# $\beta$-functions in higher dimensional field theories

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# Overview

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# Background

Part of the process of renormalizing a quantum field theory involves regularization

For analytic computations this is usually dimensional regularization with $d = D - 2\epsilon$ where $D$ is the critical dimension

Process is complete when the renormalization group functions are determined in $d = D$ dimensions

What is less apparent in the procedure is the connectivity different theories, with the same underlying symmetry, have with each other in different dimensions

The connection derives from the critical point renormalization group equation and the Wilson-Fisher (WF) fixed point

This has seen a recent revival in understanding conformal field theories beyond two dimensions; $a$-theorem, (nonperturbative) fixed points, critical exponents, relevant operators, field theories in the same universality class, ultraviolet-infrared duality across dimensions, conformal windows

# Generalities

Fixed points are defined as the zeroes of the $\beta$-function; clearly free field theory is the trivial fixed point

Value of the renormalization group functions at a fixed point correspond to critical exponents - these are renormalization group invariants and in principle can be measured physically

Various methods used to estimate exponents theoretically:

- $\epsilon$ expansion in $d = D - 2\epsilon$ dimensions
- direct evaluation in fixed (odd) dimensions
- strong coupling
- large $N$
- lattice or numerical evaluation

Aim is to extend these ideas from scalar to gauge theories

Simple example is the three dimensional Heisenberg ferromagnet. Its phase transition is described by $O(3)$ symmetric scalar field theory

Resummation of renormalization group functions for five loop four dimensional $\phi^4$ theory in $d = 4 - 2\epsilon$ dimensions are competitive with fixed dimension six loop evaluation

Higher dimensional theory is in the same universality class as the Heisenberg model at the corresponding magnetization phase transition

Work in recent years has extended the universality class to six and higher dimensions

Results complement the recent technique of the conformal bootstrap

Also apparent connection of ultraviolet stable fixed point in higher dimensional theory with infrared stable fixed point in lower dimensional theory [McKane]

Larger picture is that there is an infinite set of operators defining a universal $d$-dimensional theory such that a set of these operators are relevant at the fixed points in the critical dimension $D$

# Toy example - $\phi^4$ theory

WF is the first non-trivial fixed point in $d$-dimensions

Recall simple four dimensional scalar field theory with $O(N)$ symmetry

$$L = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i - \frac{g}{8}\left(\phi^i\phi^i\right)^2$$

It is in the same universality class as the $O(N)$ nonlinear $\sigma$ model at the Wilson-Fisher fixed point

$$L^\sigma = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}\sigma\left(\phi^i\phi^i - \frac{1}{\lambda}\right)$$

At the Lagrangian level $O(N)$ $\phi^4$ theory can be rewritten as

$$L = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}\sigma\phi^i\phi^i - \frac{\sigma^2}{2g}$$

Both Lagrangians have the same basic interaction; $[\phi^i] = d/2 - 1$ and $[\sigma] = 2$

In terms of perturbative renormalizability $L^\sigma$ is only perturbatively renormalizable in strictly two dimensions; beyond two it is perturbatively non-renormalizable

By contrast $\phi^4$ theory is perturbatively renormalizable in four dimensions and superrenormalizable below four

Evaluating the renormalization group functions at the critical point defines the critical exponents

$$\eta ~=~ \gamma_\phi(g_c) ~~,~~ \omega ~=~ \beta'(g_c)$$

which are functions of $d$ (or $\epsilon$) and $N$ and they can be expanded in powers of $\epsilon$ or $1/N$

$\epsilon$ expansion corresponds to perturbation theory; $1/N$ expansion is a reorganization of perturbation theory and exponents have been evaluated to three orders by Vasil'ev et al

Derivation of large $N$ exponents uses analytic regularization and not dimensional regularization - effectively one can study field theories in $d$-dimensions

This method is applicable to the large $N_f$ expansion in non-abelian gauge theories

# Beyond four dimensions

There *are* higher dimensional theories beyond four dimensions in the same universality class as the Heisenberg ferromagnet [Klebanov et al]

Key ingredients are renormalizability and common interaction with connectivity to lower dimensional theories

$$
\begin{aligned}
L^{(6)} &= \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 + \frac{1}{2}\left(\partial_\mu \sigma\right)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^3 \\
L^{(8)} &= \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 + \frac{1}{2}\left(\Box \sigma\right)^2 + \frac{1}{2}g_1\sigma\phi^i\phi^i + \frac{1}{6}g_2\sigma^2\Box\sigma + \frac{1}{24}g_3^2\sigma^4
\end{aligned}
$$

A price to pay is an additional coupling constant or spectator interactions

Renormalization group functions available at three loops [Klebanov et al] based on [McKane et al] as well as at four loops [Gracey] for $L^{(6)}$ and various orders for $L^{(8)}$

At the WF fixed point field anomalous dimensions, mass anomalous dimensions and $\beta$-function exponents all agree with all large $N$ exponents to the order they are available at in $d = 6 - 2\epsilon$ and $d = 8 - 2\epsilon$ dimensions

# Results

For $O(1)$ case the renormalization group functions for $L^{(6)}$ are

$$\begin{aligned}
\beta(g) &= \frac{3}{8}g^3 - \frac{125}{288}g^5 + 5[2592\zeta_3 + 6617]\frac{g^7}{41472} \\
&+ [-4225824\zeta_3 + 349920\zeta_4 + 1244160\zeta_5 - 3404365]\frac{g^9}{1492992} \\
&+ O(g^{11}) \\
\gamma_\phi(g) &= -\frac{1}{12}g^2 + \frac{13}{432}g^4 + [2592\zeta_3 - 5195]\frac{g^6}{62208} \\
&+ [10080\zeta_3 + 18144\zeta_4 - 69120\zeta_5 + 53449]\frac{g^8}{248832} + O(g^{10})
\end{aligned}$$

which are relevant for Lee-Yang edge singularity problem

Critical exponent $\sigma = 0.0747$ from four loop $\epsilon$ expansion; strong coupling expansion gives $\sigma = 0.076(2)$

# Computational technicalities

Six dimensional $\phi^3$ theory can be renormalized purely from 2-point graphs; this is because one can nullify one leg on the 3-point function without producing any infrared infinities

Laporta algorithm reduces computation down to the evaluation of a small set of master integrals which have to be determined by direct methods

Tarasov's method of relating $d$-dimensional integrals to $(d+2)$-dimensional integrals was used to and exploit known four loop 2-point masters of Baikov & Chetyrkin

With the increase in the spacetime dimension higher $n$-point functions have to be evaluated to determine the full set of $\beta$-functions for fixed point analysis

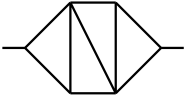
There appears to be no limit to the tower of theories

For $L^{(6)}$ conformal window can be computed

$$N_{cr} \; = \; 1038.266 - 1219.680\epsilon - 1456.693\epsilon^2 + 3621.685\epsilon^3 + O(\epsilon^4)$$

leading to $N_{cr} \approx 400$ in five dimensions

# Example - zigzag graph



Six dimensional 2-point zigzag graph $M_{44}$ produces

$$
\begin{aligned}
M_{44} \quad = \quad & \frac{7}{103680}\frac{1}{\epsilon^3} + \frac{61}{77760}\frac{1}{\epsilon^2} + \frac{32939}{7464960}\frac{1}{\epsilon} + \left[\frac{277411}{17915904} + \frac{1}{1296}\zeta_3\right] \\
& + \left[\frac{19619333}{1074954240} + \frac{1}{864}\zeta_4 + \frac{781}{155520}\zeta_3\right]\epsilon \\
& + \left[-\frac{4976176237}{12899450880} - \frac{147}{64}\zeta_7 + \frac{1999}{1728}\zeta_5 + \frac{781}{103680}\zeta_4 + \frac{113243}{93312}\zeta_3\right]\epsilon^2 \\
& + O(\epsilon^3)
\end{aligned}
$$

$\epsilon$ expansion required beyond the leading terms due to spurious poles from the integration by parts

In four dimensions $M_{44}$ is finite

# Six dimensional QCD

Can analyse non-supersymmetric six dimensional gauge theories such as QCD

Gauge invariant Lagrangian is constructed from all six dimensional operators which are independent; ignore operators which are total derivatives and not independent by Bianchi identity [Kazakov]

$$L^{(6)}_{\mathrm{GI}} = -\frac{1}{4}(D_\mu G^a_{\nu\sigma})(D^\mu G^{a\,\nu\sigma}) + \frac{g_2}{6} f^{abc} G^a_{\mu\nu} G^{b\,\mu\sigma} G^{c\,\nu}{}_{\sigma} + i\bar{\psi}^{il} \not{D} \psi^{il}$$

Only two independent dimension six gluonic operators; could have used $\left(D^\mu G^a_{\mu\sigma}\right)(D_\nu G^{a\,\nu\sigma})$ alternatively to one of the above

Four-fermi operators are dimension 10 in six dimensions

Motivation is to determine the anomalous dimension of the dimension six operator $f^{abc} G^a_{\mu\nu} G^{b\,\mu\sigma} G^{c\,\nu}{}_{\sigma}$ and gauge its effect in four dimensions from the $\epsilon$ expansion

Gauge fix in linear covariant gauge $\partial^\mu A^a_\mu = 0$ with gauge parameter $\alpha$

Gauge fixing term has to be BRST invariant and dimension 6

$$L^{(6)}_{\mathrm{GF}} = - \frac{1}{2\alpha} (\partial_\mu \partial^\nu A^a_\nu)(\partial^\mu \partial^\sigma A^a_\sigma) - \bar{c}^a \Box (\partial^\mu D_\mu c)^a$$

Propagators take a different form

$$\langle A^a_\mu(p) A^b_\nu(-p) \rangle = - \frac{\delta^{ab}}{(p^2)^2} \left[ \eta_{\mu\nu} - (1-\alpha) \frac{p_\mu p_\nu}{p^2} \right]$$
$$\langle c^a(p) \bar{c}^b(-p) \rangle = - \frac{\delta^{ab}}{(p^2)^2}$$

Structurally similar to the *eight* dimensional $O(N)$ scalar theory

Double pole gluon propagator could underly a linearly rising inter-quark confining potential in *four* dimensions but not in six

Schwinger-Dyson solution of Alekseev et al in four dimensional QCD corresponds to infrared effective Lagrangian precisely of the form of $L^{(6)}_{\rm GI}$

$L^{(6)}_{\rm GI}$ is not the most general six dimensional non-abelian gauge theory

Lower dimensional operators can be present some of which are gauge invariant

Extension of Lagrangian from massless to massive appears to have similarities to the infrared structure of *four* dimensional QCD

$$\begin{aligned} L^{(6)}_m &= L^{(6)} + m_1 \bar{\psi}^{il} \psi^{il} ~-~ \frac{1}{4} m_2^2 G^a_{\mu\nu} G^{a\,\mu\nu} ~-~ \frac{1}{2\alpha} m_3^2 (\partial^\mu A^a_\mu)^2 \\ & \quad - ~ m_3^2 \bar{c}^a \left( \partial^\mu D_\mu c \right)^a ~-~ \frac{1}{2} m_4^4 A^a_\mu A^{a\,\mu} ~+~ m_4^4 \alpha \bar{c}^a c^a \end{aligned}$$

Landau gauge propagators

$$\begin{aligned} \langle A^a_\mu(p) A^b_\nu(-p) \rangle \Big|_{\alpha=0} &= -~ \frac{\delta^{ab}}{[(p^2)^2 + m_2^2 p^2 + m_4^4]} \left[ \eta_{\mu\nu} ~-~ \frac{p_\mu p_\nu}{p^2} \right] \\ \langle c^a(p) \bar{c}^b(-p) \rangle \Big|_{\alpha=0} &= -~ \frac{\delta^{ab}}{p^2 [p^2 + m_3^2]} \end{aligned}$$

which are structurally similar to behaviour of those on the four dimensional lattice

# Renormalization

Have renormalized massless six dimensional QCD to two loops in $\overline{\text{MS}}$

Similar to four dimensional renormalization but more integration due to higher pole propagators and extra quintic gluon self-interaction

Included arbitrary linear gauge parameter to ensure that the $\overline{\text{MS}}$ $\beta$-functions are $\alpha$ independent which is a non-trivial check

Another check is that we have verified that the renormalization of $g_1$ from the triple gluon vertex is same as that from quark- and ghost-gluon ones and hence the set-up is checked to allow $g_2$ renormalization to be deduced

One loop mass mixing matrix for $m_i$ has also been constructed as well as quark mass at two loops

Final check is connection with lower dimensional gauge theories at the Wilson-Fisher fixed point via the large $N_f$ expansion

# $\beta$-functions

Two loop $\beta$-functions are computed

$$
\begin{aligned}
\beta_1(g_1,g_2) &= [-249 C_A - 16 N_f T_F] \frac{g_1^3}{120} \\
&+ \Big[ -50682 C_A^2 g_1^3 + 2439 C_A^2 g_1^2 g_2 + 3129 C_A^2 g_1 g_2^2 - 315 C_A^2 g_2^3 \\
&\quad - 1328 C_A N_f T_F g_1^3 - 624 C_A N_f T_F g_1^2 g_2 + 96 C_A N_f T_F g_1 g_2^2 \\
&\quad - 3040 C_F N_f T_F g_1^3 \Big] \frac{g_1^2}{4320} \\
\beta_2(g_1,g_2) &= \Big[ 81 C_A g_1^3 - 552 C_A g_1^2 g_2 + 135 C_A g_1 g_2^2 - 15 C_A g_2^3 + 104 N_f T_F g_1^3 \\
&\quad - 48 N_f T_F g_1^2 g_2 \Big] \frac{1}{120}
\end{aligned}
$$

Quark-gluon coupling constant is asymptotically free for all $N_f$

Dependence on $g_2$ does appear until two loops in $\beta_1(g_1,g_2)$ which is why a two loop computation is required for a non-trivial fixed point analysis as well as to check against large $N_f$ $d$-dimensional critical exponents

# Large $N_f$

Comparison of renormalization group functions at criticality in $d$-dimensions is similar to scalar $O(N)$ theories via known large $N_f$ critical exponents

In the large $N_f$ expansion QCD is equivalent to the two dimensional non-abelian Thirring model (NATM) at the Wilson-Fisher fixed point [Hasenfratz & Hasenfratz]

$$L^{\text{NATM}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i + \frac{\tilde{g}}{2}\left(\bar{\psi}^i T^a \gamma^\mu \psi^i\right)^2$$

Or using a spin-1 auxiliary field $A^a_\mu \propto \bar{\psi}^i T^a \gamma_\mu \psi^i$

$$L^{\text{NATM}} = i\bar{\psi}^i D\!\!\!\!/ \psi^i - \frac{1}{2} A^a_\mu A^{a\,\mu}$$

There are similar properties between these two models as there are between the scalar theories

Early large $N_f$ QCD work extended that of Palanques-Mestre and Tarrach for QED

In $d$-dimensions $2 < d < 4$ the interaction is common and the NATM and QCD are in the same universality class at the Wilson-Fisher fixed point

For large $N_f$ expansion use the NATM but with gauge fixing; the gluonic operators define the dimensionality of the coupling constants

Triple and higher gluon interactions emerge from integrating over closed quark loops [Hasenfratz & Hasenfratz]

Expanding known critical exponents in $d = 4 - 2\epsilon$ dimensions and comparing with known QCD renormalization group functions at two, three, four and *five* loops find precise agreement

Repeating the exercise in $d = 6 - 2\epsilon$ dimensions again find precise agreement with above two loop results

So Wilson-Fisher fixed point formulation is present in gauge theory context and the next theory in the gauge tower is known

# DIS operators in six dimensions

Evidence for the connection of the field theories across the dimensions is not just restricted to basic renormalization group functions

Can evaluate anomalous dimensions of operators such as DIS operators

First few moments of flavour non-singlet twist-2 Wilson operators have been determined to two loops

$$\begin{aligned}
\gamma_{(2)}(g_1,g_2) &= 2C_F g_1^2 \\
&+[26841 C_A g_1^2 - 1200 C_A g_1 g_2 - 600 C_A g_2^2 - 4200 C_F g_1^2 \\
&+1264 g_1^2 N_f T_F]\frac{C_F g_1^2}{1800} \\
\gamma_{(3)}(g_1,g_2) &= \frac{49}{15} C_F g_1^2 \\
&+[186321 C_A g_1^2 - 10950 C_A g_1 g_2 - 4200 C_A g_2^2 - 23564 C_F g_1^2 \\
&+9104 g_1^2 N_f T_F]\frac{7 C_F g_1^2}{54000}
\end{aligned}$$

Evaluating at the WF fixed point the $\epsilon$ expansions agree with those of the corresponding large $N_f$ critical exponent

# Fixed point analysis

Can now focus on the structure of the six dimensional renormalization group functions and search for non-trivial fixed points

In four dimensions QCD $\beta$-function has a non-trivial fixed point called the Banks-Zaks fixed point when $9 \leq N_f \leq 16$ for $SU(3)$

Can repeat the exercise in $d = 6 - 2\epsilon$ dimensions by solving numerically

$$\beta_1(g_1, g_2) ~=~ \beta_2(g_1, g_2) ~=~ 0 ~~,~~ \frac{\partial \beta_1}{\partial g_1} \frac{\partial \beta_2}{\partial g_2} ~-~ \frac{\partial \beta_1}{\partial g_2} \frac{\partial \beta_2}{\partial g_1} ~=~ 0$$

Produces three solutions for critical window; only one is real

$$N_{f\,(A)} ~=~ 2.797566 \frac{C_A}{T_F} ~+~ [2.198165 C_F - 3.432003 C_A] \frac{\epsilon}{T_F}$$

Or

$$N_{f\,(A)} \Big|_{SU(3)} ~=~ 16.785398 ~-~ 14.730246\epsilon$$

So upper bound in six dimensions is *same* as in four dimensions

# Six dimensional QED

To go to higher loop order to explore higher dimensional gauge theory connection can analyse six dimensional QED; only one coupling constant

$$L^{(6)} = -\frac{1}{4}(\partial_\mu F_{\nu\sigma})(\partial^\mu F^{\nu\sigma}) - \frac{1}{2\alpha}(\partial_\mu \partial^\nu A_\nu)(\partial^\mu \partial^\sigma A_\sigma) + i\bar{\psi}^i \not{D}\psi^i$$

Renormalized at three loops in $\overline{\text{MS}}$

$$\begin{aligned}
\gamma_A(g_1,\alpha) &= -\frac{4}{15}N_f g_1^2 - \frac{38}{27}N_f g_1^4 + 17N_f[200 - 111N_f]\frac{g_1^6}{6075} \\
\gamma_\psi(g_1,\alpha) &= [3\alpha + 5]\frac{g_1^2}{6} + 2[32N_f - 125]\frac{g_1^4}{135} \\
&\quad + [2864N_f^2 - 648000\zeta_3 N_f + 730375N_f \\
&\quad\quad + 1944000\zeta_3 - 1033000]\frac{g_1^6}{243000} \\
\beta_1(g_1) &= -\frac{2}{15}N_f g_1^3 - \frac{19}{27}N_f g_1^5 + 17N_f[200 - 111N_f]\frac{g_1^7}{12150}
\end{aligned}$$

Ward-Takahashi identity is $\beta_1(g_1) = \frac{g_1}{2}\gamma_A(g_1,\alpha)$

$\beta$-function is asymptotically free for all $N_f$ [Klebanov et al]

Type of Banks-Zaks fixed point for one electron due to change of sign at three loops

$\alpha$ only appears in $\gamma_\psi(g_1, \alpha)$ at one loop similar to four dimensions

Electron mass dimension is also available in $\overline{\text{MS}}$

$$
\begin{aligned}
\gamma_{\bar{\psi}\psi}(g_1) \quad = \quad & -\frac{5g_1^2}{3} + [-68N_f - 25]\frac{g_1^4}{135} \\
& + \Big[13456N_f^2 + 648000\zeta_3 N_f - 818575N_f \\
& \quad + 1215000\zeta_3 - 726875\Big]\frac{g_1^6}{121500}
\end{aligned}
$$

All renormalization group functions agree with large $N_f$ QED critical exponents to $O(\epsilon^3)$

# Eight dimensional QED

Can repeat analysis at next stage in the QED tower

$$L^{(8)} = -\frac{1}{4}(\partial_\mu\partial_\nu F_{\sigma\rho})(\partial^\mu\partial^\nu F^{\sigma\rho}) - \frac{1}{2\alpha}(\partial_\mu\partial^\nu A_\nu)(\partial^\mu\partial^\sigma A_\sigma)$$
$$+ i\bar{\psi}^i \not{D}\psi^i + \frac{g_2^2}{32} F_{\mu\nu}F^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} + \frac{g_3^2}{8} F_{\mu\nu}F^{\mu\sigma}F_{\nu\rho}F^{\sigma\rho}$$

which has more couplings

$$\gamma_\psi(g_i,\alpha) = [2\alpha + 7]\frac{g_1^2}{12} + [-964N_f - 13475]\frac{g_1^4}{33600}$$

$$\beta_1(g_i) = \frac{N_f g_1^3}{70} + \frac{11N_f g_1^5}{240} = \frac{g_1}{2}\gamma_A(g_i,\alpha)$$

$$\beta_2(g_i) = \left[1120g_1^4 N_f + 72g_1^2 g_2^2 N_f - 861g_2^4 - 1659g_2^2 g_3^2 - 609g_3^4\right]\frac{1}{1260}$$

$$\beta_3(g_i) = \left[-1568g_1^4 N_f + 144g_1^2 g_3^2 N_f - 21g_2^4 - 294g_2^2 g_3^2 - 1029g_3^4\right]\frac{1}{2520}$$

$$\gamma_{\bar{\psi}\psi}(g_i) = -\frac{7g_1^2}{12} + [2052N_f - 1225]\frac{g_1^4}{100800} + O(g_i^6)$$

Similar comments apply in eight dimensions to the six dimensional version

Renormalization group functions agree with large $N_f$ QED critical exponents

Spectator interactions play a key role in this comparison

Again $\alpha$ only appears in $\gamma_\psi(g_1, \alpha)$ at one loop similar to lower dimensions

Main difference is that the coupling constant $g_1$ is not asymptotically free

Under the assumption that there are no cubic purely photon interactions then $D$-dimensional QED is asymptotically free in $D = 2 + 4r$ for integer $r \geq 1$

# Conclusion

- Have reviewed recent work in six and higher dimensional scalar and gauge field theories
- Connection of fixed points in $d$-dimensions with strictly two dimensional conformal field theories gives a foundation for connecting field theories across the dimensions
- Results in gauge theories suggest parallels with scalar field theories in differing dimensions; structure of eight dimensional scalar theory similar to six dimensional QCD
- From the higher dimensional quantum field theory side the next stage is to extend loop computations to higher order to refine the fixed point structure; aim would be to gain a more insight into the operators which drive any infrared fixed points in QCD in the context of the underlying universal theory
- One question is whether there is a deeper connection of the Tarasov construction of relating $d$- and $(d+2)$-dimensional Feynman integrals and the corresponding field theories; is there a way of proceeding more fundamentally via a path integral?