The coaction on ϕ^4 periods and the six-loop beta function

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LLL (Loops & Legs & Leipzig)

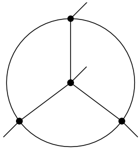
Outline

1. primitive ϕ^4 periods [with O. Schnetz]
2. ϕ^4 beta function at 6 loops [with M. Kompaniets]
   independently by O. Schnetz
3. A coaction on ϕ^4 periods?

primitive ϕ^4 periods

A graph G is called

- ϕ^4 if all vertices have degree ≤ 4
- log.-div. if edges(G) $=$ 2loops(G)
- primitive if edges(γ) $>$ 2loops(γ) for all $\emptyset \neq \gamma \subsetneq G$

$\mathrm{WS}_3 =$ 

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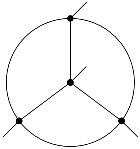
$\mathrm{WS}_3 =$ 

Definition

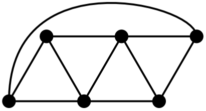
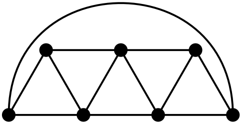
The period of a primitive log.-div. graph G is its residue at dim $\to 4$:

$$\Phi(G) = \frac{\mathcal{P}(G)}{\text{loops}(G)\varepsilon} + \mathcal{O}\left(\varepsilon^0\right) \qquad \mathcal{P}(\mathrm{WS}_3) = 6\zeta_3$$

- Their contributions to the β-function are scheme independent.
- many relations (product, completion, twist, Fourier) $\Rightarrow$ few objects
- most complicated periods (graphs with subdivergences should have "simpler" periods), but no functional dependence

History

- 7 loops (3 periods missing) [Broadhurst & Kreimer '95]
- 8 loops: census (not complete) [Schnetz '10]
- proof of the zig-zag conjecture [Brown & Schnetz '12]

$\mathrm{ZZ}_5 =$  $\mathrm{ZZ}_6 =$ 

Theorem

For every integer $n \geq 3$, the period of the zig-zag graph ZZ_n is given by

$$P(\mathrm{ZZ}_n) = 4\frac{(2n-2)!}{n!(n-1)!}\left(1 - \frac{1-(-1)^n}{2^{2n-3}}\right)\zeta_{2n-3}.$$

This remains the only infinite family of ϕ^4 periods known to all orders.

New results for ϕ^4

Without counting products, we have:

loops	3	4	5	6	7	8	9	10	11
completions	1	1	1	4	11	41	190	1182	8687
# periods	1	1	1	4	9	≤ 31	≤ 134	≤ 846	≤ 6300
known	1	1	1	4	9	23	47	88	125

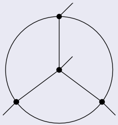
- all 7-loop periods known [Panzer & Schnetz LL2014]
  all are MZV, with one exception: $P_{7,11}$ (MDV)
- from 8 loops, there are non-mixed Tate periods [Brown, Doryn, Schnetz]
- our methods are only applicable to some (not all) of the periods expressible with MPL
- intriguing compatibility with the motivic coaction
- also: results for all primitive log.-div. non-ϕ^4 periods

Results

All our results are listed in the ancillary files `Periods` and `PeriodsNonPhi4` of arXiv:1603.04289, grouped via their completions:

Definition

The completed graph $\overline{G}$ of G is the 4-regular graph obtained by adding a vertex v to G and connecting it to all external legs:


$= K_4 \mapsto K_5$

Theorem

$$\overline{G_1} \cong \overline{G_2} \Rightarrow \mathcal{P}(G_1) = \mathcal{P}(G_2)$$

$$\text{Period}[3,1] = \Big[[\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}],$$
$$6f_3, 6\zeta_3, 7.2123\ldots, -1, p[3,1], 120\Big]$$

Method 1: Parametric integration

For linearly reducible graphs G, the α-representation of $\mathcal{P}(G)$ can be integrated [HyperInt] (open source):

$$\mathcal{P}(G) = \int_0^\infty \mathrm{d}\alpha_1 \cdots \int_0^\infty \mathrm{d}\alpha_{N-1} \frac{1}{\psi_G^2|_{\alpha_N=1}}$$

where Kirchhoff/graph/first Symanzik polynomial is

$$\psi = \mathcal{U} = \sum_{T \text{ spanning tree}} \prod_{e \notin T} \alpha_e.$$

For $P_{7,11}$ need to change variables; result is MPL at sixth roots of unity. No datamine [Blümlein, Broadhurst, Vermaseren] was available.

- exploit generalized parity theorem [Panzer 15']
- ϕ^4 seems to touch only a very special subset: multiple Deligne values (MDV) [Broadhurst 15'] (datamine for weight ≤ 11 MDV)
- datamine for general sixth roots up to weight 6 [Henn, Smirnov^2 '15]

Method 2: Graphical functions

For a graph G with 3 external vertices $0, 1, z$, the position space integral

$$f_G(x) = \left(\prod_{v \in V^{\text{int}}} \frac{\mathrm{d}^D x_v}{\pi^{D/2}} \right) \frac{1}{\prod_e \|x_{e_1} - x_{e_2}\|^2}$$

is a single-valued real analytic function on $\mathbb{C} \setminus \{0, 1\}$.

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- in some cases, f_G is expressible as single-valued MPL
- many more cases require generalized single-valued hyperlogarithms [Schnetz]
- parametric integration of small graphical functions [with M. Golz '15]
- appending edges and vertices via single-valued integration to construct bigger graphical functions [Schnetz '13] (very efficient)
- decomposition algorithm to reduce periods to a basis (f-alphabet) [Brown '11]

beta function of $O(N)$ symmetric ϕ^4 at 6 loops

Some history

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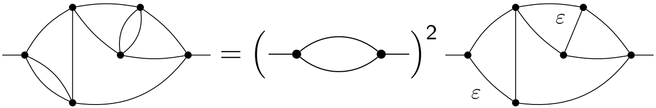
- primitives up to 7 loops: census [Schnetz '10]

Traditional methods

- infrared rearrangement: reduction to p-integrals with $L-1$ loops
- R^* [talk by Chetyrkin]
- factorization of 1-scale subgraphs



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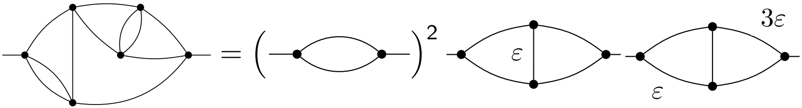

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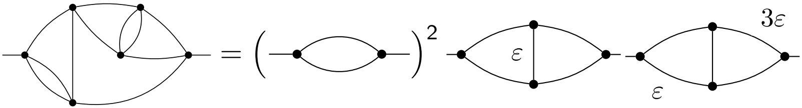

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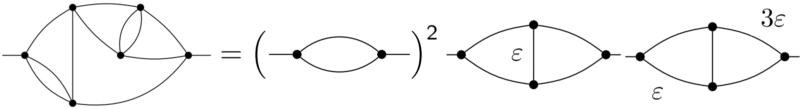

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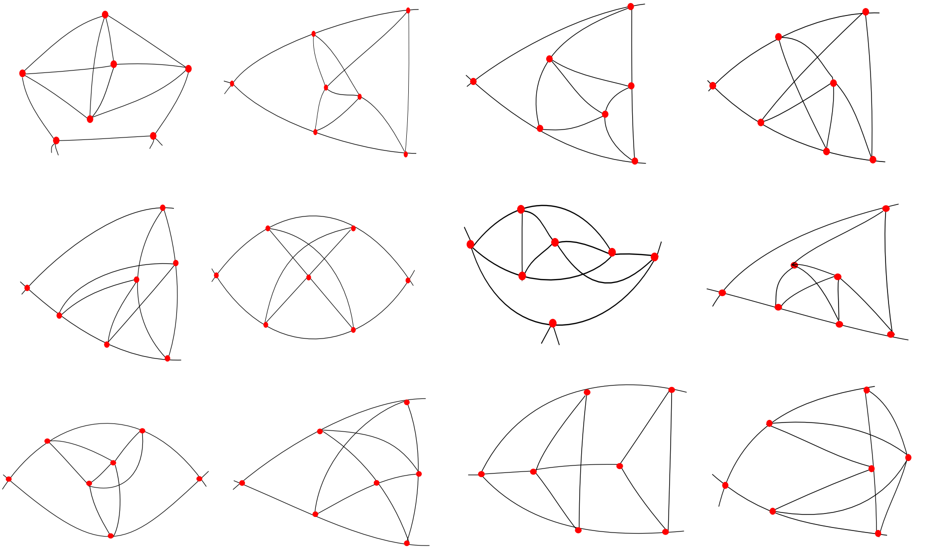
Fail at 6 loops when irreducible 5-loop p-integrals remain:

- field anomalous dimension: 2 irreducible integrals
  solved [Batkovich, Kompaniets, Chetyrkin '16]
- beta function: 12 irreducible integrals

irreducible (not 4-loop reducible) 6-loop ϕ^4 integrals



New approaches

- integration of α/Schwinger/Feynman parameters [Brown '08, Panzer '15]
  (for integrals without subdivergences)
- master integrals without subdivergences [von Manteuffel, Panzer, Schabinger '15] (not used here)
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ϕ^4 theory has only very few graphs:

# loops	1	2	3	4	5	6
# 4-point graphs	1	2	8	26	124	627

We do not use any IBP reductions and compute all Feynman integrals.

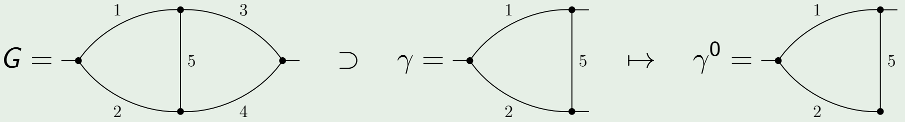
one-scale renormalization scheme

BPHZ renormalization of log. UV subdivergences via forest formula:

$$\Phi_R(G) = \sum_{F \in \mathcal{F}(G)} (-1)^F \prod_{\gamma \in G} \Phi^0(\gamma)\Phi(G/\gamma)$$

Idea [Brown & Kreimer '13]: Choose $\Phi^0(\gamma) := \Phi(\gamma^0)\big|_{p^2=1}$ to be 1-scale!

Example $(D = 4, \nu_5 = \varepsilon)$



- $\Phi_R(G)$ is a convergent integral at $\varepsilon = 0$ ($\Rightarrow$ `HyperInt`)
- except for the wanted $\Phi(G)$, all terms on the RHS are products of lower-loop p-integrals
- easy to implement

Automatization

Pipeline:

- compute the forest formula and assign infrared-safe one-scale structures $\Phi^0(\gamma)$ ($\Rightarrow$ Python)
- integrate the (convergent) $\partial_{p^2}\Phi_R(G)$ ($\Rightarrow$ `HyperInt`)
- solve the forest formula for $\Phi(G)$, using products of lower-loop integrals ($\Rightarrow$ Python)

The 5-loop calculation runs in about a day on a single thread; factor < 100 for 6 loops.

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- analytic calculation without R^*
- only a single number is put in by hand: $P_{6,4} = K_{4,4}$ (not linearly reducible)
- strong checks:
  - Schnetz' calculation
  - traditional approach $+$ 12 irreducible integrals computed with `HyperInt` via primitive linear combinations
  - sector decomposition [Kompaniets]

Result ($N = 1$), $D = 4 - 2\varepsilon$

$$
\begin{aligned}
\beta^{\overline{\mathrm{MS}}}(g) = & -2\varepsilon g + 3g^2 - \tfrac{17}{3}g^3 + \left(\tfrac{145}{8} + 12\zeta_3\right) g^4 \\
& - \left(120\zeta_5 - 18\zeta_4 + 78\zeta_3 + \tfrac{3499}{48}\right) g^5 \\
& + \left(1323\zeta_7 + 45\zeta_3^2 - \tfrac{675}{2}\zeta_6 + 987\zeta_5 - \tfrac{1189}{8}\zeta_4 + \tfrac{7965}{16}\zeta_3 + \tfrac{764621}{2304}\right) g^6 \\
& - \left(\tfrac{46112}{3}\zeta_9 + 768\zeta_3^3 + \tfrac{51984}{25}\zeta_{3,5} - \tfrac{264543}{25}\zeta_8 + 4704\zeta_3\zeta_5\right. \\
& \quad + \tfrac{63627}{5}\zeta_7 - 162\zeta_3\zeta_4 + \tfrac{8678}{5}\zeta_3^2 - \tfrac{6691}{2}\zeta_6 + \tfrac{63723}{10}\zeta_5 \\
& \quad \left. - \tfrac{16989}{16}\zeta_4 + \tfrac{779603}{240}\zeta_3 + \tfrac{18841427}{11520}\right) g^7 \\
& + \mathcal{O}\left(g^8\right)
\end{aligned}
$$

Numerical values:

$$\approx -2\epsilon g + 3g^2 - 5.7g^3 + 32.6g^4 - 271.6g^5 + 2848.6g^6 - 34776.1g^7 + \mathcal{O}\left(g^8\right)$$

A Galois coaction on ϕ^4 periods?

Theorem (Deligne)

For $N \in \{1, 2, 3, 4, '6', 8\}$, the algebra of motivic MPL at Nth roots of unity is isomorphic to a freely generated shuffle algebra. Example:

$$MZV \cong \mathbb{Q}[\pi^2] \otimes \mathbb{Q}\langle f_3, f_5, f_7, \dots \rangle \qquad MDV \cong \mathbb{Q}[i\pi] \otimes \mathbb{Q}\langle f_2, f_3, f_4, f_5, \dots \rangle$$

Example

$$\zeta_{2n+1} \mapsto f_{2n+1} \qquad \zeta_{3,5} \mapsto -5 f_5 f_3$$

Message: periods are not just numbers, but have a structure! Consider the map δ_k which clips off the first letter:

$$\delta_k(f_{n_1} \dots f_{n_r}) := \begin{cases} f_{n_2} \dots f_{n_r} & \text{if } k = n_1 \\ 0 & \text{else} \end{cases}$$

Example

$$\delta_3(\zeta_3) = 1 \qquad \delta_3(\zeta_{3,5}) = 0 \qquad \delta_5(\zeta_{3,5}) = -5\zeta_3 \qquad \delta_k \zeta_{2n} = 0$$

Coaction conjecture (O. Schnetz)

The periods of primitive log.-div. ϕ^4 graphs are closed under the action of the operators δ_k.
Other words: The cosmic Galois group acts on ϕ^4 periods.

Example

$$P_{7,11} = -\frac{332\,262}{43} f_8 f_3 + \frac{54\,918}{55} f_6 f_5 + \frac{1\,134}{13} f_4 f_7 - \frac{1\,874\,502}{3\,485} f_2 f_9 + 5\,670 f_2 f_3 f_3 f_3 - \frac{3\,216\,912\,825\,399\,005\,402\,331\,281\,812\,377\,062\,149}{14\,080\,217\,073\,343\,074\,027\,422\,017\,273\,458\,000} \left(\frac{\pi}{\sqrt{3}}\right)^{11}.$$

Note: After δ_{2k}, only odd letters survive $\Rightarrow$ MZV, in ϕ^4.

- highly non-trivial constraint on ϕ^4 periods
- proven for generalized periods [F. Brown]

Summary

- several powerful tools for massless propagators
- updated census/database
- ϕ^4 beta function at six loops
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Thanks

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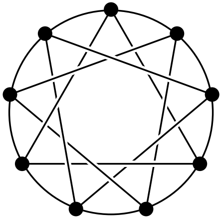
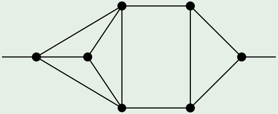


Alternative method

Given a graph G, find a linear combination X of graphs such that

1. $G - X$ is primitive (free of subdivergences) ($\Rightarrow$ `HyperInt`)
2. each term in X factorizes (has a ≥ 1 loop sub-p-integral) [Panzer '13]

Example



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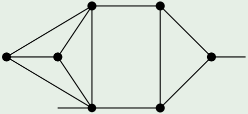
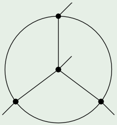
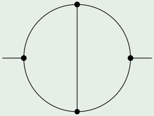


Both have the same subdivergence γ and quotient G/γ:

$\gamma =$



$G/\gamma =$



- simple: just p-integrals, no renormalization
- not straightforward to automate