Leading and next-to-leading large-n_f terms in the cusp anomalous dimension and the quark–antiquark potential

Andrey Grozin

History

1 loop

$$\Gamma(\alpha_s, \varphi) = C_F \frac{\alpha_s}{\pi} (\varphi \coth \varphi - 1)$$

Follows from the soft radiation function in classical electrodynamics

The Guinness Book of Records The anomalous dimension known for a longest time (> 100 years)

2 loops Korchemsky, Radyushkin (1987)
Kidonakis (2009)

3 loops Grozin, Henn, Korchemsky, Marquard (2014–2016)

History

γ_h

2 loops	Knauss, Scharnhorst (1984); Ji, Musolf (1991); Broadhurst, Gray, Schilcher (1991) Broadhurst, Grozin (1991); Giménez (1992)
3 loops	Melnikov, van Ritbergen (2000) Chetyrkin, Grozin (2003)

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$V(\vec{q})$

2 loops	Peter (1997) Schröder (1999)
3 loops	Smirnov, Smirnov, Steinhauser (2008, 2009) Anzai, Kiyo, Sumino (2009)

Large n_f structures

$C_F(T_F n_f)^{L-1}\alpha_s^L \ (L \geq 1)$

$C_F^2(T_F n_f)^{L-2}\alpha_s^L \ (L \geq 3)$

$C_F C_A(T_F n_f)^{L-2}\alpha_s^L \ (L \geq 2)$

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QED with n_f flavors $C_F = 1$, $C_A = 0$, $T_F = 1$, $\beta_0 = -\frac{4}{3}n_f$

$$b = \beta_0 \frac{\alpha}{4\pi} \sim 1 \qquad \frac{1}{\beta_0} \ll 1 \text{ — expansion parameter}$$

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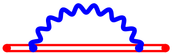
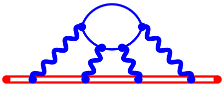
Coordinate space, Wilson line of any shape, up to NLβ_0

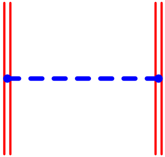
$\log W =$ 

First broken at NNLβ_0: $n_f^{L-3}\alpha^L$ $(L \geq 4)$



Quark-antiquark potential

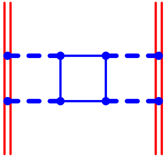
Up to NLβ_0



Coulomb gauge

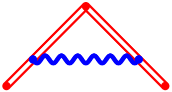
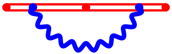
$$V(\vec{q}) = -\frac{e_0^2}{\vec{q}^2}\frac{1}{1-\Pi(-\vec{q}^2)}$$

$\Pi(q^2)$ gauge invariant
First broken at NNLβ_0



Cusp renormalization

$$\log W(t, t'; \varphi) - \log W(t, t'; 0)$$

$=$  $-$  $= \log Z + \text{finite}$

(external-leg corrections cancel)

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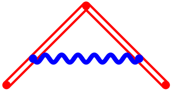
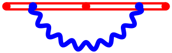
$=$  $-$  $= \log Z + \text{finite}$

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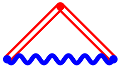
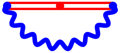
Momentum space

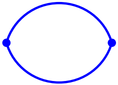
$$\log V(\omega, \omega; \varphi) - \log V(\omega, \omega; 0)$$

$=$  $-$  $= \log Z + \text{finite}$

Leading β_0 order

Photon self energy

$$\Pi_0(k^2) =$$



$$= \beta_0 \frac{e_0^2}{(4\pi)^{d/2}} e^{-\gamma\varepsilon} \frac{D(\varepsilon)}{\varepsilon} (-k^2)^{-\varepsilon} \sim 1$$

$$D(\varepsilon) = e^{\gamma\varepsilon} \frac{(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{(1-2\varepsilon)(1-\frac{2}{3}\varepsilon)\Gamma(1-2\varepsilon)} = 1 + \frac{5}{3}\varepsilon + \cdots$$

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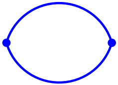

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Charge renormalization

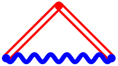
$$\beta_0 \frac{e_0^2}{(4\pi)^{d/2}} e^{-\gamma\varepsilon} = b Z_\alpha(b) \mu^{2\varepsilon}$$

$$\frac{d\log Z_\alpha}{d\log b} = -\frac{b}{\varepsilon + b} \qquad Z_\alpha = \frac{1}{1 + b/\varepsilon}$$

Vertex function

1 loop with $1/(1-\Pi_0(k^2))$

$$V(\omega,\omega;\varphi) =$$



$$= \frac{1}{\beta_0}\sum_{L=1}^{\infty}\frac{f(\varepsilon,L\varepsilon;\varphi)}{L}\Pi_0^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

Π_0 is taken at $-k^2 = (-2\omega)^2$

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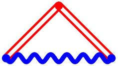
$$V(\omega,\omega;\varphi) =$$  $$= \frac{1}{\beta_0}\sum_{L=1}^{\infty}\frac{f(\varepsilon,L\varepsilon;\varphi)}{L}\Pi_0^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

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$$f(\varepsilon,u;\varphi) = -\frac{(1-\frac{2}{3}\varepsilon)\Gamma(2-2\varepsilon)\Gamma(1-u)\Gamma(1+2u)}{(1-\varepsilon)\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma(2+u-\varepsilon)}$$
$$\times\left[\Big((2+u-2\varepsilon)\cos\varphi - u\Big)\,{}_2F_1\left(\left.\begin{array}{c}1,1-u\\3/2\end{array}\right|\frac{1-\cos\varphi}{2}\right)+1\right]$$

(Landau gauge) Grozin, Kotikov (2011)

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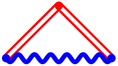

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(Landau gauge) Grozin, Kotikov (2011)

Regular at the origin

$$f(\varepsilon,u;\varphi) = \sum_{n,m=0}^{\infty} f_{nm}(\varphi)\varepsilon^n u^m$$

Renormalization

$$\log Z = \frac{Z_1(b;\varphi)}{\varepsilon} + \frac{Z_2(b;\varphi)}{\varepsilon^2} + \cdots \qquad Z_n(b;\varphi) = \mathcal{O}(b^n)$$

$$\Gamma(b;\varphi) = -2\frac{dZ_1(b;\varphi)}{d\log b}$$

(higher Z_n contain no new information)

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(higher Z_n contain no new information)

Choosing $\mu^2 = D(\varepsilon)^{-1/\varepsilon}(-2\omega)^2 \to e^{-5/3}(-2\omega)^2$

$$V(\omega,\omega;\varphi) - V(\omega,\omega;0) = \frac{1}{\beta_0}\sum_{L=1}^{\infty}\frac{\bar{f}(\varepsilon, L\varepsilon;\varphi)}{L}\left(\frac{b}{\varepsilon+b}\right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\bar{f}(\varepsilon,u;\varphi) = f(\varepsilon,u;\varphi) - f(\varepsilon,u;0)$$

Leading β_0 result

Z_1: all coefficients but f_{n0} cancel

$$Z_1(b;\varphi) = 2\frac{\varphi\cot\varphi - 1}{\beta_0}\sum_{n=0}^{\infty}\frac{\hat{f}_n}{n+1}(-b)^{n+1} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\bar{f}(\varepsilon,0;\varphi) = -2\hat{f}(\varepsilon)(\varphi\cot\varphi - 1) \qquad \hat{f}(\varepsilon) = \sum_{n=0}^{\infty}\hat{f}_n\varepsilon^n$$

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$$\Gamma(b;\varphi) = 4\frac{b}{\beta_0}\gamma_0(b)(\varphi\cot\varphi - 1) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\gamma_0(b) = \hat{f}(-b) = \frac{(1+\frac{2}{3}b)\Gamma(2+2b)}{(1+b)\Gamma^3(1+b)\Gamma(1-b)}$$

$$= 1 + \frac{5}{3}b - \frac{1}{3}b^2 - \left(2\zeta_3 - \frac{1}{3}\right)b^3 + \left(\frac{\pi^4}{30} - \frac{10}{3}\zeta_3 - \frac{1}{3}\right)b^4 + \cdots$$

Beneke, Braun (1995)

HQET field renormalization

$\varphi = 0$ Ward identity

$$V(\omega, \omega'; 0) = \frac{S^{-1}(\omega') - S^{-1}(\omega)}{\omega' - \omega} \qquad V(\omega, \omega; 0) = \frac{dS^{-1}(\omega)}{d\omega}$$

$$\log V(\omega, \omega'; 0) = -\log Z_h + \text{finite}$$

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$$\log V(\omega,\omega';0) = -\log Z_h + \text{finite}$$

$$f(\varepsilon,u;0) = -3\frac{(1-\frac{2}{3}\varepsilon)^2\Gamma(2-2\varepsilon)\Gamma(1-u)\Gamma(1+2u)}{(1-\varepsilon)\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma(2+u-\varepsilon)}$$

HQET field anomalous dimension

$$\gamma_h(b) = 2\frac{b}{\beta_0}\gamma_{h0}(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\gamma_{h0}(b) = f(-b,0;0) = \frac{\left(1+\frac{2}{3}b\right)^2 \Gamma(2+2b)}{(1+b)^2\Gamma^3(1+b)\Gamma(1-b)}$$

$$= 1 + \frac{4}{3}b - \frac{5}{9}b^2 - \left(2\zeta_3 - \frac{2}{3}\right)b^3 + \left(\frac{\pi^4}{30} - \frac{8}{3}\zeta_3 - \frac{7}{9}\right)b^4 + \cdots$$

(Landau gauge) Broadhurst, Grozin (1995)

Quark-antiquark potential

$\mu^2 = \vec{q}^2$

$$V(\vec{q}) = -\frac{(4\pi)^{d/2} e^{\gamma\varepsilon}}{\beta_0 D(\varepsilon)(\vec{q}^2)^{1-\varepsilon}} \varepsilon \sum_{L=1}^{\infty} \left(D(\varepsilon) \frac{b}{\varepsilon + b} \right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

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The sum can be written as

$$\sum_{L=1}^{\infty} f(\varepsilon, L\varepsilon) \left(\frac{b}{\varepsilon + b}\right)^L \qquad f(\varepsilon, u) = D(\varepsilon)^{u/\varepsilon} = \sum_{n,m=0}^{\infty} f_{nm} \varepsilon^n u^m$$

Quark-antiquark potential

$$\mu^2 = \vec{q}^2$$

$$V(\vec{q}) = -\frac{(4\pi)^{d/2} e^{\gamma\varepsilon}}{\beta_0 D(\varepsilon)(\vec{q}^2)^{1-\varepsilon}} \varepsilon \sum_{L=1}^{\infty} \left(D(\varepsilon) \frac{b}{\varepsilon + b} \right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

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This sum is

$$\frac{b}{\varepsilon} \sum_{n=0}^{\infty} n!\, f_{0n} b^n + \mathcal{O}(\varepsilon^0)$$

$$f(0, u) = e^{\frac{5}{3}u} \qquad f_{0n} = \frac{1}{n!}\left(\frac{5}{3}\right)^n$$

Quark-antiquark potential

$$V(\vec{q}) = -\frac{(4\pi)^2}{\vec{q}^2}\frac{b}{\beta_0}v_0(b) \qquad v_0(b) = \frac{1}{1-\frac{5}{3}b}$$

Conformal anomaly

$$\Delta \equiv \delta\Gamma(\alpha_s, \pi - \delta) - \frac{\vec{q}^{\,2} V(\vec{q}; \alpha_s(\vec{q}))}{4\pi} = 0$$

2 loops: Kilian, Mannel, Ohl (1993)

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QCD — anomaly $\Rightarrow$ β function

$$\Delta \sim C_F \beta_0 \alpha_s^3 \left(T_F n_f \Delta_f + C_F \cdot 0 + C_A \Delta_A\right)$$

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Leading β_0

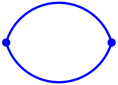
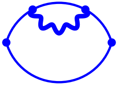
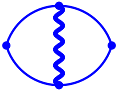
$$\Delta = 4\pi \frac{b^3}{\beta_0} \delta_0(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\delta_0(b) = \frac{v_0(b) - \gamma_0(b)}{b^2}$$

$$= \frac{28}{9} + 2\left(\zeta_3 + \frac{58}{27}\right) b - \frac{1}{3}\left(\frac{\pi^4}{10} - 10\zeta_3 - \frac{652}{27}\right) b^2 + \cdots$$

Next to leading β_0 order

Photon self energy

$\Pi(k^2) =$  $+\, 2$  $+$ 

$$= \Pi_0(k^2) + \frac{\Pi_1(k^2)}{\beta_0} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

 at the Lβ_0 order

$$\Pi_1 = 3\varepsilon \sum_{L=2}^{\infty} \frac{F(\varepsilon, L\varepsilon)}{L} \Pi_0^L$$

Palanques-Mestre, Pascual (1984); Broadhurst (1993)

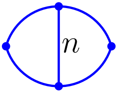
Photon self energy

$$F(\varepsilon,u)=\frac{2(1-2\varepsilon)^2(3-2\varepsilon)\Gamma^2(1-2\varepsilon)}{9(1-\varepsilon)(1-u)(2-u)\Gamma^2(1-\varepsilon)\Gamma^2(1+\varepsilon)}$$

$$\times\left[-u\frac{2-3\varepsilon-\varepsilon^2+\varepsilon(2+\varepsilon)u-\varepsilon u^2}{\Gamma^2(1-\varepsilon)}I(1+u-2\varepsilon)\right.$$

$$+2\frac{2(1+\varepsilon)(3-2\varepsilon)-(4+11\varepsilon-7\varepsilon^2)u+\varepsilon(8-3\varepsilon)u^2-\varepsilon u^3}{(1-u)(2-u)(1-u-\varepsilon)(2-u-\varepsilon)}$$

$$\left.\times\frac{\Gamma(1+u)\Gamma(1-u+\varepsilon)}{\Gamma(1-u-\varepsilon)\Gamma(1+u-2\varepsilon)}\right]=\sum_{n,m=0}^{\infty}F_{nm}\varepsilon^n u^m$$

$I(n) =$ 

$$=\frac{1}{\pi^d}\int\frac{d^dk_1\,d^dk_2}{k_1^2k_2^2(k_1+p)^2(k_2+p)^2\left[(k_1-k_2)^2\right]^n}$$

(Euclidean, $p^2=1$) ${}_3F_2(\cdots|1)$

Kotikov (1996); Broadhurst, Gracey, Kreimer (1997)

$F(\varepsilon, u)$ and F_{nm}

$$F(\varepsilon, 0) = \frac{(1+\varepsilon)(1-2\varepsilon)^2(1-\frac{2}{3}\varepsilon)^2\Gamma(1-2\varepsilon)}{(1-\varepsilon)^2(1-\frac{1}{2}\varepsilon)\Gamma(1+\varepsilon)\Gamma^3(1-\varepsilon)}$$

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$$F(0,u) = \frac{2}{3}\frac{\psi'\left(2-\frac{u}{2}\right)-\psi'\left(1+\frac{u}{2}\right)-\psi'\left(\frac{3-u}{2}\right)+\psi'\left(\frac{1+u}{2}\right)}{(1-u)(2-u)}$$

$$F_{0m} = -\frac{32}{3}\sum_{s=1}^{[(m+1)/2]} s\left(1-2^{-2s}\right)\left(1-2^{2s-m-2}\right)\zeta_{2s+1}$$

$$+\frac{4}{3}(m+1)\left(m+(m+6)2^{-m-3}\right)$$

Broadhurst (1993)

$F(\varepsilon, u)$ and F_{nm}

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Broadhurst (1993)

$$F(\varepsilon,2\varepsilon) = \frac{2}{9\varepsilon^2}\frac{3-2\varepsilon}{1-\varepsilon}\left[2\frac{(1-2\varepsilon)^2(2-2\varepsilon+\varepsilon^2)}{(1-3\varepsilon)(2-3\varepsilon)}\right.$$

$$\left.\times\frac{\Gamma(1+2\varepsilon)\Gamma^2(1-2\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(1-3\varepsilon)} - 2 + \varepsilon - 2\varepsilon^2\right]$$

Charge renormalization

$$Z_\alpha(b) = \frac{1}{1 + b/\varepsilon}\left[1 + \frac{Z_1(b)}{\beta_0} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\right]$$

$$Z_1(b) = \frac{Z_{11}(b)}{\varepsilon} + \frac{Z_{12}(b)}{\varepsilon^2} + \cdots \qquad Z_{1n} = \mathcal{O}(b^{n+1})$$

Charge renormalization

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In the abelian theory (expressed via renormalized b)

$$\log(1-\Pi) = \log Z_\alpha + \text{finite}$$

Equating the coefficients of ε^{-1} in the $1/\beta_0$ terms:
Z_{11} is the coefficient of ε^{-1} in

$$-\left(1+\frac{b}{\varepsilon}\right)\Pi_1$$

Charge renormalization

Choose $\mu^2 = D(\varepsilon)^{-1/\varepsilon}(-k^2) \to e^{-5/3}(-k^2)$

$$\Pi_1 = 3\varepsilon \sum_{L=2}^{\infty} \frac{F(\varepsilon, L\varepsilon)}{L} \left(\frac{b}{\varepsilon + b}\right)^L$$

Expand in b; expand $F(\varepsilon, u)$ in ε, u:
all coefficients but F_{n0} cancel

$$Z_{11} = -3 \sum_{n=0}^{\infty} \frac{F_{n0}(-b)^{n+2}}{(n+1)(n+2)}$$

Charge renormalization

$$\beta(b) = b + \frac{\beta_1(b)}{\beta_0} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\beta_1(b) = -\frac{dZ_{11}(b)}{d\log b} = 3\sum_{n=0}^{\infty}\frac{F_{n0}(-b)^{n+2}}{n+1}$$

$$= 3b^2 + \frac{11}{4}b^3 - \frac{77}{36}b^4 - \frac{1}{2}\left(3\zeta_3 + \frac{107}{48}\right)b^5 + \cdots$$

Palanques-Mestre, Pascual (1984); Broadhurst (1993)

Charge renormalization

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Palanques-Mestre, Pascual (1984); Broadhurst (1993)

We need the full Z_1

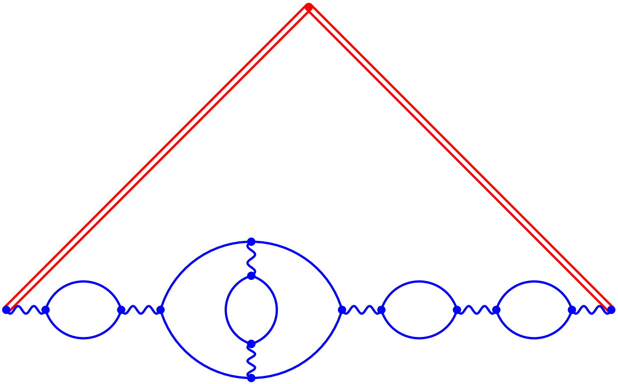
$$Z_1(b) = -\varepsilon \int_0^b \frac{\beta_1(b)\,db}{b(\varepsilon+b)^2}$$

$$= -\frac{3}{2}\frac{b^2}{\varepsilon} + \frac{1}{2}\left(4 + F_{10}\varepsilon\right)\frac{b^3}{\varepsilon^2} - \frac{1}{4}\left(9 + 3F_{10}\varepsilon + F_{20}\varepsilon^2\right)\frac{b^4}{\varepsilon^3} + \cdots$$

Next to leading β_0 order

- Photon propagator $(1 - \Pi_0 - \Pi_1/\beta_0)^{-1}$ up to $1/\beta_0$
- Charge renormalization up to $1/\beta_0$ (Z_1/β_0)

Next to leading β_0 order



$$V(\omega,\omega;\varphi) - V(\omega,\omega;0) = \frac{1}{\beta_0}\sum_{L=1}^{\infty}\frac{\bar{f}(\varepsilon, L\varepsilon;\varphi)}{L}\left(\frac{b}{\varepsilon+b}\right)^{L}$$

$$\times\left[1 + L\frac{Z_1}{\beta_0} + \frac{3\varepsilon}{\beta_0}\sum_{L'=2}^{L-1}\frac{L-L'}{L'}F(\varepsilon, L'\varepsilon)\right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)$$

Next to leading β_0 order

Expand in b; expand $F(\varepsilon, u)$ and $f(\varepsilon, u)$ in ε and u

Z_1 is the coefficient of ε^{-1}

all $\bar{f}_{nm}$ except $\bar{f}_{n0}$ cancel, i. e., at the NLβ_0 order Γ is determined by the same $\hat{f}_n$ coefficients as at Lβ_0

$$\Gamma(b;\varphi) = 4\left[\frac{b}{\beta_0}\gamma_0(b) - \frac{b^3}{\beta_0^2}\gamma_1(b)\right](\varphi\cot\varphi - 1) + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)$$

$$\begin{aligned}\gamma_1(b) = &-\frac{3}{2}\left[F_{10} + 2F_{01} - 2\hat{f}_1\right]\\ &+ \left[2F_{20} + 3(F_{11} + F_{02}) + 3F_{01}\hat{f}_1 - 6\hat{f}_2\right]b\\ &- \left[\frac{3}{4}(3F_{30} + 4(F_{21} + F_{12} + F_{03})) + (F_{20} + 3(F_{11} + F_{02}))\hat{f}_1\right.\\ &\quad\left. - \frac{3}{2}\left(F_{10} - 2F_{01}\right)\hat{f}_2 - 9\hat{f}_3\right]b^2\\ &+ \cdots\end{aligned}$$

Next to leading β_0 result

$$
\begin{aligned}
\gamma_1(b) = {} & 12\zeta_3 - \frac{55}{4} + \left[-\frac{\pi^4}{5} + 40\zeta_3 - \frac{299}{18}\right] b \\
& + \left[24\zeta_5 - \frac{2}{3}\pi^4 + \frac{233}{6}\zeta_3 + \frac{15211}{864}\right] b^2 \\
& + \left[-48\zeta_3^2 - \frac{2}{63}\pi^6 + 80\zeta_5 - \frac{167}{225}\pi^4 + \frac{1168}{15}\zeta_3 - \frac{971}{240}\right] b^3 \\
& + \left[36\zeta_7 + \frac{8}{5}\pi^4\zeta_3 - 160\zeta_3^2 - \frac{20}{189}\pi^6 + \frac{377}{3}\zeta_5 - \frac{23}{15}\pi^4 \right. \\
& \qquad \left. + \frac{929}{12}\zeta_3 - \frac{8017}{1728}\right] b^4
\end{aligned}
$$

Next to leading β_0 result

$$+\left[-240\zeta_3\zeta_5 - \frac{4}{225}\pi^8 + 120\zeta_7 + \frac{16}{3}\pi^4\zeta_3 - \frac{2776}{21}\zeta_3^2 - \frac{914}{3969}\pi^6 + \frac{6826}{21}\zeta_5 - \frac{1793}{1350}\pi^4 - \frac{31693}{315}\zeta_3 + \frac{79433}{4320}\right]b^5 + \cdots$$

Can be continued

F_{nm} with $n+m=6$, $n>0$, $m>0$ contain $\zeta_{5,3}$

but it has cancelled in γ_1

HQET field anomalous dimension

$$\gamma_h(b) = -6\left[\frac{b}{\beta_0}\gamma_{h0}(b) - \frac{b^3}{\beta_0^2}\gamma_{h1}(b)\right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)$$

$$\gamma_{h1}(b) = 3\left(4\zeta_3 - \frac{17}{4}\right) + \left(-\frac{\pi^4}{5} + 36\zeta_3 - \frac{103}{9}\right)b$$
$$+ \left(24\zeta_5 - \frac{3}{5}\pi^4 + \frac{59}{2}\zeta_3 + \frac{14579}{864}\right)b^2$$
$$+ \left(-48\zeta_3^3 - \frac{2}{63}\pi^6 + 72\zeta_5 - \frac{44}{75}\pi^4 + \frac{3229}{45}\zeta_3 - \frac{5191}{540}\right)b^3$$
$$+ \left(36\zeta_7 + \frac{8}{5}\pi^4\zeta_3 - 144\zeta_3^2 - \frac{2}{21}\pi^6 + 107\zeta_5 - \frac{946}{675}\pi^4\right.$$
$$\left. + \frac{9601}{180}\zeta_3 + \frac{22859}{8640}\right)b^4$$

HQET field anomalous dimension

$$+\left(-240\zeta_3\zeta_5 - \frac{4}{225}\pi^8 + 108\zeta_7 + \frac{24}{5}\pi^4\zeta_3 - \frac{664}{7}\zeta_3^2 - \frac{272}{1323}\pi^6 + \frac{18574}{63}\zeta_5 - \frac{119}{135}\pi^4 - \frac{6263}{63}\zeta_3 + \frac{16103}{1296}\right)b^5 + \cdots$$

$\zeta_{5,3}$ has cancelled again

Quark-antiquark potential at NLβ_0

$$V(\vec{q}) = -\frac{(4\pi)^2}{\beta_0 \vec{q}^{\,2}} \varepsilon \sum_{L=1}^{\infty} f(\varepsilon, L\varepsilon) \left(\frac{b}{\varepsilon + b} \right)^L$$

$$\times \left[1 + L \frac{Z_1}{\beta_0} + \frac{3\varepsilon}{\beta_0} \sum_{L'=2}^{L-1} \frac{L - L'}{L'} F(\varepsilon, L'\varepsilon) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3} \right)$$

$$= -\frac{(4\pi)^2}{\vec{q}^{\,2}} \left[\frac{b}{\beta_0} v_0(b) - \frac{b^3}{\beta_0^2} v_1(b) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3} \right)$$

Quark-antiquark potential at NLβ_0

$$
\begin{aligned}
v_1(b) = &-\frac{3}{2}\left[F_{10} + 2F_{01} + 2f_{01}\right] \\
&+ \frac{1}{2}\left[F_{20} - 6F_{02} - 6\left(F_{10} + 3F_{01}\right) f_{01} - 30 f_{02}\right] b \\
&- \frac{1}{4}\Big[F_{30} + 24F_{03} - 4\left(F_{20} + 12F_{02}\right) f_{01} \\
&\qquad + 36\left(F_{10} + 4F_{01}\right) f_{02} + 312 f_{03}\Big] b^2 + \cdots
\end{aligned}
$$

contains only the same coefficients $f_{0n} = (5/3)^n/n!$ as the Lβ_0 result and only F_{n0} and F_{0n}

Quark-antiquark potential at NLβ_0

$$
\begin{aligned}
v_1(b) = {} & 12\zeta_3 - \frac{55}{4} + \left(78\zeta_3 - \frac{7001}{72}\right) b \\
& + \left(60\zeta_5 + \frac{723}{2}\zeta_3 - \frac{147851}{288}\right) b^2 \\
& + \left(770\zeta_5 + \frac{\pi^4}{200} + \frac{276901}{180}\zeta_3 - \frac{70418923}{25920}\right) b^3 \\
& + \left(1134\zeta_7 + \frac{32297}{5}\zeta_5 + \frac{41}{1800}\pi^4 + \frac{402479}{60}\zeta_3 - \frac{1249510621}{77760}\right) b^4 \\
& + \left(21735\zeta_7 + \frac{\zeta_3^2}{7} + \frac{\pi^6}{1323} + \frac{5911849}{126}\zeta_5 + \frac{41}{720}\pi^4 \right. \\
& \qquad \left. + \frac{48558187}{1512}\zeta_3 - \frac{10255708489}{93312}\right) b^5 + \cdots
\end{aligned}
$$

Only ζ_n to all orders

Conformal anomaly

The b^3/β_0^2 term has cancelled

$$\Delta = 4\pi \left[\frac{b^3}{\beta_0} \delta_0(b) - \frac{b^4}{\beta_0^2} \delta_1(b) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3} \right)$$

$$\delta_1(b) = \frac{\pi^4}{5} + 38\zeta_3 - \frac{645}{8} + \left(36\zeta_5 + \frac{2}{3}\pi^4 + \frac{968}{3}\zeta_3 - \frac{114691}{216} \right) b$$
$$+ \left(48\zeta_3^2 + \frac{2}{63}\pi^6 + 690\zeta_5 + \frac{269}{360}\pi^4 + \frac{52577}{36}\zeta_3 - \frac{14062811}{5184} \right) b^2$$
$$+ \left(1098\zeta_7 - \frac{8}{5}\pi^4\zeta_3 + 160\zeta_3^2 + \frac{20}{189}\pi^6 + \frac{95006}{15}\zeta_5 + \frac{2801}{1800}\pi^4 \right.$$
$$\left. + \frac{198917}{30}\zeta_3 - \frac{39035933}{2430} \right) b^3$$

Conformal anomaly

$$+\Bigg(240\zeta_3\zeta_5+\frac{4}{225}\pi^8+21615\zeta_7-\frac{16}{3}\pi^4\zeta_3+\frac{397}{3}\zeta_3^2+\frac{131}{567}\pi^6$$
$$+\frac{838699}{18}\zeta_5+\frac{14959}{10800}\pi^4+\frac{34793081}{1080}\zeta_3-\frac{51287121209}{466560}\Bigg)b^4$$
$$+\cdots$$

Conclusion

- $C_F(T_F n_f)^{L-1}\alpha_s^L$ terms in Γ, γ_h
  $C_F(T_F n_f)^L \alpha_s^{L+1}$ terms in $V(\vec{q})$
  are known to all orders in α_s (explicit formulas)

Conclusion

- $C_F(T_F n_f)^{L-1}\alpha_s^L$ terms in Γ, γ_h
  $C_F(T_F n_f)^L \alpha_s^{L+1}$ terms in $V(\vec{q})$
  are known to all orders in α_s (explicit formulas)
- $C_F^2(T_F n_f)^{L-2}\alpha_s^L$ $(L \geq 3)$ terms in Γ, γ_h
  $C_F^2(T_F n_f)^{L-1}\alpha_s^{L+1}$ $(L \geq 2)$ terms in $V(\vec{q})$
  are known to all orders in α_s (algorithm)