Four-loop corrections to the $\overline{\mathrm{MS}}$-OS quark mass relation and to $(g-2)_\mu$

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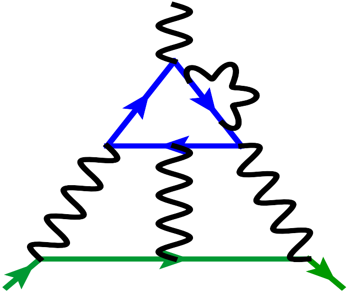
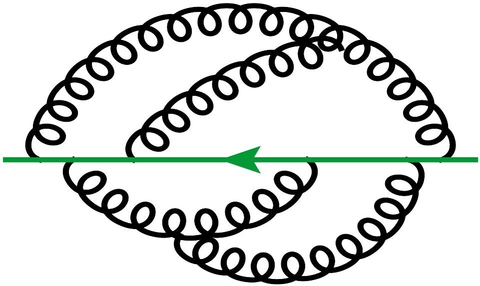
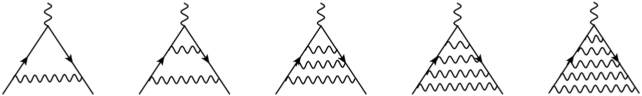




Contributions to muon $a_\mu = (g-2)_\mu/2$

$$a_\mu^{\rm exp} = 116\,592\,089(63) \times 10^{-11}$$
$$a_\mu^{\rm th} = 116\,591\,828(49) \times 10^{-11}$$

- QED: $116\,584\,718.962\,(9)(19)(7)(78) \times 10^{-11}$
  [Schwinger'48; . . . Laporta,Remiddi'96; . . . Aoyama,Hayakawa,Kinoshita,Nio'12]



- hadronic VP: $6949.1(37.2)(21.0) - 984(6)(4) + 12.4(0.1) \times 10^{-11}$
  [Davier,Höcker,Malaescu,Zhang'10; Hagiwara,Liao,Martin,Nomura,Teubner'11; Jegerlehner,Szafron'11; Benayoun,David,DelBuono,Jegerlehner'12; . . . ; Kurz,Marquard,Liu,Steinhauser'14]]
- electroweak $154(2) \times 10^{-11}$ [Czarnecki,Krause,Marciano'96; Knecht,Peris,Perrottet,de Rafale'02; Czarnecki,Marciano,Vainshtein'03; Gnendiger,Stöckinger,Stöckinger-Kim'13]
- hadronic lbl: $116(40) \times 10^{-11}$ [Nyffeler'09; Melnikov,Vainshtein'14; Bijnens,Prades'15; . . .]

- $a_\mu(\rm exp) - a_\mu(\rm SM) \sim 250(90) \times 10^{-11}$
- $a_\mu(\text{4 loop QED}) \sim 381 \times 10^{-11} = (-1.910\ldots + 132.685\ldots|_e)\,\frac{\alpha^4}{\pi^4}$

Quark mass definitions

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} G_{\mu\nu}^2 + \sum_q \bar{\psi}_q \left(\not{D} - m_q \right) \psi_q$$

- OS mass
- $\overline{\rm MS}$ mass
- PS mass [Beneke'98]
- 1S mass [Hoang,Smith,Stelzer,Willenbrock'99]
- RS mass [Pineda'01]
- ...

Quark mass definitions

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$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} G_{\mu\nu}^2 + \sum_q \bar{\psi}_q \left(\not{D} - m_q \right) \psi_q$$

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- RS mass [Pineda'01]
- ...

In this talk:

- heavy quarks: c, b, t
- $\overline{\rm MS}$-OS relation to 4 loops
- precise relation between $m^{\rm PS,1S,RS}$ and $m^{\overline{\rm MS}}$
- How precise can we determine the top pole mass?

Top and bottom quark mass

- top quark mass
  - Tevatron/LHC March 2014: $m_t^{\rm OS} = 173.34 \pm 0.27 \pm 0.71$ GeV
    CMS September 2015: $m_t^{\rm OS} = 172.44 \pm 0.13 \pm 0.47$ GeV
    ➪ convert to $\overline{\rm MS}$ top mass
  - threshold scan at ILC
    ➪ determine in a first step m_t^{PS} or m_t^{1S} or ...
    ➪ convert to $\overline{\rm MS}$ top mass
- bottom quark mass
  Example: m_b from sum rules: $\mathcal{M}_n \equiv \int {\rm d}s \frac{R_b(s)}{s^{(n+1)}}$
  - $\overline{\rm MS}$: m_b from low-moments SRs [Chetyrkin et al.'09, ...]
    $m_b^{\overline{\rm MS}}(m_b) = 4.163 \pm 0.016$ GeV
  - PS mass: m_b from Υ SRs [Penin,Zerf'14; Beneke,Maier,Piclum,Rauh'15; ...]
    $m_b^{\rm PS}(2GeV) = 4.532^{+0.013}_{-0.039}$ GeV
    ➪ convert to $\overline{\rm MS}$ bottom mass

Example for threshold mass: PS mass

1. defined via relation to pole mass

[Beneke'98]

$$m^{\mathrm{PS}}(\mu_f) = m^{\mathrm{OS}} - \delta m(\mu_f)$$

$$\delta m(\mu_f) = -\frac{1}{2}\int_{|\vec{q}\,|<\mu_f} \frac{\mathrm{d}^3 q}{(2\pi)^3} V(\vec{q}\,)$$

$$= \mu_f \frac{C_F \alpha_s}{\pi} \left\{ 1 + \frac{\alpha_s}{4\pi} a_1 + \left(\frac{\alpha_s}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s}{4\pi}\right)^3 a_3 \right\}$$

$V(\vec{q}\,)$: static potential
known to 3 loops
[Smirnov,Smirnov,Steinhauser'09; Anzai,Kiyo,Sumino'09]

➪ $m^{\mathrm{PS}} - m^{\mathrm{OS}}$ relation known to N^{3}LO (top: $\mu_f = 80$ GeV)

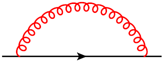
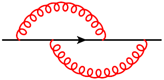
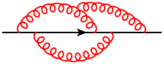
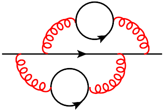
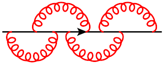
2. use $m^{\mathrm{OS}} - m^{\overline{\mathrm{MS}}}$ relation

$$m^{\mathrm{OS}} = m^{\overline{\mathrm{MS}}} \left(1 + \frac{\alpha_s}{\pi} c_m^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 c_m^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 c_m^{(3)} + \left(\frac{\alpha_s}{\pi}\right)^4 c_m^{(4)} + \ldots \right)$$

needed to 4 loops

$Z_m^{\rm OS}$: known results

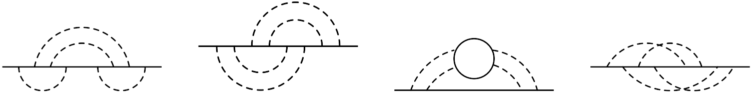
$$m^{\rm bare} = Z_m^{\rm OS} m^{\rm OS} = Z_m^{\overline{\rm MS}} m^{\overline{\rm MS}}$$

-  [Tarrach'81]
-  [Gray,Broadhurst,Grafe,Schilcher'90]
-  [Chetyrkin,Steinhauser'99; Melnikov, v. Ritbergen'00; Marquard,Mihaila,Piclum,Steinhauser'07]
- n_l^2  [Lee,Marquard,Smirnov,Smirnov,Steinhauser'13]
- full  [Marquard,Smirnov,Smirnov,Steinhauser'15]

- $Z_m^{\overline{\rm MS}}$ known to 4 loops [Chetyrkin'97; Larin,van Ritbergen,Vermaseren'97] (5 loops: [Baikov,Chetyrkin,Kühn'14])
- electroweak corrections [Hempfling,Kniehl'94; Jegerlehner,Kalmykov'03; Faisst,Kühn,Veretin'04; Martin'05; Eiras,Steinhauser'05; Kniehl,Pikelner,Veretin'15; Martin'16]

Some technical details

- generation of amplitudes: `qgraf` [Nogueira'91]
  manipulation/transformation to `FORM`: `q2e/exp`
  [Harlander,Seidelsticker,Steinhauser'97;Seidelsticker'97]
- map to $\sim$ 100 families (topologies)

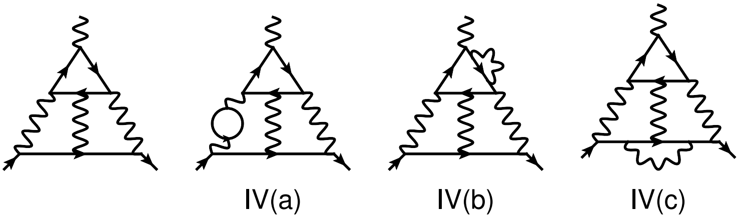


- reduce to master integrals (MIs): `FIRE5` [Smirnov'14] and `crusher` [Marquard,Seidel]
- minimize MIs over all topologies: `tsort` [Pak'11]
  ➪ 386 4-loop OS MIs
- compute MIs (analytic, numerical with `FIESTA` [Smirnov'14'15], Mellin Barnes)

Note: $m^{\mathrm{OS}} - m^{\overline{\mathrm{MS}}}$ relation in terms of MIs known analytically
➪ systematic improvement possible

Additional complication for $(g-2)_\mu$

"light-by-light"-type



- $m_e = 0$ not possible ➪ apply methods of region [Beneke,Smirnov'98; Smirnov]

Additional complication for $(g-2)_\mu$

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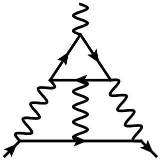
“light-by-light”-type



- $m_e = 0$ not possible ➪ apply methods of region [Beneke,Smirnov’98; Smirnov]
- example: 3-loop diagram
  loop momenta $k, l, r \sim m_\mu$ (“hard”) or $k, l, r \sim m_e$ (“soft”)

  e.g.: k soft, $p^2 = m_\mu^2$ ➪ $\frac{1}{m_\mu^2 - (p-k)^2} = \frac{1}{2pk - k^2} \to \frac{1}{2pk}$

➪

3-loop OS integrals	($3\times$ hard)
3-loop “linear” integrals	($3\times$ soft)
2-loop OS $\times$ 1-loop tadpole	($2\times$ hard, $1\times$ soft)
1-loop OS $\times$ 2-loop tadpole	($1\times$ hard, $2\times$ soft)
1-loop OS $\times$ 2-loop “linear”	($1\times$ hard, $2\times$ soft)

Additional complication for $(g-2)_\mu$

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"light-by-light"-type

- asymptotic expansion for $x \equiv m_e/m_\mu \ll 1$

3 loops

$$
\begin{aligned}
a_\mu = \; & a_\mu^{(0,0)} + a_\mu^{(0,1)} \ln \frac{m_\mu}{m_e} \\
& + \frac{m_e}{m_\mu} \left(a_\mu^{(1,0)} + a_\mu^{(1,1)} \ln \frac{m_\mu}{m_e} \right) \\
& + \frac{m_e^2}{m_\mu^2} \left(a_\mu^{(2,0)} + a_\mu^{(2,1)} \ln \frac{m_\mu}{m_e} + a_\mu^{(2,2)} \ln^2 \frac{m_\mu}{m_e} \right. \\
& \left. \quad + a_\mu^{(2,3)} \ln^3 \frac{m_\mu}{m_e} \right) \\
& + \ldots
\end{aligned}
$$

$$\ln \frac{m_\mu}{m_e} \approx 5.332$$

$$\frac{m_e}{m_\mu} \approx 0.005 \qquad \frac{m_e}{m_\mu} \ln \frac{m_\mu}{m_e} \approx 0.026 \qquad \frac{m_e}{m_\mu} \ln^2 \frac{m_\mu}{m_e} \approx 0.137$$

$$\frac{m_e^2}{m_\mu^2} \ln^3 \frac{m_\mu}{m_e} \approx 0.004 \qquad \frac{m_e^2}{m_\mu^2} \ln^4 \frac{m_\mu}{m_e} \approx 0.019$$

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& + \frac{m_e}{m_\mu}\left(a_\mu^{(1,0)} + a_\mu^{(1,1)}\ln\frac{m_\mu}{m_e}\right) \\
& + \frac{m_e^2}{m_\mu^2}\left(a_\mu^{(2,0)} + a_\mu^{(2,1)}\ln\frac{m_\mu}{m_e} + a_\mu^{(2,2)}\ln^2\frac{m_\mu}{m_e} + a_\mu^{(2,3)}\ln^3\frac{m_\mu}{m_e}\right) \\
& + \ldots \\
=\; & (20.52801865 + 0.42645778 + 0.00655695 + \ldots)\left(\frac{\alpha}{\pi}\right)^3
\end{aligned}
$$

$\ln\frac{m_\mu}{m_e} \approx 5.332$

$\frac{m_e}{m_\mu} \approx 0.005$ $\quad$ $\frac{m_e}{m_\mu}\ln\frac{m_\mu}{m_e} \approx 0.026$ $\quad$ $\frac{m_e}{m_\mu}\ln^2\frac{m_\mu}{m_e} \approx 0.137$

$\frac{m_e^2}{m_\mu^2}\ln^3\frac{m_\mu}{m_e} \approx 0.004$ $\quad$ $\frac{m_e^2}{m_\mu^2}\ln^4\frac{m_\mu}{m_e} \approx 0.019$

Additional complication for $(g-2)_\mu$

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- asymptotic expansion for $x \equiv m_e/m_\mu \ll 1$

4 loops

$$
\begin{aligned}
a_\mu = \; & a_\mu^{(0,0)} + a_\mu^{(0,1)} \ln\frac{m_\mu}{m_e} + a_\mu^{(0,2)} \ln^2\frac{m_\mu}{m_e} \\
& + \frac{m_e}{m_\mu}\left(a_\mu^{(1,0)} + a_\mu^{(1,1)} \ln\frac{m_\mu}{m_e} + a_\mu^{(1,2)} \ln^2\frac{m_\mu}{m_e} \right) \\
& + \frac{m_e^2}{m_\mu^2}\left(a_\mu^{(2,0)} + a_\mu^{(2,1)} \ln\frac{m_\mu}{m_e} + a_\mu^{(2,2)} \ln^2\frac{m_\mu}{m_e} \right. \\
& \qquad \left. + a_\mu^{(2,3)} \ln^3\frac{m_\mu}{m_e} + a_\mu^{(2,4)} \ln^4\frac{m_\mu}{m_e} \right) \\
& + \ldots
\end{aligned}
$$

$$\ln\frac{m_\mu}{m_e} \approx 5.332$$

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Techniques

- generation of amplitudes with `qgraf` [Nogueira'91]
- traces, projectors, … `FORM` [Vermaseren,…]
- asymptotic expansion: `asy` [Pak,Smirnov'10] and in-house program
  basic idea: alpha parametrization
- all 3-loop integrals are known analytically
  ➪ all 4-loop CTs known analytically
- each individual Feynman amplitude:
  ➪ analytic linear combination of master integrals
- 4-loop on-shell integrals:
  $\approx 70 = 40_{\text{ana/high prec.}} + 30_{\text{num}}$ [Marquard,Smirnov,Smirnov,Steinhauser'15]
- 4-loop "linear" integrals:
  $\approx 70 = 20_{\text{ana/high prec.}} + 50_{\text{num}}$
- numerical integration with `FIESTA` [Smirnov'13]

Results

4-loop coefficient

$$m^{\overline{\rm MS}} = m^{\rm OS}\left(1 + \frac{\alpha_s}{\pi} z_m^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 z_m^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 z_m^{(3)} + \left(\frac{\alpha_s}{\pi}\right)^4 z_m^{(4)} + \ldots\right)$$

[Marquard,Smirnov,Smirnov,Steinhauser'15]

charm

$$z_m^{(4)}\Big|_{n_l=3} = -1744.8 \pm 21.5 - 703.48\, l_{\rm OS} - 122.97\, l_{\rm OS}^2 - 14.234\, l_{\rm OS}^3 - 0.75043\, l_{\rm OS}^4$$

bottom

$$z_m^{(4)}\Big|_{n_l=4} = -1267.0 \pm 21.5 - 500.23\, l_{\rm OS} - 83.390\, l_{\rm OS}^2 - 9.9563\, l_{\rm OS}^3 - 0.514033\, l_{\rm OS}^4$$

top

$$z_m^{(4)}\Big|_{n_l=5} = -859.96 \pm 21.5 - 328.94\, l_{\rm OS} - 50.856\, l_{\rm OS}^2 - 6.4922\, l_{\rm OS}^3 - 0.33203\, l_{\rm OS}^4$$

$$l_{\rm OS} = \ln(\mu^2/M^2)$$

- $\ln(\mu^2/M^2)$ known from RGE
- constant term: uncertainty $< 3\%$

$\overline{\mathrm{MS}}$−OS relation for top and bottom

$$
\begin{aligned}
m_t^{\mathrm{OS}} &= m_t^{\overline{\mathrm{MS}}} \left[1 + 0.4244\,\alpha_s + 0.8345\,\alpha_s^2 + 2.375\,\alpha_s^3 + (8.49 \pm 0.25)\,\alpha_s^4\right] \\
&= 163.643 + 7.557 + 1.617 + 0.501 + (0.195 \pm 0.005)\ \mathrm{GeV}
\end{aligned}
$$

$\overline{\mathrm{MS}}$−OS relation for top and bottom

$$m_t^{\mathrm{OS}} = m_t^{\overline{\mathrm{MS}}}\left[1 + 0.4244\,\alpha_s + 0.8345\,\alpha_s^2 + 2.375\,\alpha_s^3 + (8.49 \pm 0.25)\,\alpha_s^4\right]$$
$$= 163.643 + 7.557 + 1.617 + 0.501 + (0.195 \pm 0.005)\ \mathrm{GeV}$$

$$m_b^{\mathrm{OS}} = m_b^{\overline{\mathrm{MS}}}\left[1 + 0.4244\,\alpha_s + 0.9401\,\alpha_s^2 + 3.045\,\alpha_s^3 + (12.57 \pm 0.38)\,\alpha_s^4\right]$$
$$= 4.163 + 0.401 + 0.201 + 0.148 + (0.138 \pm 0.004)\ \mathrm{GeV}$$

$$m_c^{\mathrm{OS}} = m_c^{\overline{\mathrm{MS}}}(3\ \mathrm{GeV})$$
$$\times\left(1 + 1.133\,\alpha_s + 3.119\,\alpha_s^2 + 10.98\,\alpha_s^3 + (51.29 \pm 0.52)\,\alpha_s^4\right)$$
$$= 0.986 + 0.286 + 0.202 + 0.182 + (0.217 \pm 0.002)\ \mathrm{GeV}$$

- top: good/reasonable convergence
- bottom, charm: no convergence

top quark: renormalon ambiguity

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$$m_t^{\mathrm{OS}} = m_t^{\overline{\mathrm{MS}}} \left(1 + \sum_{k \geq 1} r_{k-1} \alpha_s^k \right)$$

$$r_k \stackrel{k\to\infty}{\longrightarrow} N\,(2\beta_0)^k \Gamma(k+1+b) \left(1 + \frac{s_1}{k+b} + \frac{s_2}{(k+b)(k+b+1)} + \ldots \right)$$

$b = \beta_1/(2\beta_0)$

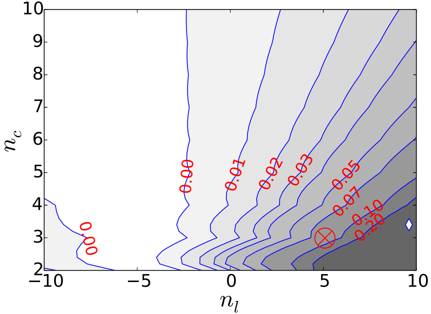
[Beneke,Braun'94; Beneke'95]

1. determine N from exact (3- and) 4-loop coefficient

top quark: renormalon ambiguity

$$m^{\mathrm{OS}} = m^{\overline{\mathrm{MS}}}\left(1+\sum r_k \alpha^k\right)$$

$\Delta_{34}(n_c, n_l)$: consistency of determination of N from 3 and 4 loops



top: $n_c = 3, n_l = 5$

PRELIMINARY

[Nason'15; Beneke,Marquard,Nason,Steinhauser'16]

top quark: renormalon ambiguity

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$$r_k \stackrel{k\to\infty}{\longrightarrow} N\,(2\beta_0)^k\Gamma(k+1+b)\left(1+\frac{s_1}{k+b}+\frac{s_2}{(k+b)(k+b+1)}+\ldots\right)$$

1. determine N from exact (3- and) 4-loop coefficient
2. consider $m_t^{\rm OS}(n) = m_t^{\overline{\rm MS}}\left(1+\sum_{k=1}^{n} r_{k-1}\alpha_s^k\right)$
   and determine n such that $m_t^{\rm OS}(n+1) - m_t^{\rm OS}(n)$ is minimal
3. $$m_t^{\rm OS} = m_t^{\overline{\rm MS}}\left(1+\sum_{k=1}^{4} r_{k-1}\alpha_s^k\right) + \delta^{(5+)}m_t^{\rm OS}$$

$$\delta^{(5+)}m_t^{\rm OS} = 0.2xx \pm (10-20)\%(N) \pm 0.07(\text{last term})\ \text{GeV}$$

➪ final uncertainty about (below?) 100 MeV!

PRELIMINARY

[Nason'15; Beneke,Marquard,Nason,Steinhauser'16]

$\overline{\mathrm{MS}}-$ threshold bottom mass relation

	$m_b^{\overline{\mathrm{MS}}}(m_b^{\overline{\mathrm{MS}}})$			
input	$m^{\mathrm{PS}} =$	$m^{\mathrm{1S}} =$	$m^{\mathrm{RS}} =$	
#loops	4.483	4.670	4.365	
1	4.266	4.308	4.210	
2	4.191	4.190	4.172	$\lesssim$ 110 MeV

$\overline{\mathrm{MS}}-$ threshold bottom mass relation

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1	4.266	4.308	4.210	
2	4.191	4.190	4.172	$\lesssim$ 110 MeV
3	4.161	4.154	4.158	$\lesssim$ 40 MeV

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2	4.191	4.190	4.172	$\lesssim$ 110 MeV
3	4.161	4.154	4.158	$\lesssim$ 40 MeV
4	4.163	4.163	4.163	$\lesssim$ 9 MeV

$\overline{\rm MS}-$ threshold bottom mass relation

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	$m_b^{\overline{\rm MS}}(m_b^{\overline{\rm MS}})$			
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#loops	4.483	4.670	4.365	
1	4.266	4.308	4.210	
2	4.191	4.190	4.172	$\lesssim$ 110 MeV
3	4.161	4.154	4.158	$\lesssim$ 40 MeV
4	4.163	4.163	4.163	$\lesssim$ 9 MeV
4 ($\times$1.03)	4.159	4.159	4.159	

- 3 loops: $\approx$ 40 MeV
- 4 loops: $\{2, 9, 5\}$ MeV
- 3% uncertainty: 4 MeV

$\Rightarrow$ $\{4, 6, 5\}$ MeV uncertainty in the $m_b^{\overline{\rm MS}} - m_b^{\rm thr}$ relation

- N[3]LO extractions: $\delta m_b \approx 10 - 20$ MeV
- "OS-$\overline{\rm MS}$" – "PS-OS"

$$m_b^{\rm PS} = 4.163 + (0.401 - 0.192) + (0.201 - 0.121) + (0.148 - 0.115) + (0.138 - 0.140)$$

$\overline{\text{MS}}-$ threshold top mass relation

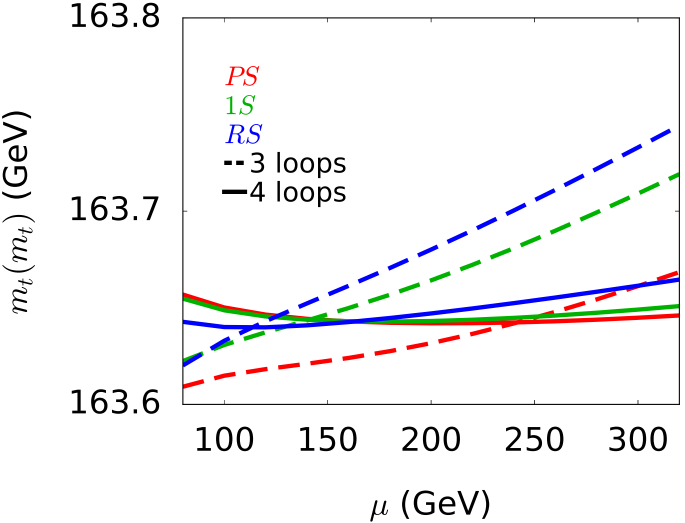
input	$m^{\text{PS}} =$	$m^{1\text{S}} =$	$m^{\text{RS}} =$	
#loops	168.204	172.227	171.215	
	$m_t^{\overline{\text{MS}}}(m_t^{\overline{\text{MS}}})$			
1	164.311	165.045	164.847	
2	163.713	163.861	163.853	1-2 GeV
3	163.625	163.651	163.663	$\lesssim$ 200 MeV
4	163.643	163.643	163.643	$\lesssim$ 20 MeV
4 ($\times$1.03)	163.637	163.637	163.637	

- 3 loops: $\lesssim$ 200 MeV
- 4 loops: $\{18, 8, 20\}$ MeV
- 3% uncertainty $\widehat{=}$ 6 MeV
- $\delta m_t^{\text{ILC}} \lesssim$ 50 MeV

$\Rightarrow$ $\{9, 7, 11\}$ MeV uncertainty in the $m_t^{\overline{\text{MS}}} - m_t^{\text{thr}}$ relation

$\overline{\text{MS}}-$ threshold top mass relation

$$m^{\text{thr}} \longrightarrow m_t^{\overline{\text{MS}}}(\mu) \longrightarrow m_t^{\overline{\text{MS}}}(m_t^{\overline{\text{MS}}})$$



$(g-2)_\mu$: results

Karlsruhe Institute of Technology

$A_2^{(8)}(m_\mu/$	[Kurz et al]	literature	
I(a0)	7.223076	7.223077 ± 0.000029	[Kinoshita et al.'04]
		7.223076	[Laporta'93]
I(a1)	0.494072	0.494075 ± 0.000006	[Kinoshita et al.'04]
		0.494072	[Laporta'93]
I(a2)	0.027988	0.027988 ± 0.000001	[Kinoshita et al.'04]
		0.027988	[Laporta'93]
I(a)	7.745136	7.74547 ± 0.00042	[Aoyama et al.'12]
I(bc0)	8.56876 ± 0.00001	8.56874 ± 0.00005	[Kinoshita et al.'04]
I(bc1)	0.1411 ± 0.0060	0.141184 ± 0.000003	[Kinoshita et al.'04]
I(bc2)	0.4956 ± 0.0004	0.49565 ± 0.00001	[Kinoshita et al.'04]
I(bc)	9.2054 ± 0.0060	9.20632 ± 0.00071	[Aoyama et al.'12]
I(d)	-0.2303 ± 0.0024	-0.22982 ± 0.00037	[Aoyama et al.'12]
		-0.230362 ± 0.000005	[Baikov et al.'95]
II(a)	-2.77885	-2.77888 ± 0.00038	[Aoyama et al.'12]
		-2.77885	[Laporta'93]
II(bc0)	-12.212631	-12.21247 ± 0.00045	[Kinoshita et al.'04]
II(bc1)	-1.683165 ± 0.000013	-1.68319 ± 0.00014	[Kinoshita et al.'04]
II(bc)	-13.895796 ± 0.000013	-13.89457 ± 0.00088	[Aoyama et al.'12]
III	10.800 ± 0.022	10.7934 ± 0.0027	[Aoyama et al.'12]
IV(a0)	116.76 ± 0.02	116.759183 ± 0.000292	[Kinoshita et al.'04]
		111.1 ± 8.1	[Calmet et al.'75]
		117.4 ± 0.5	[Chlouber et al'77]
IV(a1)	2.69 ± 0.14	2.697443 ± 0.000142	[Kinoshita et al.'04]
IV(a2)	4.33 ± 0.17	4.328885 ± 0.000293	[Kinoshita et al.'04]
IV(a)	123.78 ± 0.22	123.78551 ± 0.00044	[Aoyama et al.'12]
IV(b)	-0.38 ± 0.08	-0.4170 ± 0.0037	[Aoyama et al.'12]
IV(c)	2.94 ± 0.30	2.9072 ± 0.0044	[Aoyama et al.'12]
IV(d)	-4.32 ± 0.30	-4.43243 ± 0.00058	[Aoyama et al.'12]

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		0.230362 ± 0.000005	[Baikov et al.'05]
II(a)	− 2.7788		
II(bc0)	−12.2126		
II(bc1)	− 1.6831		
II(bc)	−13.8957		
III	10.800 ±		
IV(a0)	116.76 ±		
IV(a1)	2.69 ±		
IV(a2)	4.33 ± 0.17	4.328885 ± 0.000293	[Kinoshita et al.'04]
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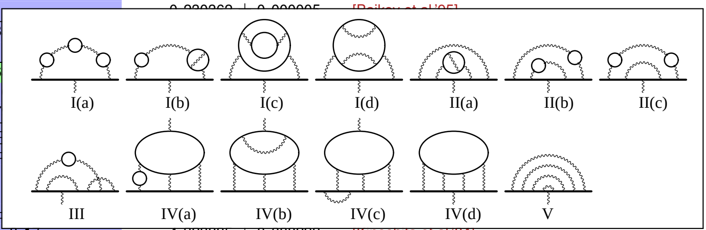

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II(a)			
II(bc			
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III			
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IV(b			
IV(c			
IV(d)	− 4.32 ± 0.30	− 4.43243 ± 0.00058	[Aoyama et al.'12]

			e^-	τ	$e^- + \tau$	
$a_\mu^{(8)}$	$=$	$a_\mu^{(8)}\vert_{\rm univ.}$	$+132.86(48)$	$+0.0424941(53)$	$+0.062722(10)$	[Kurz et al.]
$a_\mu^{(8)}$	$=$	$-1.9106(20)$	$+132.6852(60)$	$+0.04234(12)$	$+0.06272(4)$	[Aoyama et al.12]

final uncertainty

$0.48 \times (\alpha/\pi)^4 \approx 1.4 \times 10^{-11}$

much smaller than uncertainty of

$a_\mu(\mathrm{exp}) - a_\mu(\mathrm{SM}) \sim 250(90) \times 10^{-11}$

even after a projected improvement by a factor 4

Conclusions

- 4-loop $\overline{\rm MS}$-OS relation for heavy quarks
- 4-loop contribution to pole mass: 200 MeV
- Precise $m^{\overline{\rm MS}}$-$m^{\rm thr}$ relations to N^{3}LO
- Renormalon analysis for top quark
  ➪ uncertainty (probably) below 100 MeV
- all fermionic 4-loop contributions to $(g-2)_\mu$

[Aoyama,Hayakawa,Kinoshita,Nio'12]

group	$A_2^{(8)}(m_\mu/m_e)$	$A_2^{(8)}(m_\mu/m_\tau)$	$A_3^{(8)}$
I(a)	7.74547 (42)	0.000032 (0)	0.003209 (0)
I(b)	7.58201 (71)	0.000252 (0)	0.002611 (0)
I(c)	1.624307 (40)	0.000737 (0)	0.001807 (0)
I(d)	-0.22982 (37)	0.000368 (0)	0
II(a)	-2.77888 (38)	-0.007329 (1)	0
II(b)	-4.55277 (30)	-0.002036 (0)	-0.009008 (1)
II(c)	-9.34180 (83)	-0.005246 (1)	-0.019642 (2)
III	10.7934 (27)	0.04504 (14)	0
IV(a)	123.78551 (44)	0.038513 (11)	0.083739 (36)
IV(b)	-0.4170 (37)	0.006106 (31)	0
IV(c)	2.9072 (44)	-0.01823 (11)	0
IV(d)	-4.43243 (58)	-0.015868 (37)	0