Evaluating multi-loop Feynman integrals using differential equations: automatizing the transformation to a canonical basis

CHRISTOPH MEYER (HU BERLIN)

AG PHENOMENOLOGY OF ELEMENTARY PARTICLE PHYSICS

LOOPS AND LEGS IN QUANTUM FIELD THEORY

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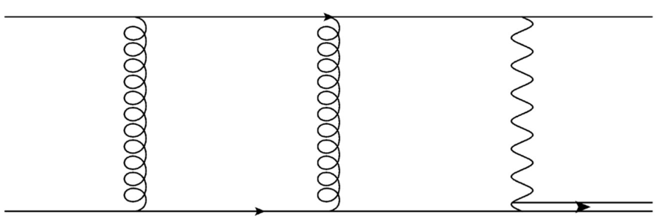
Goal: Compute scalar Feynman integrals

$$I[a_1,\ldots,a_n] = \int \frac{\mathrm{d}^d l_1}{\mathrm{i}\pi^{d/2}} \cdots \frac{\mathrm{d}^d l_L}{\mathrm{i}\pi^{d/2}} \frac{P_{t+1}^{-a_{t+1}} \cdots P_n^{-a_n}}{P_1^{a_1} \cdots P_t^{a_t}}$$

Example: Single top quark production @ NNLO QCD



$$\int \frac{\mathrm{d}^d l_1}{\mathrm{i}\pi^{d/2}} \frac{\mathrm{d}^d l_2}{\mathrm{i}\pi^{d/2}} \frac{[(l_1-p_2)^2]^{-a_8}[(l_2+p_3+p_1)^2]^{-a_9}}{[l_2^2]^{a_1}[l_1^2-m_w^2]^{a_2}[(l_1+p_3)^2]^{a_3}[(l_2+p_2)^2]^{a_4}[(l_1-p_4)^2]^{a_5}[(l_2-p_1)^2]^{a_6}[(l_1+l_2-p_1+p_3)^2]^{a_7}}$$

Kinematics: $p_1^2=0, \quad p_2^2=0, \quad p_3^2=0, \quad p_4^2=m_t^2 \quad s=(p_1+p_2)^2, \quad t=(p_2-p_3)^2$

Turn to dimensionless integrals by using:

$$x_1 = \frac{s}{m_w^2}, \quad x_2 = \frac{t}{m_w^2}, \quad x_3 = \frac{m_t^2}{m_w^2}$$

Existing approach: Method of differential equations

[A. V. Kotikov ‘91; E. Remiddi ‘97]

- Integrals depend on M dimensionless invariants and the dimensional regulator ϵ
- Take total derivative of $\vec{f}$ and express in terms of master integrals

$$\mathrm{d}\vec{f}(\epsilon,\{x_i\}) = \sum_{j=1}^{M} \frac{\partial \vec{f}}{\partial x_j}\mathrm{d}x_j = a(\epsilon,\{x_i\})\vec{f}(\epsilon,\{x_i\})$$

$$a(\epsilon,\{x_i\}) = \sum_{j=1}^{M} a_j(\epsilon,\{x_i\})\mathrm{d}x_j$$

- $a_j(\epsilon,\{x_i\})$ are $m \times m$ matrices of rational functions
- Differential equation determines the master integrals up to integration constants
- Solve differential equation for $\vec{f}(\epsilon,\{x_i\})$ ⇒ in general very hard!

Recent progress: Change of basis to simplify solution

[J.M. Henn '13]

- New basis: $\vec{f} = T(\epsilon, \{x_i\})\vec{f'}$

$$\mathrm{d}\vec{f'} = a'\vec{f'}$$

$$\Rightarrow \qquad a' = T^{-1}aT - T^{-1}\mathrm{d}T$$

- Idea: Use a basis such that the differential equation is in ϵ- *form*:

$$a(\epsilon, \{x_i\}) = \epsilon\, \mathrm{d}\tilde{A} \qquad \text{with} \qquad \tilde{A} = \sum_{l=1}^{N} \tilde{A}_l \log(L_l(\{x_i\}))$$

with $\tilde{A}_l$ being constant $m \times m$ matrices and $L_l(\{x_i\})$ polynomials

➡ Solution in terms of iterated integrals

Integration of the differential equation becomes a combinatorial task!

Current status

- New method has been very successfull: e.g.

Two-loop non-leptonic B decays [G. Bell, T. Huber '14]
Two-loop Bhabha scattering [J. M. Henn, A. V. Smirnov, V. A. Smirnov '14]
Two-loop VV production [J. M. Henn, K. Melnikov, V. A. Smirnov '14; F. Caola, J. M. Henn, K. Melnikov, V. A. Smirnov '14; T. Gehrmann, A. v. Manteuffel, L. Tancredi, E. Weihs '14]
Three-loop ladder boxes [J. M. Henn, V. A. Smirnov '13]
Three-loop $gg \to H$ [M. Höschele, J. Hoff, T. Ueda '14]
Two-loop $H \to Z\gamma$ [R. Bonciani, V. Del Duca, H. Frellesvig, J. M. Henn, F. Moriello, V. A. Smirnov '15; T. Gehrmann, S. Guns, D. Kara '15]
many more...

How to find such a basis?

Special cases solved:

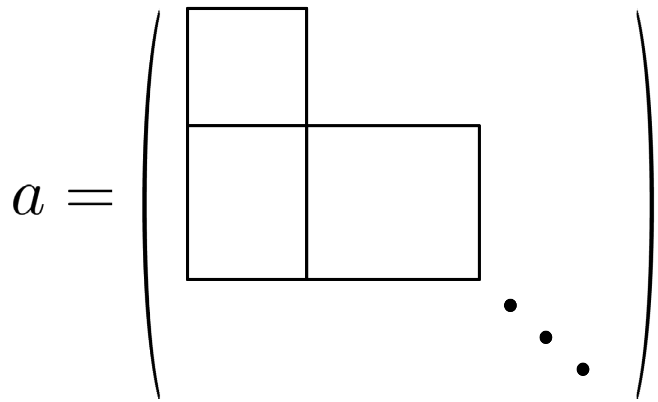
- Algorithms available for case of one variable [R. Lee '14; J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Von Manteuffel, C. Schneider '15]

- $a(\epsilon, \{x_i\})$ linear in ϵ: Magnus/Dyson series approach [M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk, U. Schubert, L. Tancredi '14]

- Diagonal blocks of $a(\epsilon, \{x_i\})$ linear in ϵ : Algorithm partially described [T. Gehrmann, A. von Manteuffel, L. Tancredi, E. Weihs '14]

- Derivative of Masterintegral $=$ Sum of Masterintegrals of the same or lower sectors

➡ Differential Equation in block-triangular form:



➡ Compute transformation recursively:

- Derivative of Masterintegral $=$ Sum of Masterintegrals of the same or lower sectors

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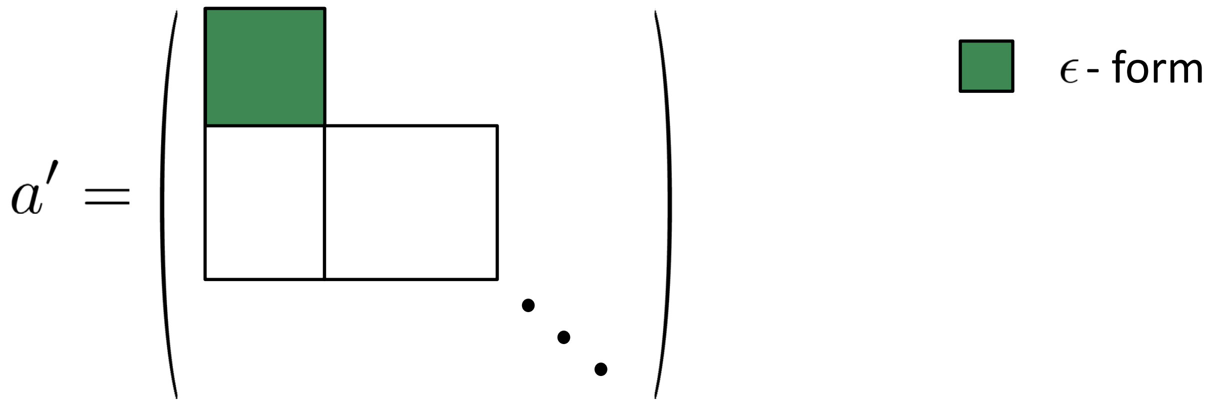


➡ Compute transformation recursively:

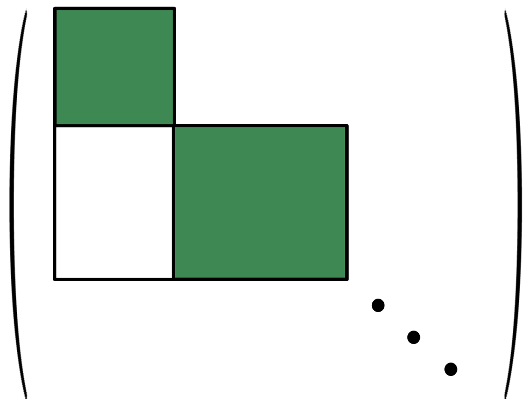
$$\begin{pmatrix} T_1 & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

Structure of the problem: the block triangular form

- Derivative of Masterintegral $=$ Sum of Masterintegrals of the same or lower sectors

➡ Differential Equation in block-triangular form:

$a'' =$



ϵ - form

➡ Compute transformation recursively:

$$\begin{pmatrix} T_1 & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & T_2 \end{pmatrix}$$

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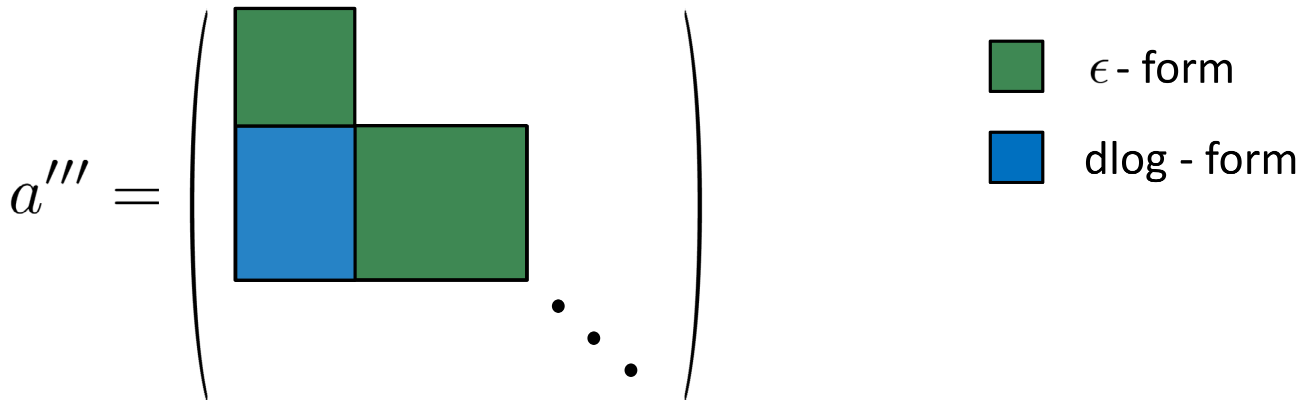
➡ Differential Equation in block-triangular form:



➡ Compute transformation recursively:

$$\begin{pmatrix} T_1 & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ D & \mathbb{I} \end{pmatrix}$$

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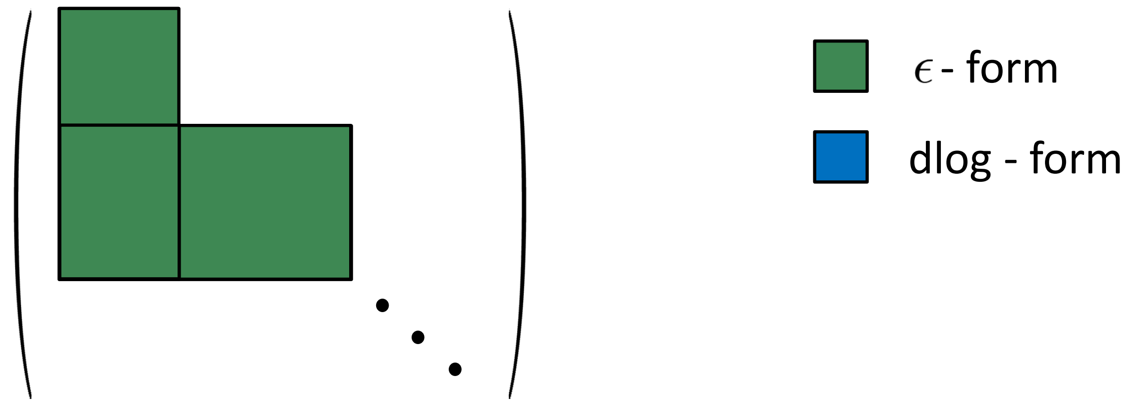
➔ Differential Equation in block-triangular form:

$$a'''' =$$



➔ Compute transformation recursively:

$$\begin{pmatrix} T_1 & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ D & \mathbb{I} \end{pmatrix} \begin{pmatrix} K(\epsilon) & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

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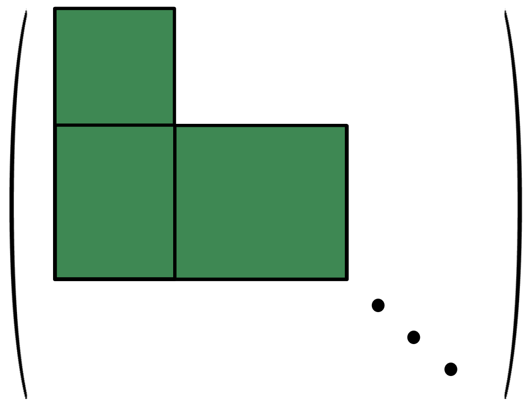
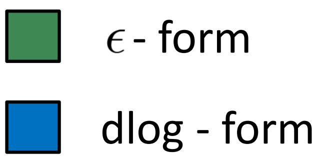
➡ Differential Equation in block-triangular form:

$a'''' =$ 



➡ Compute transformation recursively:

$$\begin{pmatrix} T_1 & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ D & \mathbb{I} \end{pmatrix} \begin{pmatrix} K(\epsilon) & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

Part 1 (T_2) | Part 2 (D) | [R. Lee'14] ($K(\epsilon)$)

Part 1: Transformation law

- Assume $a(\epsilon, \{x_i\})$ to be rational in ϵ and the invariants
- For $a' = \epsilon\,\mathrm{d}\tilde{A}$ the transformation T has to satisfy:

$$\mathrm{d}T - aT + \epsilon T \mathrm{d}\tilde{A} = 0$$

Idea: Solve for T order by order in ϵ

$\mathrm{d}\tilde{A}$ also unknown

Problem: In general T and a have infinite ϵ - expansions:

Simple example:

$$a = \left(\frac{1}{\epsilon - x} + \frac{\epsilon}{x}\right)\mathrm{d}x \xrightarrow{T = (\epsilon - x)^{-1}} \mathrm{d}\tilde{A} = \frac{1}{x}\mathrm{d}x$$

⇒ ϵ - expansion would lead to infinite number of equations

Solution: Find polynomial $h(\epsilon, \{x_i\})$ such that $\hat{a} = ah$ has a finite expansion

In the example: $h(\epsilon, x) = \epsilon - x$

$$\Rightarrow \quad \hat{a} = ah = (\epsilon - x)\left(\frac{1}{\epsilon - x} + \frac{\epsilon}{x}\right)\mathrm{d}x = \left(1 + \frac{\epsilon(\epsilon - x)}{x}\right)\mathrm{d}x$$

finite ϵ - expansion

Observation:

$$\hat{T} = Th = \frac{1}{\epsilon - x}(\epsilon - x) = 1$$

has a finite expansion as well!

This holds in general:

If ϵ - form exists $\Longrightarrow$ Can always find T such that $\hat{T} = Th$ has finite expansion

Part 1: Expansion of the transformation law

In terms of $\hat{T}$ and $\hat{a}$ the transformation law reads:

$$-\hat{T}\mathrm{d}h + h\mathrm{d}\hat{T} - \hat{a}\hat{T} + \epsilon h\hat{T}\mathrm{d}\tilde{A} = 0$$

Expand in ϵ and solve order by order ➡ terminates after a finite number of steps!

Example:

$$\hat{a} = \left(1 - \epsilon + \frac{\epsilon^2}{x}\right)\mathrm{d}x \qquad h = \epsilon - x \qquad \hat{T} = \sum_{n=0}^{n_{\max}} \epsilon^n \hat{T}^{(n)}$$

ϵ^0: $\quad \mathrm{d}\hat{T}^{(0)} = 0 \qquad \hat{T}^{(0)} = \beta^{(0)}$ ← constant

ϵ^1: $\quad \mathrm{d}\hat{T}^{(1)} + \beta^{(0)}\left(\frac{\mathrm{d}x}{x} - \mathrm{d}\tilde{A}\right) = 0 \qquad \hat{T}^{(1)} = 0, \quad \mathrm{d}\tilde{A} = \frac{\mathrm{d}x}{x}$

ϵ^2: $\quad \mathrm{d}\hat{T}^{(2)} = 0 \qquad \hat{T}^{(2)} = 0$

All higher order equations trivial

$$\Rightarrow \quad T = \frac{\hat{T}}{h} = \frac{\beta^{(0)}}{\epsilon - x}$$

Systematic approach: Solve with rational ansatz

- Only interested in rational solutions ⇒ make rational ansatz for Taylor coefficients

$$\hat{T}^{(n)} = \sum_{j=1}^{K} \beta_j^{(n)} r_j(\{x_i\})$$

complex parameters $\beta_j^{(n)}$

set of rational functions $\mathcal{R} = \{r_1(\{x_i\}), \ldots, r_K(\{x_i\})\}$

Note: We have to solve for $\mathrm{d}\tilde{A}$ as well!

- Extract the $L_l(\{x_i\})$ from $\hat{a}$ and make ansatz for $\tilde{A}$

$$\tilde{A} = \sum_{l=1}^{N} \alpha_l \log(L_l(\{x_i\}))$$

complex parameters α_l

⇒ Solve algebraic equations in the parameters α and β

Systematic approach: Solve with rational ansatz

Example reloaded: $$\hat{a} = \left(1 - \epsilon + \frac{\epsilon^2}{x}\right) \mathrm{d}x$$

x is the only irreducible denominator factor in $\hat{a}$

➡ Ansatz for $\mathrm{d}\tilde{A}$, with $L_1 = x$: $\mathrm{d}\tilde{A} = \alpha_1 \frac{\mathrm{d}x}{x}$

Choose set of rational functions: $\mathcal{R} = \left\{1,\, x^{-1}\right\}$

➡ Ansatz for $\hat{T}^{(n)}$: $\hat{T}^{(n)} = \beta_1^{(n)} + \beta_2^{(n)} x^{-1}$

Insert Ansatz:

$$\mathrm{d}\hat{T}^{(1)} + \beta_1^{(0)} \left(\frac{\mathrm{d}x}{x} - \mathrm{d}\tilde{A}\right) = 0 \quad \Longrightarrow \quad \left(-\beta_2^{(1)} x^{-2} + \beta_1^{(0)} (1 - \alpha_1) x^{-1}\right) \mathrm{d}x = 0$$

$$\Longrightarrow \quad \beta_2^{(1)} = 0, \quad \beta_1^{(0)} (1 - \alpha_1) = 0$$

Decomposition of rational functions

How do we choose the ansatz?

Idea: decompose $\hat{a}$ into linear combination of *simple* rational functions

➡ set of *simple* functions ➡ starting point to generate ansatz

How should this decomposition look like?

Univariate case ➡ partial fractioning of denominator polynomial ✓

Multivariate case ➡ simple generalization of partial fractioning fails:

$$\frac{1}{x(x+y)} = \frac{1}{xy} - \frac{1}{y(x+y)}$$

partial fraction w.r.t. x

$$= \frac{1}{xy} - \left[\frac{1}{xy} - \frac{1}{x(x+y)}\right] = \frac{1}{x(x+y)}$$

partial fraction w.r.t. y

➡ Runs into infinite loop!

Leinartas decomposition

Need a more careful generalization:

[E. K. Leinartas '78, A. Raichev '12]

→ **Leinartas decomposition**

Two steps: Nullstellensatz decomposition and algebraic independence decomposition

1. Nullstellensatz decomposition:

Finite set of polynomials $\{f_1, \dots f_m\}$ with no common zero

(weak) Nullstellensatz $\Downarrow$

Exist polynomials $\{h_1, \dots h_m\}$ such that $\sum_{i=1}^{m} h_i f_i = 1$

Example: $\{x+y, 1+x+y\}$ has no common zero $\Longrightarrow$ $h_1 = -1,\ h_2 = 1$

Yields decomposition:

multiplication by one $\sum_{i=1}^{m} h_i f_i = 1$

$$\frac{1}{(x+y)(1+x+y)} = \frac{(-1)(x+y)+(1)(1+x+y)}{(x+y)(1+x+y)} = \frac{-1}{1+x+y} + \frac{1}{x+y}$$

2. Algebraic independence decomposition:

Finite set of algebraically dependent polynomials $\{f_1, \dots f_m\}$

$$\Updownarrow$$

Exists polynomial κ in m variables $\kappa(f_1, \dots, f_m) = 0$

Example:

$\{x,\ y,\ x+y\}$ is algebraically dependent $\Longrightarrow$ $\kappa(Y_1, Y_2, Y_3) = Y_1 + Y_2 - Y_3$

$$\kappa(x, y, x+y) = 0 \qquad \Longrightarrow \qquad \frac{(y) - (x+y)}{-x} = 1$$

Yields decomposition:

$$\frac{1}{xy(x+y)} = \frac{(y) - (x+y)}{-x}\frac{1}{xy(x+y)} = -\frac{1}{x^2(x+y)} + \frac{1}{x^2 y}$$

Leinartas decomposition: Apply step 1 and step 2 repeatedly

$\Longrightarrow$ algebraically independent denominator polynomials with no common zero

Constraining the ansatz further: Information from the trace

Additional information:

$$\det(T) = C(\epsilon) \exp\left(\int_\gamma \mathrm{Tr}\left[a^{(0)}\right]\right)$$

$$\mathrm{Tr}[\mathrm{d}\tilde{A}] = \mathrm{Tr}[a^{(1)}]$$

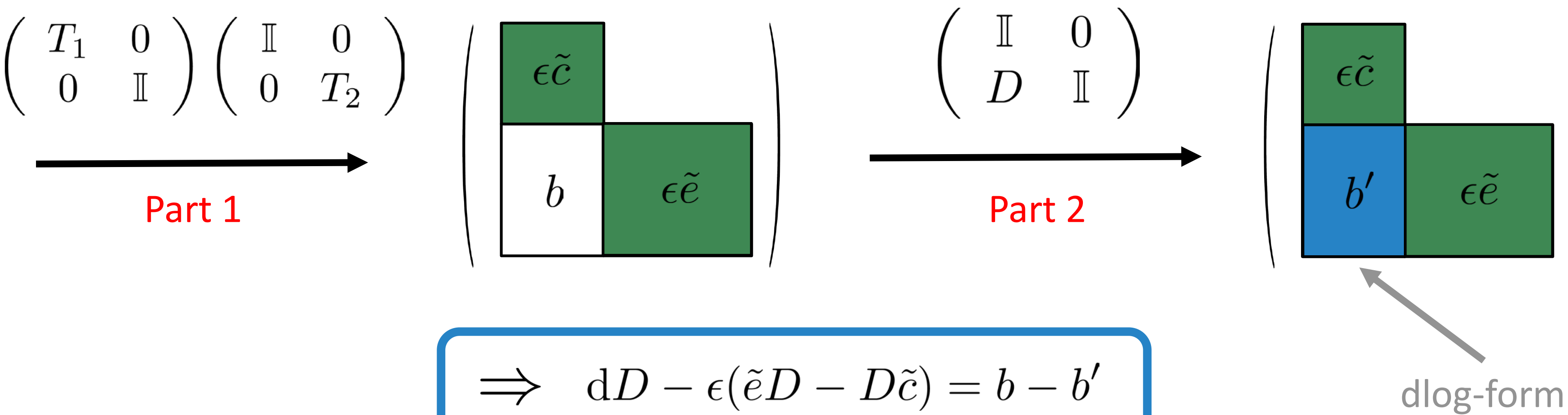
Example: $a = \left(\frac{1}{1-x} + \frac{\epsilon}{x}\right) \mathrm{d}x \quad \Rightarrow \quad \mathrm{Tr}[a^{(0)}] = \frac{\mathrm{d}x}{1-x}, \quad \mathrm{Tr}[a^{(1)}] = \frac{\mathrm{d}x}{x}$

$$\Rightarrow \quad \det(T) = C(\epsilon)\frac{1}{1-x}, \quad \mathrm{Tr}[\mathrm{d}\tilde{A}] = \frac{\mathrm{d}x}{x}$$

Since we have 1-dim. sector: $T = C(\epsilon)\frac{1}{1-x}, \quad \mathrm{d}\tilde{A} = \frac{\mathrm{d}x}{x}$

- Fully determines transformation of 1-dim. sectors (up to irrelevant constant)
- Formulas also hold for higher dimensional sectors
- Provide useful information for the ansatz

Remember:



$$\Rightarrow \quad \mathrm{d}D - \epsilon(\tilde{e}D - D\tilde{c}) = b - b'$$

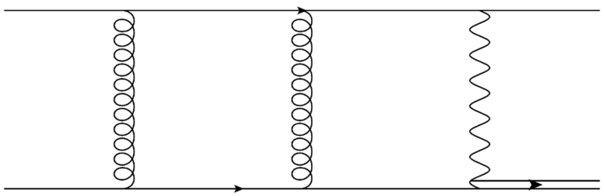
Analogous strategy:

- Factorize for finite expansion
- Extract rational ansatz for D from b
- b' in dlog-form ➡ only alphabet needed for ansatz

All parts completely automated, implementation in *Wolfram Mathematica*

Example:

Apply current implementation to



- Mathematica implementation needs 65 min. to compute ϵ- form on desktop computer
- Resulting ϵ- form contains the following set of letters:

$$\mathcal{A} = \{x_1, x_2, x_1+x_2, x_1-x_3, x_2-x_3, x_1+x_2-x_3, 1+x_1+x_2-x_3, -1+x_3, x_3, -1-x_2+x_3, x_1(-1+x_3)+(1+x_2-x_3)x_3\}$$

- Sample Output for the highest sector:

$$\tilde{A}_{\text{top}} = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \log(x_1) + \begin{pmatrix} 0 & -6 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \log(1+x_1+x_2-x_3) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \log(x_2-x_3)$$
$$+ \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \log(x_2) + \begin{pmatrix} -2 & -4 & 2 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{pmatrix} \log(-1-x_2+x_3) + \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \log(x_1+x_2-x_3)$$

Conclusions and Outlook

Conclusions:

- New algorithm applicable to the general case of rational $a(\epsilon, \{x_i\})$
- Implemented in *Wolfram Mathematica*
- Tested for non-trivial examples

Outlook:

- Study further applications
- Better understanding of how to automatically choose ansatz
- Extension to letters with square roots

Note: Not all Feynman integrals admit an ϵ- form

Thank You!