} \quad \text{(preliminary)}$$

($\overline{\text{MS}}$ scheme)

Heavy flavour contributions to deep-inelastic scattering

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Structure functions: $F_2(x, Q^2, m^2) = x \sum_j \mathbb{C}_{2,j}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes f_j(x, \mu^2)$

Wilson coefficients (perturbative) — PDFs (non-perturbative)

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Wilson coefficients (perturbative)

PDFs (non-perturbative)

x- and N-space are connected by a Mellin transformation:

$$M[f(x)](N) = \int_0^1 \mathrm{d}x \, x^{N-1} f(x)$$

Representation simplifies in Mellin space.

Heavy flavour contributions to deep-inelastic scattering

Hadronic tensor: $W_{\mu\nu} = (\dots)_{\mu\nu} F_L(x, Q^2) + (\dots)_{\mu\nu} \boxed{F_2(x, Q^2)}$

Structure functions: $$\boxed{F_2(N-1, Q^2, m^2) = \sum_j \mathbb{C}_{2,j}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \cdot f_j(N, \mu^2)}$$

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$C_{2,j}$: massless Wilson coefficients
NNLO: [Moch, Vermaseren, Vogt '05]

$H_{2,j}$: heavy-flavor Wilson coefficients

Heavy flavour contributions to deep-inelastic scattering

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For F_2 and $Q^2/m^2 \gtrsim 10$ the heavy flavor Wilson coefficients factorise:
[Buza, Matiounine, Smith, Migneron, van Neerven '96]

Heavy flavor Wilson coefficients: $H_{2,j}(N) = \sum_i A_{ij}(N) C_{2,i}(N)$

massive operator matrix elements (OMEs)

LO: [Witten '76; Babcock, Sievers '78; Shifman, Vainshtein, Zakharov '78; Leveille, Weiler '79; Glück, Reya '79; Glück, Hoffmann, Reya '82]
NLO: [Laenen, van Neerven, Riemersma, Smith '93; Buza, Matiounine, Smith, Migneron, van Neerven '96; Bierenbaum, Blümlein, Klein '07a, '07b, '08, '09a]

massless Wilson coefficients

Heavy flavour contributions to deep-inelastic scattering

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Heavy flavor Wilson coefficients: $H_{2,j}(N) = \sum_i A_{ij}(N) C_{2,i}(N)$

OMEs A_{ij} also essential to define the variable flavor number scheme
$\rightarrow$ describe transition $N_F \rightarrow N_F + 1$ massless quarks
$\rightarrow$ transitions relevant for the PDFs at the LHC

Massive operator matrix elements

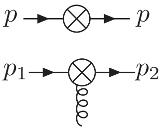
Definition of the OMEs A_{ij}

$$A_{ij} := \langle j|O_i|j\rangle$$

$|j\rangle$: partonic states (massless, on-shell)
O_i: local light-cone operators
Example:

$$O^{\rm NS}_{q,a;\mu_1,\ldots,\mu_N} = i^{N-1} S[\overline{\Psi}\gamma_{\mu_1} D_{\mu_2}\ldots D_{\mu_N}\frac{\lambda^a}{2}\Psi] - \text{trace terms}$$

Feynman rules for operators



$\propto (\Delta.p)^{N-1}$

$\propto \sum_{j=0}^{N-2}(\Delta.p_1)^j(\Delta.p_2)^{N-2-j}$

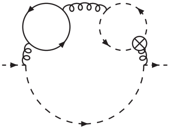
⋮

Depend on integer variable N (Mellin variable)

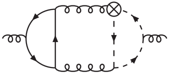
Massive operator matrix elements at NNLO

Fixed moments for OMEs: $N = 2 \ldots 10(14)$ ✓ [Bierenbaum, Blümlein, Klein, '09b]

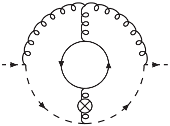
All logarithmic terms from renormalisation ✓ [Behring et al. '14]



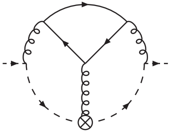
$A_{qq,Q}^{\mathrm{PS}}$
8 diagrams
✓ [Ablinger et al. '10]



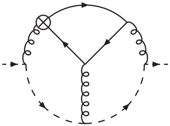
$A_{qg,Q}$
132 diagrams
✓ [Ablinger et al. '10]



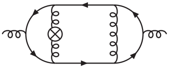
$A_{gq,Q}$
89 diagrams
✓ [Ablinger et al. '14a]



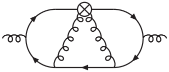
$A_{qq,Q}^{\mathrm{NS}}$ & $A_{qq,Q}^{\mathrm{TR}}$
112 diagrams
✓ [Ablinger et al. '14b]



A_{Qq}^{PS}
125 diagrams
✓ [Ablinger et al. '14c]



$A_{gg,Q}$
642 diagrams
✓



A_{Qg}
1233 diagrams
in progress
(1003 diags. done)

Feynman diagrams drawn using `axodraw` [Vermaseren '94]

Wilson coefficients at $Q^2 \gg m^2$ in terms of OMEs

- Factorisation into massive OMEs and massless Wilson coefficients
  Example: [Buza, Matiounine, Smith, Migneron, van Neerven '96] [Bierenbaum, Blümlein, Klein, '09b]

$$
\begin{aligned}
H_{q,2}^{\mathrm{PS}}(N_F+1) = a_s^2 \Bigg[& A_{Qq}^{\mathrm{PS},(2)}(N_F+1) + \frac{C_{q,2}^{\mathrm{PS},(2)}(N_F+1)}{N_F+1} \Bigg] \\
+ a_s^3 \Bigg[& A_{Qq}^{\mathrm{PS},(3)}(N_F+1) + \frac{C_{q,2}^{\mathrm{PS},(3)}(N_F+1)}{N_F+1} \\
& + A_{gq,Q}^{(2)}(N_F+1) \frac{C_{g,2}^{(1)}(N_F+1)}{N_F+1} \\
& + A_{Qq}^{\mathrm{PS},(2)}(N_F+1) C_{q,2}^{\mathrm{NS},(1)}(N_F+1) \Bigg]
\end{aligned}
$$

- Similar relations for other coefficients
- Status of Wilson coefficients:

$L_{q,2}^{\mathrm{PS}}$	$(\propto A_{qq,Q}^{\mathrm{PS},(3)})$	✓ [Ablinger et al. '10] [Behring et al. '14]		$H_{q,2}^{\mathrm{PS}}$	$(\propto A_{Qq}^{\mathrm{PS},(3)})$	✓ [Ablinger et al. '14c]
$L_{g,2}^{\mathrm{S}}$	$(\propto A_{qg,Q}^{(3)})$	✓ [Ablinger et al. '10] [Behring et al. '14]		$H_{g,2}^{\mathrm{S}}$	$(\propto A_{Qg}^{(3)})$	in progress
$L_{q,2}^{\mathrm{NS}}$	$(\propto A_{qq,Q}^{\mathrm{NS},(3)})$	✓ [Ablinger et al. '14b]				

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$L_{q,2}^{\mathrm{NS}}$	$(\propto A_{qq,Q}^{\mathrm{NS},(3)})$	✓	[Ablinger et al. '14b]
$H_{q,2}^{\mathrm{PS}}$	$(\propto A_{Qq}^{\mathrm{PS},(3)})$	✓	[Ablinger et al. '14c]
$H_{g,2}^{\mathrm{S}}$	$(\propto A_{Qg}^{(3)})$	in progress	

Variable flavour number scheme (VFNS)

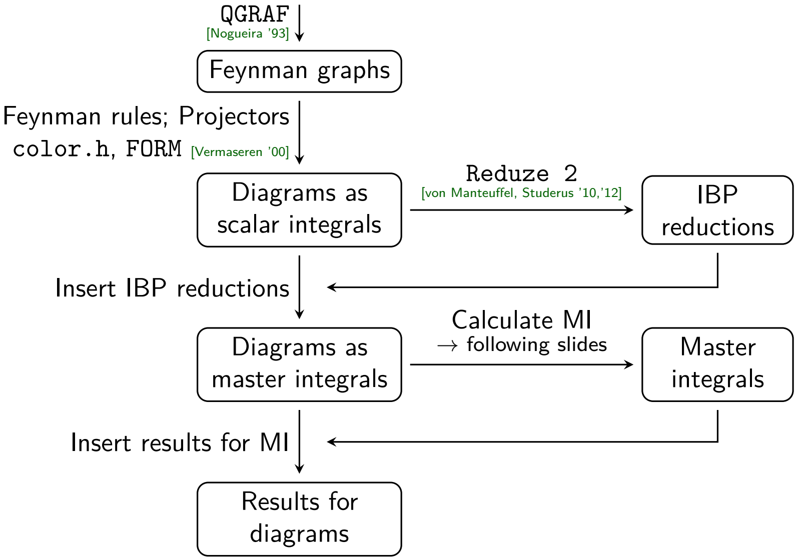
- Transition from scheme with N_F massless and 1 massive flavour to scheme with N_F+1 effectively massless flavours
- Massive OMEs appear in the matching conditions of the PDFs

NNLO matching relations:
[Buza et al. '96] [Bierenbaum, Blümlein, Klein, '09a, '09b]

$$
\begin{aligned}
f_k(N_F+1,\mu^2)+\bar{f}_k(N_F+1,\mu^2) &= A^{\mathrm{NS}}_{qq,Q}\otimes[f_k(N_F,\mu^2)+\bar{f}_k(N_F,\mu^2)] \\
&\quad+\frac{1}{N_F}\left[A^{\mathrm{PS}}_{qq,Q}\otimes\Sigma(N_F,\mu^2)+A_{qg,Q}\otimes G(N_F,\mu^2)\right] \\
f_{Q+\bar{Q}}(N_F+1,\mu^2) &= A^{\mathrm{PS}}_{Qq}\otimes\Sigma(N_F,\mu^2)+A_{Qg}\otimes G(N_F,\mu^2) \\
G(N_F+1,\mu^2) &= A_{gq,Q}\otimes\Sigma(N_F,\mu^2)+A_{gg,Q}\otimes G(N_f,\mu^2) \\
\Sigma(N_F+1,\mu^2) &= \left[A^{\mathrm{NS}}_{qq,Q}+A^{\mathrm{PS}}_{qq,Q}+A^{\mathrm{PS}}_{Qq}\right]\otimes\Sigma(N_F,\mu^2) \\
&\quad+\left[A_{qg,Q}+A_{Qg}\right]\otimes G(N_F,\mu^2)
\end{aligned}
$$

with the singlet combination $\Sigma(N_F,\mu^2)=\sum_{k=1}^{N_F}[f_k(N_F,\mu^2)+f_k(N_F,\mu^2)]$

Outline of the calculation



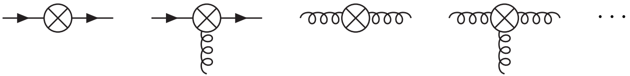
Statistics

- Long-term effort to calculate 3-loop OMEs

	Diagrams	Scalar integrals	Master integrals
$A_{gq,Q}^{(3)}$	89	12253	41
$A_{qq,Q}^{\mathrm{NS},(3)}$	112	5636	35
$A_{Qq}^{\mathrm{PS},(3)}$	125	4824	66
$A_{gg,Q}^{(3)}$	642	67212	139
$A_{Qg}^{(3)}$	1233	33198	340

- 2.1 TB of IBP relations
- Requires large computing resources $\to$ 12 computers with overall
  - 348 cores
  - 3.2 TB RAM
  - 97 TB local disk space
  - large number of `Mathematica` licences

Dealing with operator insertions



- Large number of scalar integrals ($\sim 10^5$) requires using integration-by-parts reductions to master integrals (474)
- Problem: Operators prevent straightforward application of Laporta's algorithm (N in exponents of scalar products)
- Solution: Introduce generating functions for operators

$$\sum_{N=0}^{\infty} t^N (\Delta.k)^N = \frac{1}{1 - t(\Delta.k)}$$

and similar expressions for more complex operators

$\Rightarrow$ treat them as linear propagators

- Allows to use `Reduze 2` to obtain IBP reductions
- Additional advantage: Allows to derive differential equations in t
- Result in N is recovered as Nth coefficient of expansion in t at the end of the calculation

Calculation of master integrals

Master integrals are calculated using a range of techniques:

- Hypergeometric function techniques
- Mellin-Barnes representations

⇒ Yields multi-sum representations

⇒ Simplify using summation algorithms based on ΣΠ fields/rings implemented in `Sigma` [Schneider '01-], `EvaluateMultiSums` and `SumProduction` [Ablinger, Blümlein, Hasselhuhn, Schneider'10-] and special function tools from `HarmonicSums` [Ablinger, Blümlein, Schneider '10,'13]

Moreover, we use

- Coupled systems of differential equations/difference equations [Ablinger et al. '15]
  `SolveCoupledSystem` → C. Schneider's talk
- Almkvist-Zeilberger algorithm [Almkvist, Zeilberger '90; Apagodu, Zeilberger '06]
  → `MultiIntegrate` [Ablinger '12]

⇒ Yields scalar recurrences for the integrals

⇒ Solve using the packages listed above

Nested sums and iterated integrals

Results require mathematical objects of increasing complexity:

$A_{qq,Q}^{\mathrm{PS}}$, $A_{qg,Q}$, $A_{qq,Q}^{\mathrm{NS}}$, $A_{gq,Q}$

Harmonic sums
[Vermaseren '98] [Blümlein, Kurth '98]

$$\sum_{i=1}^{N} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}$$

HPLs
[Remiddi, Vermaseren '99]

$$\int_0^x \frac{\mathrm{d}y}{y} \int_0^y \frac{\mathrm{d}z}{1-z}$$

A_{Qq}^{PS}

Generalised harmonic sums
[Moch, Uwer, Weinzierl '01]
[Ablinger, Blümlein, Schneider '13]

$$\sum_{i=1}^{N} \frac{2^{-i}}{i^2} \sum_{j=1}^{i} \frac{2^j}{j}$$

(Here:) HPLs at $1-2x$

$$\int_0^{1-2x} \frac{\mathrm{d}y}{y} \int_0^y \frac{\mathrm{d}z}{1-z}$$

$A_{gg,Q}$, A_{Qg} (so far)

Cyclotomic & binomial sums
[Ablinger, Blümlein, Schneider '11]
[Ablinger, Blümlein, Raab, Schneider '14]

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \binom{2j}{j} \frac{(-1)^j}{j^3}}{\binom{2i}{i}(2i+1)}$$

Cyclotomic HPLs
[Ablinger, Blümlein, Schneider '11]

& iterated integrals over root-valued letters
[Ablinger, Blümlein, Raab, Schneider '14]

$$\int_0^x \frac{\mathrm{d}y}{y\sqrt{y+\frac{1}{4}}} \int_0^y \frac{\mathrm{d}z}{z} \int_0^z \frac{\mathrm{d}w}{w}$$

Anomalous dimensions

- Renormalisation of the OMEs [Bierenbaum, Blümlein, Klein, '09b] involves the NNLO anomalous dimensions [Moch, Vermaseren, Vogt '04a, '04b]
  Example: $(\hat{\gamma}_{ij} = \gamma_{ij}(N_F + 1) - \gamma_{ij}(N_F))$

$$\hat{\hat{A}}^{\mathrm{NS},(3)}_{qq,Q} = \frac{1}{\varepsilon^3}\cdots + \frac{1}{\varepsilon^2}\cdots + \frac{1}{\varepsilon}\Bigg[\frac{\hat{\gamma}^{\mathrm{NS},(2)}_{qq}}{3} - 4a^{\mathrm{NS},(2)}_{qq,Q}\left[\beta_0 + \beta_{0,Q}\right] + \beta^{(1)}_{1,Q}\gamma^{(0)}_{qq} + \frac{\gamma^{(0)}_{qq}\beta_0\beta_{0,Q}\zeta_2}{2} - 2\delta m_1^{(0)}\beta_{0,Q}\gamma^{(0)}_{qq} - \delta m_1^{(-1)}\hat{\gamma}^{\mathrm{NS},(1)}_{qq}\Bigg] + \mathcal{O}\left(\varepsilon^0\right)$$

$\Rightarrow$ $\mathcal{O}(N_F)$ contributions to anomalous dimensions

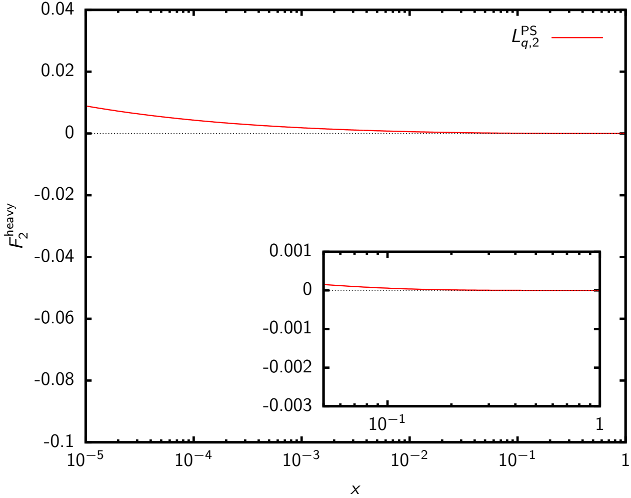
$A_{gq,Q} \to \gamma^{(2)}_{gq}$ [Ablinger et al. '14a] $A_{gg,Q} \to \gamma^{(2)}_{gg}$

$A^{\mathrm{NS}}_{qq,Q} \to \gamma^{\mathrm{NS},(2)}_{qq}$ [Ablinger et al. '14b] $A_{Qg} \to \gamma^{(2)}_{qg}$

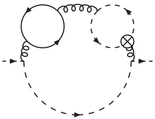
$A^{\mathrm{PS}}_{Qq} \to \gamma^{\mathrm{PS},(2)}_{qq}$ [Ablinger et al. '14c] complete PS anom. dim.

- First independent calculation in a massive setting

Contributions to the structure function F_2



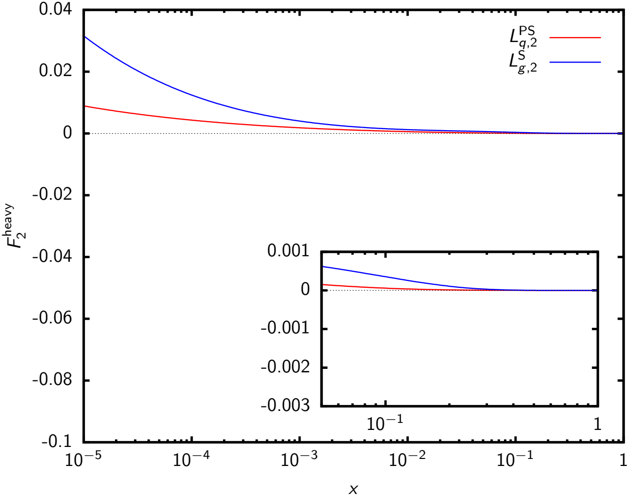
$L_{q,2}^{\text{PS}}$ [Ablinger et al. '10] [Behring et al. '14]



$\mathcal{O}(\alpha_s^3)$; $Q^2 = 100\,\text{GeV}^2$; $\mu^2 = Q^2$; $m_c^{\text{pole}} = 1.59\,\text{GeV}$; ABM13 $N_F = 3$ PDFs

Renormalisation: α_s in $\overline{\text{MS}}$ scheme, m_c in on-shell scheme

Contributions to the structure function F_2



$L_{q,2}^{\text{PS}}$ [Ablinger et al. '10] [Behring et al. '14]

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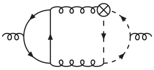

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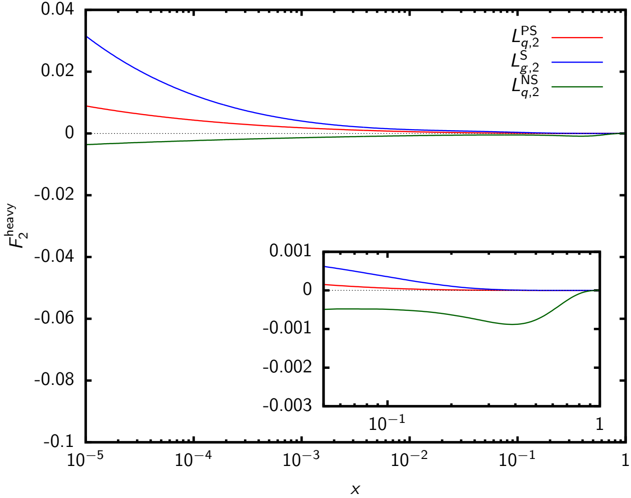

$L_{q,2}^{\text{PS}}$ [Ablinger et al. '10] [Behring et al. '14]

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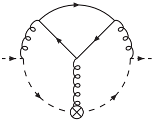
$L_{q,2}^{\text{NS}}$ [Ablinger et al. '14b]



$\mathcal{O}(\alpha_s^3)$; $Q^2 = 100\,\text{GeV}^2$; $\mu^2 = Q^2$; $m_c^{\text{pole}} = 1.59\,\text{GeV}$; ABM13 $N_F = 3$ PDFs
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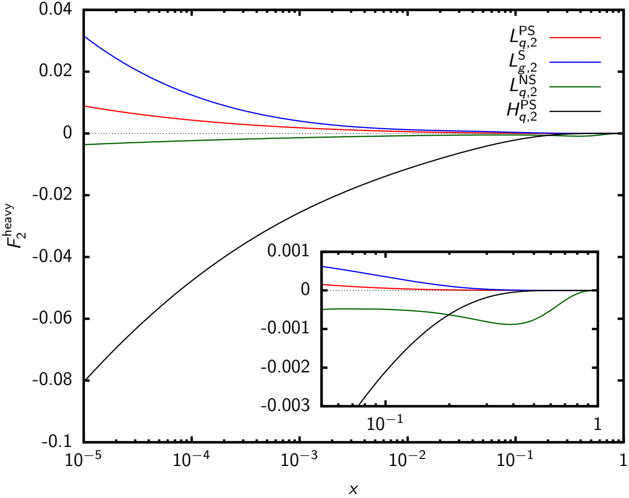
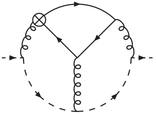


$L_{q,2}^{\text{PS}}$ [Ablinger et al. '10] [Behring et al. '14]

$L_{g,2}^{\text{S}}$ [Ablinger et al. '10] [Behring et al. '14]

$L_{q,2}^{\text{NS}}$ [Ablinger et al. '14b]

$H_{q,2}^{\text{PS}}$ [Ablinger et al. '14c]



$\mathcal{O}(\alpha_s^3)$; $Q^2 = 100\,\text{GeV}^2$; $\mu^2 = Q^2$; $m_c^{\text{pole}} = 1.59\,\text{GeV}$; ABM13 $N_F = 3$ PDFs
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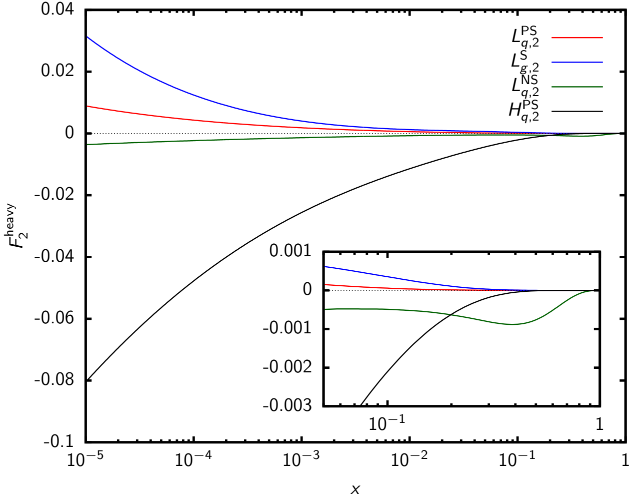

$L_{q,2}^{\text{PS}}$ [Ablinger et al. '10] [Behring et al. '14]

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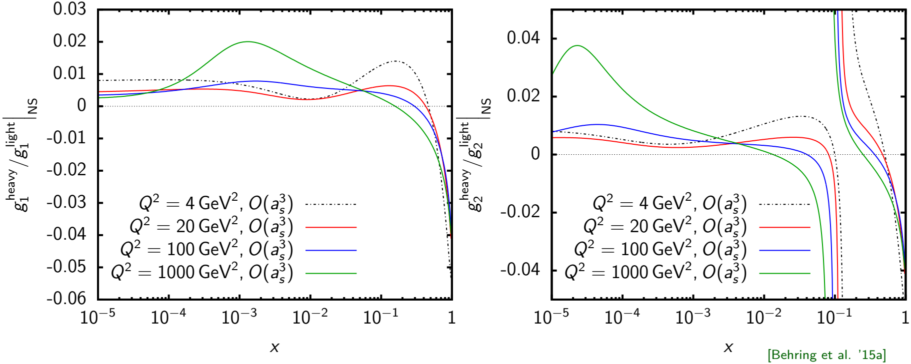
$L_{q,2}^{\text{NS}}$ [Ablinger et al. '14b]

$H_{q,2}^{\text{PS}}$ [Ablinger et al. '14c]

$H_{g,2}^{\text{S}}$ not yet known at $\mathcal{O}(\alpha_s^3)$
$\rightarrow$ work in progress

$\mathcal{O}(\alpha_s^3)$; $Q^2 = 100\,\text{GeV}^2$; $\mu^2 = Q^2$; $m_c^{\text{pole}} = 1.59\,\text{GeV}$; ABM13 $N_F = 3$ PDFs
Renormalisation: α_s in $\overline{\text{MS}}$ scheme, m_c in on-shell scheme

Non-singlet part of polarised structure functions g_1 & g_2



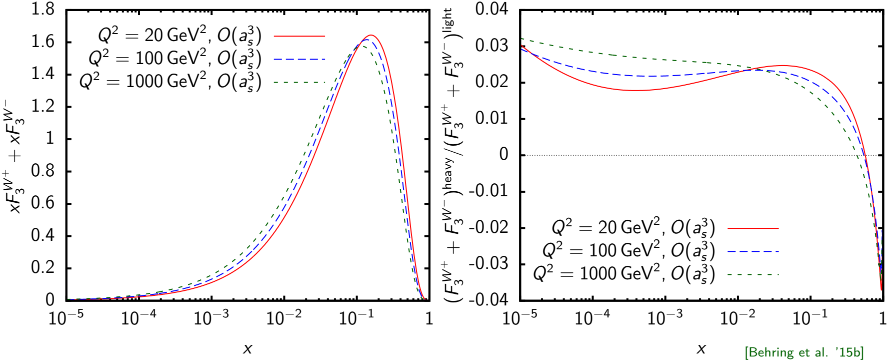
[Behring et al. '15a]

- Odd moments of $A_{qq,Q}^{\text{NS}}$ calculated as well [Ablinger et al. '14b]
- They enter the non-singlet contribution to g_1
- Twist-2 part of g_2 determined via Wandzura-Wilczek relation:

$$g_2(x, Q^2) = -g_1(x, Q^2) + \int_x^1 \frac{\mathrm{d}y}{y} g_1(y, Q^2)$$

- Power corrections at 2-loops: → G. Falcioni's talk

Charged current function xF_3



- Odd moments of $A_{qq,Q}^{\text{NS}}$ enter also $xF_3^{W^+} + xF_3^{W^-}$
- Two non-singlet Wilson coefficients:
  - $L_{q,3}^{\text{NS}}$: W couples to light quarks ($u \to d, \dots$)
  - $H_{q,3}^{\text{NS}}$: W couples to heavy quark ($s \to c, \dots$)
- Power corrections at 2-loops: $\to$ G. Falcioni's talk

Variable flavour number scheme (VFNS)

NNLO matching condition for non-singlet case:
[Buza et al. '96] [Bierenbaum, Blümlein, Klein, '09a, '09b]

$$\begin{aligned} f_k(N_F+1,\mu^2) + \bar{f}_k(N_F+1,\mu^2) \\ &= A_{qq,Q}^{\mathrm{NS}} \otimes [f_k(N_F,\mu^2) + \bar{f}_k(N_F,\mu^2)] \\ &+ \frac{1}{N_F}\left(A_{qq,Q}^{\mathrm{PS}} \otimes \Sigma(N_F,\mu^2) + A_{qg,Q} \otimes G(N_F,\mu^2)\right) \end{aligned}$$

- Ingredients at NNLO are now complete for the above relation
- $A_{qq,Q}^{\mathrm{PS}}$ and $A_{qg,Q}$ start at NNLO

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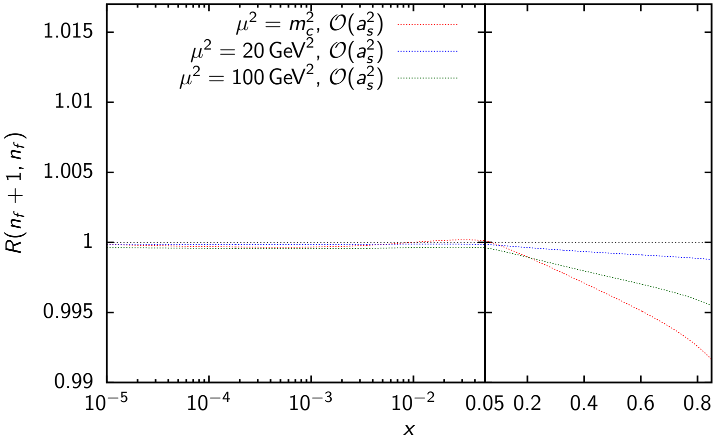
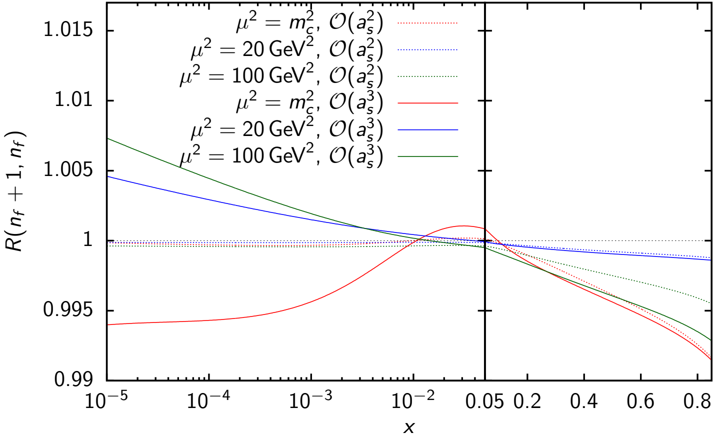


[Ablinger et al. '14b]

$$R(N_F+1, N_F) = \frac{u(N_F+1,\mu^2) + \bar{u}(N_F+1,\mu^2)}{u(N_F,\mu^2) + \bar{u}(N_F,\mu^2)}, \qquad \text{here } N_F = 3$$

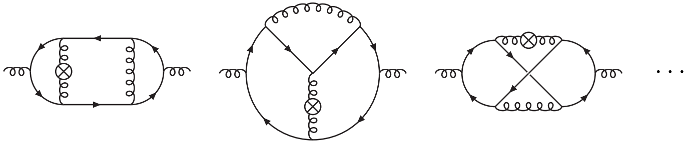
Variable flavour number scheme (VFNS)



[Ablinger et al. '14b]

$$R(N_F+1, N_F) = \frac{u(N_F+1,\mu^2) + \bar{u}(N_F+1,\mu^2)}{u(N_F,\mu^2) + \bar{u}(N_F,\mu^2)}, \qquad \text{here } N_F = 3$$

Gluonic operator matrix element $A_{gg,Q}$



- Important building block for the VFNS
  $\rightarrow$ enters the matching relation of the gluon PDF
  [Buza et al. '96] [Bierenbaum, Blümlein, Klein, '09a, '09b]

$$G(N_F + 1, \mu^2) = A_{gq,Q} \otimes \Sigma(N_F, \mu^2) + A_{gg,Q} \otimes G(N_F, \mu^2)$$

- 642 diagrams $\rightarrow$ 67212 scalar integrals $\rightarrow$ 139 master integrals
- 2 crossed-box diagrams
- MI partly overlap with earlier calculations ($\sim$ 25%)
- Remaining MI calculated mainly via differential/difference equations

$\Rightarrow$ Diagrams are all done

$\Rightarrow$ Unrenormalised OME is known for all even N; vanishes for odd N

Constant term of the gluonic OME $A_{gg,Q}$

$$
\begin{aligned}
a_{gg,Q}^{(3)} = \frac{1+(-1)^N}{2}\Bigg\{ & C_F^2 T_F \Bigg[\frac{16(N^2+N+2)}{N^2(N+1)^2}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2} - \frac{4P_{69}S_1^2}{3(N-1)N^4(N+1)^4(N+2)} \\
& + \tilde{\gamma}_{gq}^{(0)}\left(\frac{128\,(S_{-4}-S_{-3}S_1+S_{-3,1}+2S_{-2,2})}{3N(N+1)(N+2)} + \frac{4(5N^2+5N-22)S_1^2S_2}{3N(N+1)(N+2)}+\cdots\right)+\cdots\Bigg] \\
& + C_A C_F T_F \Bigg[\frac{16P_{42}}{3(N-1)N^2(N+1)^2(N+2)}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2} + \frac{32P_2S_{-2,2}}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{64P_{14}S_{-2,1,1}}{3(N-1)N^2(N+1)^2(N+2)} - \frac{16P_{23}S_{-4}}{3(N-1)N^2(N+1)^2(N+2)} + \frac{4P_{63}S_4}{3(N-2)(N-1)N^2(N+1)^2(N+2)}+\cdots\Bigg] \\
& + C_A^2 T_F \Bigg[-\frac{4P_{46}}{3(N-1)N^2(N+1)^2(N+2)}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2} + \frac{256P_5S_{-2,2}}{9(N-1)N^2(N+1)^2(N+2)} \\
& + \frac{32P_{30}S_{-2,1,1}+16P_{35}S_{-3,1}+16P_{44}S_{-4}}{9(N-1)N^2(N+1)^2(N+2)} + \frac{16P_{52}S_{-2}^2}{27(N-1)N^2(N+1)^2(N+2)} + \frac{8P_{36}S_2^2}{9(N-1)N^2(N+1)^2}+\cdots\Bigg] \\
& + C_F T_F^2 \Bigg[-\frac{16P_{48}\binom{2N}{N}4^{-N}\left(\sum_{i=1}^{N}\frac{4^i S_1(i-1)}{\binom{2i}{i}i^2}-7\zeta_3\right)}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} - \frac{32P_{86}S_1}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \\
& + \frac{16P_{45}S_1^2}{27(N-1)N^3(N+1)^3(N+2)} - \frac{16P_{45}S_2}{9(N-1)N^3(N+1)^3(N+2)}+\cdots\Bigg]+\cdots\Bigg\}
\end{aligned}
$$

Constant term of the gluonic OME $A_{gg,Q}$

$$a_{gg,Q}^{(3)} = \frac{1+(-1)^N}{2}\Bigg\{C_F^2 T_F\Bigg[\frac{16(N^2+N+2)}{N^2(N+1)^2}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2} - \frac{4P_{69}S_1^2}{3(N-1)N^4(N+1)^4(N+2)}$$

$$+\tilde{\gamma}_{gq}^{(0)}\left(\frac{128\,(S_{-4}-S_{-3}S_1+S_{-3,1}+2S_{-2,2})}{3N(N+1)\ldots} + \frac{4(5N^2+5N-22)S_1^2S_2}{\ldots} + \cdots\right) + \cdots\Bigg]$$

$$+C_AC_FT_F\Bigg[\frac{1\ldots}{3(N-1)N^2(\ldots} \ldots \frac{32P_2S_{-2,2}}{\ldots-1)N^2(N+1)^2(N+2)}$$

$$-\frac{64P_{14}S_{-2,1,1}}{3(N-1)N^2(N+1)^2(\ldots} \ldots \frac{4P_{63}S_4}{\ldots(N-1)N^2(N+1)^2(N+2)}+\cdots\Bigg]$$

$$\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2}$$

Binomial sums

- Two objects involving binomial weights appear
- One of them already occurred in the T_F^2 colour factor [Ablinger et al. '14d]

More about binomial sums $\rightarrow$ J. Ablinger's talk

$$+\frac{16P_{45}S_1}{27(N-1)N^3(N+1)^3(N+2)} - \frac{16P_{45}S_2}{9(N-1)N^3(N+1)^3(N+2)}+\cdots\Bigg]+\cdots\Bigg\}$$

Conclusions

- Heavy quark corrections yield important contributions to DIS
  $\rightarrow$ essential for precision measurements
  of α_s (1%) and m_c (3%). [Alekhin et al. '12]
- New mathematical and computer-algebraic methods required for analytic calculation of the 3-loop corrections
  $\rightarrow$ includes new classes of higher transcendental functions
  and function spaces
- Completed massive OMEs and Wilson coefficients:
  - $A_{qq,Q}^{\mathrm{PS}}$, $A_{qg,Q}$, $A_{qq,Q}^{\mathrm{NS}}$, $A_{qq,Q}^{\mathrm{TR}}$, A_{Qq}^{PS}, $A_{gq,Q}$, $A_{gg,Q}$,
  - $L_{q,2}^{\mathrm{PS}}$, $L_{g,2}^{\mathrm{S}}$, $L_{q,2}^{\mathrm{NS}}$, $H_{q,2}^{\mathrm{PS}}$, L_{q,g_1}^{NS}, $L_{q,3}^{\mathrm{NS}}$
- Ingredients for first matching relation of the VFNS are complete.
- Calculation of the remaining massive OME A_{Qg}
  and Wilson coefficient $H_{g,2}^{\mathrm{S}}$ is in progress.

Related publications: Physics

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Nucl. Phys. B755 (2006) 272

Bierenbaum, Blümlein, Klein
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Blümlein, Klein, Tödtli
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Bierenbaum, Blümlein, Klein, Schneider
Nucl. Phys. B803 (2008) 1

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Nucl. Phys. B844 (2011) 26

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Nucl. Phys. B864 (2012) 52; Nucl. Phys. B882 (2014) 263; Nucl. Phys. B885 (2014) 280; Nucl. Phys. B885 (2014) 409; Nucl. Phys. B886 (2014) 733; Nucl. Phys. B890 (2014) 48; Comp. Phys. Commun. 202 (2016) 33

Behring et al.
Eur. Phys.J. C74 (2014) 9, 3033; Nucl. Phys. B897 (2015) 612; Phys. Rev. D92 (2015) 11405

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DESY 15-171

Related publications: Mathematics

Blümlein
Comput. Phys. Commun. 159 (2004) 19

Blümlein
Comput. Phys. Commun. 180 (2009) 2143; arXiv:0901.0837

Blümlein, Broadhurst, Vermaseren
Comput. Phys. Commun. 181 (2010) 582

Blümlein, Kauers, Schneider
Comput. Phys. Commun. 180 (2009) 2143

Blümlein, Klein, Schneider, Stan
J. Symbolic Comput. 47 (2012) 1267

Ablinger, Blümlein, Schneider
J. Math. Phys. 52 (2011) 102301; J. Math. Phys. 54 (2013) 082301

Ablinger, Blümlein
arXiv:1304.7071 [Contr. to a Book: Springer, Wien]

Ablinger, Blümlein, Raab, Schneider
J. Math. Phys. 55 (2014) 112301

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Comp. Phys. Commun. 202 (2016) 33