Gluon form factor decompositions from the worldline formalism

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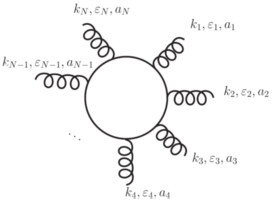
Loops and Legs in Quantum Field Theory
Leipzig 2016

The QCD N-gluon vertices



One-loop off-shell 1PI N-gluon functions ("vertices") $\Gamma^{a_1 a_2 \cdots a_N}_{s\,\mu_1 \ldots \mu_N}[k_1, \ldots, k_N]$
$s = 0, \frac{1}{2}, 1$ for scalar, spinor, gluon loop.

- Building blocks for higher-loop amplitudes.
- Input for the Dyson-Schwinger equations.
- Important for the RG group.
- IR properties of QCD.
- ...

The off-shell Ward identities

Off-shell, the Ward identities for the gluon amplitudes are inhomogeneous and map N - point to $N-1$ - point:

$$
\begin{aligned}
k_1^{\mu_1}\Gamma^{a_1 a_2 \cdots a_N}_{s\,\mu_1 \ldots \mu_N}[k_1,\ldots,k_N] \quad = \quad & -igf_{a_1 a_2 c}\Gamma^{c a_3 a_4 \cdots a_N}_{s\,\mu_2 \ldots \mu_N}[k_1+k_2, k_3, \cdots, k_N] \\
& -igf_{a_1 a_3 c}\Gamma^{a_2 c a_4 \cdots a_N}_{s\,\mu_2 \ldots \mu_N}[k_1, k_2+k_3, \cdots, k_N] \\
& - \ldots . \\
& (+\,\text{possible ghost terms})
\end{aligned}
$$

At the one-loop level:

- These identities hold for the scalar and spinor loop without ghost terms.
- For the gluon loop, there are two equivalent ways of constructing the vertex that avoid ghosts:
  - The background field method with quantum Feynman gauge (M. Binger and S. J. Brodsky 2006)
  - The pinch technique (J. M. Cornwall and J. Papavassiliou 1989; J. Papavassiliou 1993)

Ball-Chiu decomposition of the three-gluon vertex

J. S. Ball and T. W. Chiu 1980:

$$
\begin{aligned}
\Gamma_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) \quad = \quad & f^{abc}\Big\{A(k_1^2,k_2^2;k_3^2)g_{\mu_1\mu_2}(k_1-k_2)_{\mu_3} + B(k_1^2,k_2^2;k_3^2)g_{\mu_1\mu_2}(k_1+k_2)_{\mu_3} \\
& -C(k_1^2,k_2^2;k_3^2)[(k_1k_2)g_{\mu_1\mu_2} - k_{1\mu_2}k_{2\mu_1}](k_1-k_2)_{\mu_3} \\
& +\frac{1}{3}S(k_1^2,k_2^2,k_3^2)(k_{1\mu_3}k_{2\mu_1}k_{3\mu_2} + k_{1\mu_2}k_{2\mu_3}k_{3\mu_1}) \\
& +F(k_1^2,k_2^2;k_3^2)[(k_1k_2)g_{\mu_1\mu_2} - k_{1\mu_2}k_{2\mu_1}][k_{1\mu_3}(k_2k_3) - k_{2\mu_3}(k_1k_3)] \\
& +H(k_1^2,k_2^2,k_3^2)\Big(-g_{\mu_1\mu_2}[k_{1\mu_3}(k_2k_3) - k_{2\mu_3}(k_1k_3)] + \frac{1}{3}(k_{1\mu_3}k_{2\mu_1}k_{3\mu_2} - k_{1\mu_2}k_{2\mu_3}k_{3\mu_1})\Big) \\
& +[\text{cyclic permutations of } (k_1,\mu_1),(k_2,\mu_2),(k_3,\mu_3)]\Big\}
\end{aligned}
$$

- Universal tensor decomposition, valid for scalar, spinor and gluon loop, and also for higher loop corrections. Only the coefficient functions A, B, C, F, H, S change.
- From an analysis of the Ward identities.
- A, B, C: two-point kinematics, not transversal.
- F, H: three-point kinematics, transversal.
- At tree-level, $A = 1$, the other functions vanish. $S = 0$ even at one-loop.

The Bern-Kosower formalism

Bern-Kosower master formula (Z. Bern and D. Kosower 1991)

$$\begin{aligned}\Gamma^{a_1\dots a_N}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] \quad = \quad & (-ig)^N \mathrm{tr}(T^{a_1}\dots T^{a_N}) \int_0^\infty dT (4\pi T)^{-D/2} e^{-m^2 T} \\ & \times \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \dots \int_0^{\tau_{N-2}} d\tau_{N-1} \\ & \times \exp\left\{ \sum_{i,j=1}^N \left[\frac{1}{2} G_{Bij} k_i \cdot k_j - i\dot{G}_{Bij}\varepsilon_i \cdot k_j + \frac{1}{2}\ddot{G}_{Bij}\varepsilon_i \cdot \varepsilon_j \right] \right\}\Big|_{\mathrm{lin}(\varepsilon_1\dots\varepsilon_N)}\end{aligned}$$

As it stands, this is a parameter integral representation for the (color-ordered) N - gluon vertex, with momenta k_i and polarizations ε_i, induced by a scalar loop, in D dimensions.
Here m and T are the loop mass and proper-time, τ_i the location of the ith gluon, and

$$G_{Bij} = |\tau_i - \tau_j| - \frac{(\tau_i - \tau_j)^2}{T}, \; \dot{G}_B(\tau_1,\tau_2) = \mathrm{sign}(\tau_1 - \tau_2) - 2\frac{(\tau_1 - \tau_2)}{T}, \; \ddot{G}_B(\tau_1,\tau_2) = 2\delta(\tau_1 - \tau_2) - \frac{2}{T}.$$

The Bern-Kosower rules

In the Bern-Kosower formalism, the master formula is a generating functional for the full on-shell N - gluon amplitudes for the scalar, spinor and gluon loop, through the

Bern-Kosower rules:

1. For fixed N, expand the generating exponential.
2. Use suitable integrations-by-parts (IBPs) to remove all second derivatives $\ddot{G}_{Bij}$.
3. Apply two types of pattern-matching rules:
   - The "tree replacement rules" generate the contributions of the missing reducible diagrams.
   - The "loop replacement rules" generate the integrands for the spinor and gluon loop from the one for the scalar loop.

Strassler's worldline path integral approach

M. J. Strassler, NPB 385 (1992) 145:

- Rederived the master formula and the loop replacement rules using worldline path integral representations of the gluonic effective actions. E.g. for the scalar loop

$$\Gamma[A] = \mathrm{tr}\int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int \mathcal{D}x(\tau)\, \mathcal{P}e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ig\dot{x}\cdot A(x(\tau))\right)}$$

  where $A_\mu = A^a_\mu T^a$ and $\mathcal{P}$ denotes path ordering.
- This also shows that the master formula and the loop replacement rules hold off-shell.
- Reducible contributions have to be calculated separately.

M. J. Strassler, SLAC-PUB-5978 (unpubl.): noted that the IBP generates automatically

- abelian field strength tensors $f_i^{\mu\nu} \equiv k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu$ in the bulk and
- color commutators $[T^{a_i}, T^{a_j}]$ as boundary terms.
- Those fit together to produce full nonabelian field strength tensors

$$F_{\mu\nu} \equiv F^a_{\mu\nu} T^a = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) T^a + ig[A^b_\mu T^b, A^c_\nu T^c] \qquad (1)$$

  in the low-energy effective action.

Thus we see the emergence of gauge invariant tensor structures at the integrand level.

Ball-Chiu from the master formula

N. Ahmadiniaz, C. Schubert, NPB 869 (2013) 417:
For $N=3$, the master formula yields

$$\Gamma_0^{a_1a_2a_3}[k_1,\varepsilon_1;k_2,\varepsilon_2;k_3,\varepsilon_3] = (-ig)^3\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3})\int_0^\infty dT(4\pi T)^{-D/2}e^{-m^2T}$$
$$\times\int_0^T d\tau_1\int_0^{\tau_1} d\tau_2\,(-i)^3 P_3\, e^{(G_{B12}k_1\cdot k_2+G_{B13}k_1\cdot k_3+G_{23}k_2\cdot k_3)}\,,$$

where

$$P_3 = \dot G_{B1i}\varepsilon_1\cdot k_i\dot G_{B2j}\varepsilon_2\cdot k_j\dot G_{B3k}\varepsilon_3\cdot k_k-\ddot G_{B12}\varepsilon_1\cdot\varepsilon_2\dot G_{B3k}\varepsilon_3\cdot k_k$$
$$-\ddot G_{B13}\varepsilon_1\cdot\varepsilon_3\dot G_{B2j}\varepsilon_2\cdot k_j-\ddot G_{B23}\varepsilon_2\cdot\varepsilon_3\dot G_{B1i}\varepsilon_1\cdot k_i\,,$$

(repeated indices $i,j,k,\ldots$are to be summed). To remove the term involving $\ddot G_{B12}\dot G_{B31}$, add the total derivative

$$-\frac{\partial}{\partial\tau_2}\Big(\dot G_{B12}\varepsilon_1\cdot\varepsilon_2\dot G_{B31}\varepsilon_3\cdot k_1 e^{(G_{B12}k_1\cdot k_2+G_{B13}k_1\cdot k_3+G_{23}k_2\cdot k_3)}\Big)\,. \tag{2}$$

In the abelian case this total derivative term would integrate to zero, but here due to the color ordering it produces (one half of) the term

$$\mathrm{tr}\,(T^{a_1}[T^{a_2},T^{a_3}])\varepsilon_3\cdot f_1\cdot\varepsilon_2\dot G_{B12}\dot G_{B21}\,e^{G_{B12}k_1\cdot(k_2+k_3)}\,. \tag{3}$$

This term involves only a two-point integral, with "pinched" momenta k_2+k_3.

The three-gluon vertex in the Q-representation

At this stage have

$$\Gamma_0 = \frac{g^3}{(4\pi)^{\frac{D}{2}}} \mathrm{tr}(T^{a_1}[T^{a_2}, T^{a_3}])(\Gamma_0^{\rm bulk} + \Gamma_0^{\rm bound})$$

$$\Gamma_0^{\rm bulk} = -\int_0^\infty \frac{dT}{T^{\frac{D}{2}}} e^{-m^2 T} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \, (Q_3^3 + Q_3^3) \exp\Big\{\sum_{i,j=1}^3 \frac{1}{2} G_{Bij} k_i \cdot k_j\Big\}$$

$$\Gamma_0^{\rm bound} = -\int_0^\infty \frac{dT}{T^{\frac{D}{2}}} e^{-m^2 T} \int_0^T d\tau_1 \dot G_{B12} \dot G_{B21} \Big[\varepsilon_3 \cdot f_1 \cdot \varepsilon_2 \, e^{G_{B12} k_1 \cdot (k_2+k_3)} + \mathrm{cycl.}\Big]$$

$$Q_3^3 = \dot G_{B12} \dot G_{B23} \dot G_{B31} \mathrm{tr}\,(f_1 f_2 f_3)$$

$$Q_3^2 = \frac{1}{2} \dot G_{B12} \dot G_{B21} \mathrm{tr}\,(f_1 f_2) \dot G_{B3i} \varepsilon_3 \cdot k_i + 2\,\mathrm{perm.}$$

This is not yet Ball-Chiu: Q_3^3 corresponds to the form factor H, but Q_3^2 not to F; it is not even transversal.

Second integration-by-parts

To make Q_3^2 transversal, add another total derivative:

$$-\frac{r_3\cdot\varepsilon_3}{r_3\cdot k_3}\frac{1}{2}\mathrm{tr}(f_1f_2)\frac{\partial}{\partial\tau_3}\Bigl(\dot G_{B12}\dot G_{B21}e^{(\cdot)}\Bigr)\,.$$

Here r_3 is a *reference momentum* such that $r_3\cdot k_3\neq 0$. This transforms Q_3^2 into

$$\begin{aligned} S_3^2 \quad := \quad & \dot G_{B12}\dot G_{B21}\frac{1}{2}\mathrm{tr}(f_1f_2)\dot G_{B3k}\frac{r_3\cdot f_3\cdot k_k}{r_3\cdot k_3}+\dot G_{B13}\dot G_{B31}\frac{1}{2}\mathrm{tr}(f_1f_3)\dot G_{B2j}\frac{r_2\cdot f_2\cdot k_j}{r_2\cdot k_2}\\ & +\dot G_{B23}\dot G_{B32}\frac{1}{2}\mathrm{tr}(f_2f_3)\dot G_{B1i}\frac{r_1\cdot f_1\cdot k_i}{r_1\cdot k_1}\,. \end{aligned}$$

which is transversal. With the cyclic choice of reference vectors

$$r_1=k_2-k_3\,,\,r_2=k_3-k_1\,,\,r_3=k_1-k_2$$

S_3^2 becomes the Ball-Chiu form factor F. The boundary terms match with the form factors A, B, C.

Loop replacement rules for the three-gluon vertex

Scalar to Spinor Loop:

$$\begin{aligned} \dot{G}_{Bij}\dot{G}_{Bji} &\rightarrow \dot{G}_{Bij}\dot{G}_{Bji} - G_{Fij}G_{Fji} \\ \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} &\rightarrow \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} - G_{F12}G_{F23}G_{F31} \end{aligned}$$

where $G_{Fij} = \mathrm{sign}(\tau_i - \tau_j)$.

Scalar to Gluon Loop:

$$\begin{aligned} \dot{G}_{Bij}\dot{G}_{Bji} &\rightarrow \dot{G}_{Bij}\dot{G}_{Bji} - 4G_{Fij}G_{Fji} \\ \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} &\rightarrow \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31} - 4G_{F12}G_{F23}G_{F31} \end{aligned}$$

The generated integrand for the gluon loop corresponds to the background field method with quantum Feynman gauge (M. Reuter, C. Schubert, M.G. Schmidt 1996).

Generalization to the N-gluon case

N. Ahmadiniaz, C. Schubert, V.M. Villanueva, JHEP 1301 (2013) 132:
Various integration-by-parts algorithms for the general N - gluon case.
Two preferred representations emerged:

- The Q - representation uses *only local total derivative terms* and relates directly to the effective action.
- The S - representation uses *both local and non-local total derivative terms* and is Ball-Chiu like in that all bulk terms are manifestly transversal.

The four-gluon vertex

N. Ahmadiniaz, C. Schubert (in preparation):
$N = 4$ is much more challenging - at four points, a priori one can construct 138 tensors!

- Our S - representation yields, up to permutations, a decomposition in terms of 19 tensors.
- Only 14 of those involve true four-point tensors.
- The remaining five are just the Ball-Chiu form factors reappearing with pinched momenta as boundary (resp. double-boundary) terms.

The 14 true four-point tensors

$$
\begin{aligned}
T_P^4 &= \mathrm{tr}\,(f_1 f_2 f_3 f_4), \quad T_{NP}^4 = \mathrm{tr}\,(f_1 f_3 f_2 f_4)\,, \\
T_P^{22} &= \frac{1}{4}\mathrm{tr}\,(f_1 f_2)\mathrm{tr}\,(f_3 f_4), \quad T_{NP}^{22} = \frac{1}{4}\mathrm{tr}\,(f_1 f_3)\mathrm{tr}\,(f_2 f_4)\,, \\
T_P^3 &= \mathrm{tr}\,(f_1 f_2 f_3)\frac{r_4 f_4 k_1}{r_4 k_4}, \quad T_{NP}^3 = T_P^3(k_1 \to k_2), \\
T_{\mathrm{quart}}^{2adj} &= \frac{1}{2}\mathrm{tr}\,(f_1 f_2)\frac{r_3 f_3 k_1}{r_3 k_3}\,\frac{r_4 f_4 k_1}{r_4 k_4}, \quad T_{\mathrm{quart}}^{2opp} = \frac{1}{2}\mathrm{tr}\,(f_1 f_3)\frac{r_2 f_2 k_1}{r_2 k_2}\,\frac{r_4 f_4 k_1}{r_4 k_4}\,, \\
T_P^{2adj} &= \frac{1}{2}\mathrm{tr}\,(f_1 f_2)\frac{r_3 f_3 k_2}{r_3 k_3}\,\frac{r_4 f_4 k_1}{r_4 k_4}, \quad T_{NP}^{2adj} = \frac{1}{2}\mathrm{tr}\,(f_1 f_2)\frac{r_3 f_3 k_1}{r_3 k_3}\,\frac{r_4 f_4 k_2}{r_4 k_4}, \\
T_C^{2adj} &= \frac{1}{2}\mathrm{tr}\,(f_1 f_2)\frac{r_3 f_3 k_4 r_4 f_4 k_1 + \frac{1}{2} r_3 f_3 f_4 r_4\; k_4 k_1}{r_3 k_3\; r_4 k_4}, \quad T_Z^{2adj} = T_C^{2adj}(k_1 \to k_2)\,, \\
T_P^{2opp} &= \frac{1}{2}\mathrm{tr}\,(f_1 f_3)\frac{r_2 f_2 k_3}{r_2 k_2}\,\frac{r_4 f_4 k_1}{r_4 k_4}\,, \\
T_{NP}^{2opp} &= \frac{1}{2}\mathrm{tr}\,(f_1 f_3)\frac{r_2 f_2 k_4 r_4 f_4 k_1 + \frac{1}{2} r_2 f_2 f_4 r_4\; k_4 k_1}{r_2 k_2\; r_4 k_4}\,.
\end{aligned}
$$

Comparison with the effective action

The low energy expansion of the one-loop QCD effective action induced by a loop particle of mass m has the form

$$\Gamma[F] = \int_0^\infty \frac{dT}{T} \frac{\mathrm{e}^{-m^2 T}}{(4\pi T)^{D/2}} \,\mathrm{tr} \int dx_0 \sum_{n=2}^{\infty} \frac{(-T)^n}{n!} O_n[F]$$

where $O_n(F)$ is a Lorentz and gauge invariant expression of mass dimension $2n$. To lowest orders,

$$\begin{aligned}
O_2 &= -\frac{1}{6} g^2 F_{\mu\nu} F_{\mu\nu} \\
O_3 &= -\frac{2}{15} ig^3 \, F_{\kappa\lambda} F_{\lambda\mu} F_{\mu\kappa} - \frac{1}{20} g^2 D_\lambda F_{\mu\nu} D^\lambda F^{\mu\nu} \\
O_4 &= +\frac{2}{35} g^4 F_{\kappa\lambda} F_{\lambda\kappa} F_{\mu\nu} F_{\nu\mu} + \frac{4}{35} g^4 F_{\kappa\lambda} F_{\lambda\mu} F_{\kappa\nu} F_{\nu\mu} - \frac{1}{21} g^4 F_{\kappa\lambda} F_{\lambda\mu} F_{\mu\nu} F_{\nu\kappa} \\
&\quad - \frac{8}{105} ig^3 F_{\kappa\lambda} D_\lambda F_{\mu\nu} D_\kappa F_{\nu\mu} - \frac{6}{35} ig^3 F_{\kappa\lambda} D_\mu F_{\lambda\nu} D^\mu F_{\nu\kappa} \\
&\quad + \frac{11}{420} g^4 F_{\kappa\lambda} F_{\mu\nu} F_{\lambda\kappa} F_{\nu\mu} + \frac{1}{70} g^2 D_\kappa D_\lambda F_{\mu\nu} D^\lambda D^\kappa F_{\nu\mu}
\end{aligned}$$

As a check, we have reproduced this effective action from the off-shell amplitude (up to the four-point level).

Off-shell one-loop four-gluon vertex in $\mathcal{N} = 4$ SYM

In $\mathcal{N} = 4$ SYM the one-loop two - and three - gluon amplitudes vanish because of the finiteness of the theory. The one-loop four-gluon vertex becomes extremely simple: all boundary terms cancel out, and the bulk term involves only the scalar box integral:

$$\Gamma^{a_1 a_2 a_3 a_4} = 4g^4 \mathrm{tr}\,(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) T_8 B(1234) + \mathrm{non-cyclic\ permutations}$$

Here $B(1234)$ is the off-shell scalar box integral with momenta $p_1, \ldots, p_4$, and

$$\begin{aligned} T_8 &= \mathrm{tr}\,(f_1 f_2 f_3 f_4) + \mathrm{tr}\,(f_1 f_2 f_4 f_3) + \mathrm{tr}\,(f_1 f_3 f_2 f_4) \\ &\quad - \frac{1}{4}\mathrm{tr}\,(f_1 f_2)\mathrm{tr}\,(f_3 f_4) - \frac{1}{4}\mathrm{tr}\,(f_1 f_3)\mathrm{tr}\,(f_2 f_4) - \frac{1}{4}\mathrm{tr}\,(f_1 f_4)\mathrm{tr}\,(f_2 f_3) \end{aligned}$$

The tensor T_8 is known from string theory.

Summary and Outlook

- The string-inspired formalism makes it possible to generate form factor decompositions of the N - gluon vertex without analyzing the Ward identities.
- At the one-loop level, the parameter integrals appearing in the form factors for the scalar, spinor and gluon loop cases are all obtained directly from the Bern-Kosower master formula.
- We have carried out this program explicitly for the three- and four-point cases.
- In particular, we have obtained a natural four-point generalization of the Ball-Chiu decomposition, and the corresponding one-loop parameter integrals for the scalar, spinor and gluon loop.
- Main limitation: cannot treat the gluon loop in more general gauges (yet).
- In the abelian case: only 6 instead of 14 tensors.

Feynman-Schwinger parameter integrals (scalar loop)

$$
\begin{aligned}
P^{4}_{0,P} &= (1-2\alpha_1)(1-2\alpha_2)(1-2\alpha_3)(1-2\alpha_4)\,, \\
P^{4}_{0,NP} &= -(1-2\alpha_1)(1-2\alpha_3)(1-2\alpha_2-2\alpha_3)(1-2\alpha_3-2\alpha_4)\,, \\
P^{22}_{0,P} &= (1-2\alpha_2)^2(1-2\alpha_4)^2\,, \\
P^{22}_{0,NP} &= (1-2\alpha_2-2\alpha_3)^2(1-2\alpha_3-2\alpha_4)^2\,, \\
P^{3}_{0,P} &= -(1-2\alpha_1)(1-2\alpha_2)(1-2\alpha_3)(1-2\alpha_2-2\alpha_3)\,, \\
P^{3}_{0,NP} &= (1-2\alpha_2)(1-2\alpha_3)(1-2\alpha_2-2\alpha_3)(1-2\alpha_3-2\alpha_4)\,, \\
P^{2adj}_{0,\mathrm{quart}} &= (1-2\alpha_1)(1-2\alpha_2-2\alpha_3)(1-2\alpha_2)^2\,, \\
P^{2adj}_{0,P} &= (1-2\alpha_1)(1-2\alpha_3)(1-2\alpha_2)^2\,, \\
P^{2adj}_{0,NP} &= -(1-2\alpha_2-2\alpha_3)(1-2\alpha_3-2\alpha_4)(1-2\alpha_2)^2\,, \\
P^{2adj}_{0,C} &= -(1-2\alpha_1)(1-2\alpha_4)(1-2\alpha_2)^2\,, \\
P^{2adj}_{0,Z} &= (1-2\alpha_4)(1-2\alpha_3-2\alpha_4)(1-2\alpha_2)^2\,, \\
P^{2opp}_{0,\mathrm{quart}} &= (1-2\alpha_2)(1-2\alpha_1)(1-2\alpha_2-2\alpha_3)^2\,, \\
P^{2opp}_{0,P} &= -(1-2\alpha_1)(1-2\alpha_3)(1-2\alpha_2-2\alpha_3)^2\,, \\
P^{2opp}_{0,NP} &= -(1-2\alpha_1)(1-2\alpha_3-2\alpha_4)(1-2\alpha_2-2\alpha_3)^2\,.
\end{aligned}
$$