Loops and legs master integrals for splitting functions from differential equations in QCD

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Loops and Legs in Quantum Field Theory
Leipzig (Germany)
26 April 2016

Splitting functions in perturbative QCD

- process independent quantities
- govern collinear evolution of hard scattering processes with hadrons
  - *parton distribution functions* in the *initial state* (a space-like hard scale $Q^2 = -q^2 > 0$)

$$\frac{d}{d \ln q^2} f_a^h(x, q^2) = \int_x^1 \frac{dz}{z} P_{ab}^S\left(z, \alpha_s(q^2)\right) f_b^h\left(\frac{x}{z}, q^2\right)$$

  - *fragmentation functions* in the *final state* (a time-like hard scale $Q^2 = q^2 > 0$)

$$\frac{d}{d \ln q^2} D_a^h(x, q^2) = \int_x^1 \frac{dz}{z} P_{ba}^T\left(z, \alpha_s(q^2)\right) D_b^h\left(\frac{x}{z}, q^2\right)$$

- expanded in perturbation theory, e.g.,

$$P_{ab}^T(x, \alpha_s) = \frac{\alpha_s}{4\pi} P_{ab}^{T(0)}(x) + \left(\frac{\alpha_s}{4\pi}\right)^2 P_{ab}^{T(1)}(x) + \left(\frac{\alpha_s}{4\pi}\right)^3 \boldsymbol{P_{ab}^{T(2)}(x)} + \dots$$

Analytical expression for $P_{qg}^{T(2)}$ is known with a small uncertainty.

Splitting functions at NLO

Space-like

- Axial gauge
  - Principal Value
    Curci Furmanski Petronzio '80
    Ellis Vogelsang '98
  - Mandelstam-Leibbrandt
    Bassetto Heinrich Kunszt Vogelsang '98
  - New Principal Value
    OG Jadach Skrzypek Kusina '14
- Feynman gauge
  - x space
    Floratos Kounnas Lacaze '81
  - Mellin space Moch Vermaseren '99

Time-like

- Axial gauge
  - Principal Value
    Furmanski Petronzio '80
- Feynman gauge
  - x space
    Floratos Kounnas Lacaze '81
    Rijken, van Neerven '96
  - Mellin space Mitov Moch '06
- analytic cont. $-q^2 \to +q^2$
  Stratmann Vogelsang '96
  Blumlein, Ravindran, van Neerven '00
  Moch Vogt '07
  - Drell-Yan-Levy relation
    Drell Levy Yan '70
  - Gribov-Lipatov relation
    Gribov Lipatov '72

Splitting functions at NNLO

Space-like

- Axial gauge*
- Feynman gauge
  - Mellin space

    Moch Vermaseren Vogt '04

Time-like

- Axial gauge*
- analytic continuation $P_{ab}^{S(2)} \to P_{ba}^{T(2)}$
  - NNLO non-singlet

    Mitov Moch Vogt '06
  - NNLO singlet $P_{qq}^{T(2)}$ and $P_{gg}^{T(2)}$

    Moch Vogt '07
  - NNLO singlet $P_{gq}^{T(2)}$ and $\boldsymbol{P_{qg}^{T(2)}}$

    Almasy Moch Vogt '11

* Spurious poles of axial gauge are too complex at NNLO.

— How can we find a missing contribution to $\boldsymbol{P_{qg}^{T(2)}}$ splitting function?
— Well, let us try to calculate it explicitly!

In this talk

- I will briefly discuss how to extract $P_{qg}^{T(NNLO)}$ from $e^+e^- \to 3$ jets at NNLO.
- We will see that to solve this (and many other problems in QFT) we need tools for automatic calculation of master integrals.
- I will introduce `Fuchsia` — program for reducing differential equations for master integrals to canonical form based on the Lee algorithm.
- We will consider $P_{gq}^{T(NLO)}$ as a demonstration example.
- At the very end I will discuss current status of the `Fuchsia` project.

$e^+e^- \to \gamma^* \to 3$ jets

$$\frac{d^2\sigma}{dx\, d\cos\theta} = \frac{3}{8}(1+\cos^2\theta)\,\mathcal{F}_T(x,\epsilon) + \frac{3}{4}\sin^2\theta\,\mathcal{F}_L(x,\epsilon) + \frac{3}{4}\cos\theta\,\mathcal{F}_A(x,\epsilon)$$

- Transverse fragmentation functions

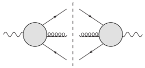
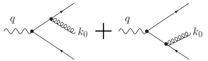
$$\mathcal{F}_T(x,\epsilon) \simeq \left(x^2 g^{\mu\nu} + 4k_0^\mu k_0^\nu\right) W_{\mu\nu}(x,\epsilon), \qquad x = 2q\cdot k_0$$

- Hadronic tensor

$$W_{\mu\nu}(x,\epsilon) \simeq \int \mathrm{d}^m\mathrm{PS}^{(n)}\; M_\mu^{(n)}\; M_\nu^{(n)*}$$

where $\mathrm{d}^m\mathrm{PS}^{(n)}$ is n-particle phase-space in $m = 4-2\epsilon$ dimensions and $M_\mu^{(n)}$ is amplitude of the process

Example: LO contribution

$\mathcal{F}_T^{(1)}(x,\epsilon) \equiv$  $\simeq \left(x^2 g^{\mu\nu} + 4k_0^\mu k_0^\nu\right)\int \mathrm{d}^m\mathrm{PS}^{(3)}\;\Big($  $\Big)^2$

$$\mathrm{d}^m\mathrm{PS}^{(3)} = \underbrace{\mathrm{d}^m k_0\,\delta^+(k_0^2)}_{\text{gluon}}\,\underbrace{\mathrm{d}^m k_1\,\delta^+(k_1^2)}_{\text{quark}}\,\underbrace{\mathrm{d}^m k_2\,\delta^+(k_2^2)}_{\text{anti-quark}}\;\;\delta\left(x-2q\cdot k_0\right)\;\delta^m(q-k_0-k_1-k_2)$$

Mass factorization for $\mathcal{F}_T(x, \epsilon)$

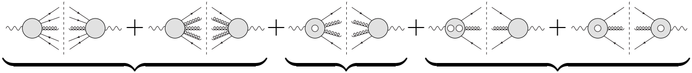
Vermaseren, Vogt, Moch '05

We can extract splitting functions on the rhs ($P_{gq}^{(0)}, P_{gq}^{(1)}, P_{gq}^{(2)}$) when we know the lhs of the following expressions

- $$\mathcal{F}_T^{(1)}(x,\epsilon) = \frac{1}{\epsilon} \boldsymbol{P_{gq}^{(0)}(x)} + c_{T,g}^{(1)}(x) + \epsilon\, a_{T,g}^{(1)}(x) + \epsilon^2 b_{T,g}^{(1)}(x)$$
- $$\mathcal{F}_T^{(2)}(x,\epsilon) = \frac{1}{\epsilon^2}\left\{\frac{1}{2}P_{gi}^{(0)}P_{iq}^{(0)} + \frac{1}{2}\beta_0 P_{gq}^{(0)}\right\} - \frac{1}{\epsilon}\left\{\frac{1}{2}\boldsymbol{P_{gq}^{(1)}} + P_{gi}^{(0)}c_i^{(1)}\right\} + \left\{c_g^{(2)} - P_{gi}^{(0)}a_i^{(1)}\right\} + \epsilon\left\{a_g^{(2)} - P_{gi}^{(0)}b_i^{(1)}\right\}$$
- $$\begin{aligned}\mathcal{F}_T^{(3)}(x,\epsilon) = {} & \frac{1}{\epsilon^3}\left\{\frac{1}{6}P_{gi}^{(0)}P_{ij}^{(0)}P_{jq}^{(0)} + \frac{1}{2}\beta_0 P_{gi}^{(0)}P_{iq}^{(0)} + \frac{1}{3}\beta_0^2 P_{gq}^{(0)}\right\} \\ & + \frac{1}{\epsilon^2}\left\{\frac{1}{6}P_{gi}^{(0)}P_{iq}^{(1)} + \frac{1}{3}P_{gi}^{(1)}P_{iq}^{(0)} + \frac{1}{3}\beta_1 P_{gq}^{(0)} + \frac{1}{2}P_{gi}^{(0)}P_{ij}^{(0)}c_j^{(1)} + \beta_0\left(\frac{1}{3}P_{gq}^{(1)} + \frac{1}{2}P_{gi}^{(0)}c_i^{(1)}\right)\right\} \\ & - \frac{1}{\epsilon}\left\{\frac{1}{3}\boldsymbol{P_{gq}^{(2)}} + \frac{1}{2}P_{gi}^{(1)}c_i^{(1)} + P_{gi}^{(0)}c_i^{(2)} - \frac{1}{2}P_{gi}^{(0)}P_{ij}^{(0)}a_j^{(1)} - \frac{1}{2}\beta_0 P_{gi}^{(0)}a_i^{(1)}\right\} \\ & + \left\{c_g^{(3)} - P_{gi}^{(0)}a_i^{(2)} - \frac{1}{2}P_{gi}^{(1)}a_i^{(1)} + \frac{1}{2}P_{gi}^{(0)}P_{ij}^{(0)}b_j^{(1)} + \frac{1}{2}\beta_0 P_{gi}^{(0)}b_i^{(1)}\right\}\end{aligned}$$

Feynman integrals for $\mathcal{F}_T(x,\epsilon)$

$\mathcal{F}_T^{(NNLO)}(x,\epsilon) =$



$$\mathrm{d}^m\mathrm{PS}^{VV} = \underbrace{\mathrm{d}^m l_1}_{\text{loop \#1}} \; \underbrace{\mathrm{d}^m l_2}_{\text{loop \#2}} \; \underbrace{\mathrm{d}^m k_0 \delta(k_0^2)}_{\text{leg \#1}} \; \underbrace{\mathrm{d}^m k_1 \delta(k_1^2)}_{\text{leg \#2}} \; \underbrace{\mathrm{d}^m k_2 \delta(k_2^2)}_{\text{leg \#3}}$$

$$\times\, \delta^m(q - k_0 - k_1 - k_2)\, \delta\,(x - 2q\cdot k_0)$$

Perfectly suits for **IBP reduction**:

- Loops and Legs integrals are reduced simultaneously
- $\delta(x - 2q\cdot k_0)$ and $\delta(k_i^2)$ are replaced by cut propagators, i.e., according to Cutkosky's rules
- `LiteRed` and `Reduze2` have support for cut propagators
- define system of differential equations in x-space

Fuchsia project

Gituliar, Magerya '16 (in preparation)

Fuchsia is a program for reducing differential equations for master integrals to the **canonical form** Henn '13:

- based on the **Lee algorithm** Lee '14
- open-source and free (no proprietary software dependencies)
- implemented in SageMath (Python, Maxima, GiNaC)

The idea is to find a **rational transformation** in three reduction steps:

1. **Fuchsification** decrease Poincaré rank to 0 at all singular points (i.e. get rid of irregular singularities)
2. **Normalization**: balance eigenvalues to $n\epsilon$ form
3. **Factorization**: reduce to canonical form

Notation and definitions

Let us consider a system of ODEs

$$\frac{\mathrm{d}\bar{f}}{\mathrm{d}x} = \mathbb{A}(x,\epsilon)\,\bar{f},$$

where $\bar{f}(x,\epsilon)$ is a vector of n unknown functions (e.g., master integrals).

$$\mathbb{A}(x,\epsilon) = \sum_k \frac{1}{(x-x_k)^{1+p_k}} \sum_{i=0}^{\infty} A_{ik}(\epsilon)(x-x_k)^i$$

For any system we can define an integer number

$$\boxed{m_{x=x_k}(\mathbb{A}) = p_k \geq 0}$$

as a **Poincaré rank** of $\mathbb{A}$ at $x = x_k$.

For $\mathbb{B} = \begin{pmatrix} \frac{\epsilon}{x} & 0 \\ -\frac{\epsilon}{x^2} & \frac{\epsilon}{1+x} \end{pmatrix}$ we have $m_{x=0} = 1$ and $m_{x=-1} = 0$.

Step I: Fuchsification

We say that such matrix has **Fuchsian form**[1] if its Poincaré rank is 0 at every singular point (including ∞).

For example, we can transform

$$\mathbb{B} = \begin{pmatrix} \frac{\epsilon}{x} & 0 \\ -\frac{\epsilon}{x^2} & \frac{\epsilon}{1+x} \end{pmatrix} \text{ to } \begin{pmatrix} \frac{\epsilon}{x} & 0 \\ -\frac{1}{x} & \frac{\epsilon}{1+x} + \frac{1}{x} \end{pmatrix} \text{ with } m(\mathbb{B}) = 0 \text{ at any point.}$$

Not every system can be transformed to the Fuchsian form, however

- due to the analyticity of S-matrix; and
- structure of the Feynman integrals

every system for master integrals should, in principle, be reducible to Fuchsian form.

`Fuchsia` **finds Fuchian form and transformation matrix of rational functions by analyzing Jordan form of the input matrix.**

[1] after German mathematician Lazarus Fuchs (1833–1902)

Example: differential equations for $P_{gq}^{T(1)}$ at NLO

Input

$$
\begin{pmatrix}
\frac{(2\epsilon-1)(2x-1)}{x(1-x)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3\epsilon-2}{x(1-x)} & \frac{1-3\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{(2\epsilon-1)}{\epsilon^2 x(1-x)(x+1)} & 0 & \frac{1-6\epsilon}{x+1} & \frac{2}{x+1} & 0 & 0 & 0 & 0 \\
\frac{(2\epsilon-1)}{\epsilon x^2(x+1)} & -\frac{(2\epsilon-1)(3\epsilon-1)}{x^2} & \frac{2(6\epsilon-1)}{x(x+1)} & \frac{2\epsilon(x^2+3x-2)}{(1-x)x(x+1)} & 0 & 0 & 0 & 0 \\
\frac{2(x^2+4x+1)}{\epsilon^2(1-x)x^3(x+1)^3} & \frac{2(2\epsilon-1)(x-1)}{\epsilon x^3(x+1)^2} & \frac{2(6\epsilon-1)(x-1)}{x^2(x+1)^3} & \frac{4(x^2+1)}{x^2(x+1)^3} & -\frac{(2\epsilon+1)(2x+1)}{x(x+1)} & 0 & 0 & 0 \\
0 & -\frac{(2\epsilon-1)}{\epsilon(1-x)x} & 0 & 0 & 0 & \frac{2\epsilon}{1-x} & 0 & 0 \\
-\frac{4}{\epsilon^2(1-x)^3x^3(x+1)} & -\frac{2(2\epsilon-1)(x-2)}{\epsilon(1-x)^2x^3} & -\frac{2(6\epsilon-1)}{x^2(1-x)(x+1)} & \frac{4(x^2+1)}{(1-x)^2x^2(x+1)} & 0 & -\frac{4\epsilon}{(1-x)^2x} & \frac{(2\epsilon+1)(2x-1)}{(1-x)x} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(2\epsilon+1)(2x-1)}{(1-x)x}
\end{pmatrix}
$$

Output

$$
\begin{pmatrix}
\frac{(2\epsilon-1)(2x-1)}{x(1-x)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{x(1-x)} & \frac{1-3\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{1-x} & \frac{1+(2\epsilon-1)x}{x(1-x)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{(1+2\epsilon)(2x-1)}{x(1-x)} & 0 & 0 & 0 & 0 \\
\frac{\epsilon}{x(1+x)} & \frac{\epsilon^2}{x} & 0 & 0 & \frac{(2\epsilon-1)x^2+6\epsilon x-4\epsilon+1}{x(1-x)(x+1)} & \frac{\epsilon}{x(1+x)} & 0 & 0 \\
\frac{2(6\epsilon-1)}{(1-x)(1+x)} & 0 & 0 & 0 & \frac{4(6\epsilon-1)}{1+x} & \frac{1-2(3\epsilon-1)x}{x(1+x)} & 0 & 0 \\
\frac{(14\epsilon-3)x+22\epsilon-3}{2\epsilon(1-x)x(1+x)} & \frac{\epsilon x-5\epsilon+2}{x(1-x)} & \frac{4\epsilon^2}{x} & 0 & \frac{(2\epsilon-3)x^2-18\epsilon+8x+11}{\epsilon x(1-x)(1+x)} & \frac{(4\epsilon-3)x^2-2(2\epsilon-3)x+9}{4\epsilon x(1-x)(1+x)} & \frac{(4\epsilon-1)x-2\epsilon}{x(1-x)} & \frac{4\epsilon+x-3}{2x(1-x)} \\
\frac{4}{x(1-x)(1+x)} & \frac{4\epsilon}{x} & 0 & 0 & \frac{8(x^2+2x-1)}{x(1-x)(1+x)} & \frac{x+3}{x(1+x)} & 0 & \frac{(2\epsilon+1)(2x+1)}{x(1+x)}
\end{pmatrix}
$$

Step II: Normalization

We say that matrix $\mathbb{A}(\boldsymbol{x},\boldsymbol{\epsilon})$ **is normalized** if eigenvalues of all its residues have form $m\epsilon$, where m is some number.

We assume that initial eigenvalues have form $n+m\epsilon$, where ***n* is integer**

A key idea for Lee's normalization algorithm is a **balance transformation** between two points x_1 and x_2

$$\mathbb{T}(x)=\mathcal{B}(\mathbb{P},x_1,x_2;x)=1-\mathbb{P}(\epsilon)+\frac{x-x_2}{x-x_1}\mathbb{P}(\epsilon)$$

where $\mathbb{P}(\epsilon)$ is some projector matrix, i.e. $\mathbb{P}^2=\mathbb{P}$.
We choose

- points x_1 and x_2 by analyzing eigenvalues
- projector $\mathbb{P}(\epsilon)$ by analyzing eigenvectors

Example: eigenvalues for x_1 and x_2:

$x_1=0:$

$[1-4\epsilon,\ 1-3\epsilon,\ -1-2\epsilon,\ -2\epsilon,\ 1-2\epsilon,\ 1-2\epsilon,\ 1,\ 1]$

$x_2=1:$

$[-1-2\epsilon,\ -2\epsilon,\ -2\epsilon,\ 1-2\epsilon,\ 1-2\epsilon,\ 0,\ 0,\ 0]$

Input

$$
\begin{pmatrix}
\frac{(2\epsilon-1)(2x-1)}{x(1-x)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{x(1-x)} & \frac{1-3\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{1-x} & \frac{1+(2\epsilon-1)x}{x(1-x)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{(1+2\epsilon)(2x-1)}{x(1-x)} & 0 & 0 & 0 & 0 \\
\frac{\epsilon}{x(1+x)} & \frac{\epsilon^2}{x} & 0 & 0 & \frac{(2\epsilon-1)x^2+6\epsilon x-4\epsilon+1}{x(1-x)(x+1)} & \frac{\epsilon}{x(1+x)} & 0 & 0 \\
\frac{2(6\epsilon-1)}{(1-x)(1+x)} & 0 & 0 & 0 & \frac{4(6\epsilon-1)}{1+x} & \frac{1-2(3\epsilon-1)x}{x(1+x)} & 0 & 0 \\
\frac{(14\epsilon-3)x+22\epsilon-3}{2\epsilon(1-x)x(1+x)} & \frac{\epsilon x-5\epsilon+2}{x(1-x)} & \frac{4\epsilon^2}{x} & 0 & \frac{(2\epsilon-3)x^2-18\epsilon+8x+11}{\epsilon x(1-x)(1+x)} & \frac{(4\epsilon-3)x^2-2(2\epsilon-3)x+9}{4\epsilon x(1-x)(1+x)} & \frac{(4\epsilon-1)x-2\epsilon}{x(1-x)} & \frac{4\epsilon+x-3}{2x(1-x)} \\
\frac{4}{x(1-x)(1+x)} & \frac{4\epsilon}{x} & 0 & 0 & \frac{8(x^2+2x-1)}{x(1-x)(1+x)} & \frac{x+3}{x(1+x)} & 0 & \frac{(2\epsilon+1)(2x+1)}{x(1+x)}
\end{pmatrix}
$$

Output

$$
\begin{pmatrix}
\frac{2\epsilon}{1-x}-\frac{2\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{x} & -\frac{3\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{1-x} & \frac{2\epsilon}{1-x} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2\epsilon}{1-x}-\frac{2\epsilon}{x} & 0 & 0 & 0 & 0 \\
\frac{3}{x} & \frac{\epsilon}{x} & 0 & 0 & \frac{2\epsilon}{1-x}+\frac{4\epsilon}{x} & -\frac{1}{x} & 0 & 0 \\
\frac{6\epsilon}{1+x}+\frac{6\epsilon}{1-x}+\frac{16\epsilon}{x} & \frac{8\epsilon^2}{x}-\frac{4\epsilon^2}{1+x} & 0 & 0 & \frac{12\epsilon^2}{1+x}+\frac{12\epsilon^2}{1-x}+\frac{32\epsilon^2}{x} & -\frac{2\epsilon}{1+x}-\frac{8\epsilon}{x} & 0 & 0 \\
\frac{4}{x}-\frac{2}{1-x} & \frac{4\epsilon}{1-x}+\frac{8\epsilon}{x} & \frac{4\epsilon^2}{1-x}-\frac{4\epsilon^2}{x} & 0 & \frac{16\epsilon}{x}-\frac{4\epsilon}{1-x} & -\frac{3}{x} & \frac{2\epsilon}{1-x}-\frac{2\epsilon}{x} & 0 \\
\frac{6}{1+x}-\frac{8}{x} & \frac{4\epsilon}{1+x}-\frac{4\epsilon}{x} & 0 & 0 & \frac{8\epsilon}{1+x}-\frac{4\epsilon}{1-x}-\frac{16\epsilon}{x} & \frac{3}{x}+\frac{2}{1+x} & 0 & -\frac{2\epsilon}{1+x}-\frac{2\epsilon}{x}
\end{pmatrix}
$$

Output

$$\begin{pmatrix}
\frac{2\epsilon}{1-x}-\frac{2\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{x} & -\frac{3\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{1-x} & \frac{2\epsilon}{1-x} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2\epsilon}{1-x}-\frac{2\epsilon}{x} & 0 & 0 & 0 & 0 \\
\frac{3}{x} & \frac{\epsilon}{x} & 0 & 0 & \frac{2\epsilon}{1-x}+\frac{4\epsilon}{x} & -\frac{1}{x} & 0 & 0 \\
\frac{6\epsilon}{1+x}+\frac{6\epsilon}{1-x}+\frac{16\epsilon}{x} & \frac{8\epsilon^2}{x}-\frac{4\epsilon^2}{1+x} & 0 & 0 & \frac{12\epsilon^2}{1+x}+\frac{12\epsilon^2}{1-x}+\frac{32\epsilon^2}{x} & -\frac{2\epsilon}{1+x}-\frac{8\epsilon}{x} & 0 & 0 \\
\frac{4}{x}-\frac{2}{1-x} & \frac{4\epsilon}{1-x}+\frac{8\epsilon}{x} & \frac{4\epsilon^2}{1-x}-\frac{4\epsilon^2}{x} & 0 & \frac{16\epsilon}{x}-\frac{4\epsilon}{1-x} & -\frac{3}{x} & \frac{2\epsilon}{1-x}-\frac{2\epsilon}{x} & 0 \\
\frac{6}{1+x}-\frac{8}{x} & \frac{4\epsilon}{1+x}-\frac{4\epsilon}{x} & 0 & 0 & \frac{8\epsilon}{1+x}-\frac{4\epsilon}{1-x}-\frac{16\epsilon}{x} & \frac{3}{x}+\frac{2}{1+x} & 0 & -\frac{2\epsilon}{1+x}-\frac{2\epsilon}{x}
\end{pmatrix}$$

Eigenvalues before and after normalization:

$x = 0$:

$[1-4\epsilon,\ 1-3\epsilon,\ -1-2\epsilon,\ -2\epsilon,\ 1-2\epsilon,\ 1-2\epsilon,\ 1,\ 1] \to [-4\epsilon, -3\epsilon, -2\epsilon, -2\epsilon, -2\epsilon, -2\epsilon,\ 0,\ 0]$

$x = 1$:

$[-1-2\epsilon,\ -2\epsilon,\ -2\epsilon,\ 1-2\epsilon,\ 1-2\epsilon,\ 0,\ 0,\ 0] \to [-2\epsilon, -2\epsilon, -2\epsilon, -2\epsilon, -2\epsilon, 0, 0, 0]$

$x = -1$:

$[1-2\epsilon,\ 1-2\epsilon,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0] \to [-2\epsilon, -2\epsilon, 0, 0, 0, 0, 0, 0]$

$x = \infty$:

$[-1+2\epsilon, -1+2\epsilon, -1+3\epsilon, -2+4\epsilon, -2+4\epsilon, -1+4\epsilon, 1+4\epsilon, -2+6\epsilon] \to [2\epsilon, 2\epsilon, 3\epsilon, 4\epsilon, 4\epsilon, 4\epsilon, 4\epsilon, 6\epsilon]$

Step III: Factorization

Now we can find an x-independent transformation $\mathbb{T}(\epsilon)$ for any point $x = x_k$ such that

$$\mathbb{T}^{-1}(\epsilon)\mathbb{A}_k(\epsilon)\mathbb{T}(\epsilon) = \epsilon\mathbb{C}_k$$

Since matrix $\mathbb{C}_k$ is constant, for every residue of $\mathbb{A}$ we write

$$\frac{\mathbb{T}^{-1}(\epsilon)\mathbb{A}_k(\epsilon)\mathbb{T}(\epsilon)}{\epsilon} = \mathbb{C}_k = \frac{\mathbb{T}^{-1}(\mu)\mathbb{A}_k(\mu)\mathbb{T}(\mu)}{\mu}, \quad \mu = \text{any number}$$

We treat components of $\mathbb{T}$ as unknown variables and solve linear system of equations for them. That gives unknown transformation $\mathbb{T}(\epsilon)$.

Input

$$
\begin{pmatrix}
\frac{2\epsilon}{1-x}-\frac{2\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{x} & -\frac{3\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{1-x} & \frac{2\epsilon}{1-x} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2\epsilon}{1-x}-\frac{2\epsilon}{x} & 0 & 0 & 0 & 0 \\
\frac{3}{x} & \frac{\epsilon}{x} & 0 & 0 & \frac{2\epsilon}{1-x}+\frac{4\epsilon}{x} & -\frac{1}{x} & 0 & 0 \\
\frac{6\epsilon}{1+x}+\frac{6\epsilon}{1-x}+\frac{16\epsilon}{x} & \frac{8\epsilon^2}{x}-\frac{4\epsilon^2}{1+x} & 0 & 0 & \frac{12\epsilon^2}{1+x}+\frac{12\epsilon^2}{1-x}+\frac{32\epsilon^2}{x} & -\frac{2\epsilon}{1+x}-\frac{8\epsilon}{x} & 0 & 0 \\
\frac{4}{x}-\frac{2}{1-x} & \frac{4\epsilon}{1-x}+\frac{8\epsilon}{x} & \frac{4\epsilon^2}{1-x}-\frac{4\epsilon^2}{x} & 0 & \frac{16\epsilon}{x}-\frac{4\epsilon}{1-x} & -\frac{3}{x} & \frac{2\epsilon}{1-x}-\frac{2\epsilon}{x} & 0 \\
\frac{6}{1+x}-\frac{8}{x} & \frac{4\epsilon}{1+x}-\frac{4\epsilon}{x} & 0 & 0 & \frac{8\epsilon}{1+x}-\frac{4\epsilon}{1-x}-\frac{16\epsilon}{x} & \frac{3}{x}+\frac{2}{1+x} & 0 & -\frac{2\epsilon}{1+x}-\frac{2\epsilon}{x}
\end{pmatrix}
$$

Output

$$
\boxed{\epsilon}
\begin{pmatrix}
\frac{2}{1-x}-\frac{2}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{19x} & -\frac{3}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{19(1-x)} & \frac{2}{1-x} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{1-x}-\frac{2}{x} & 0 & 0 & 0 & 0 \\
\frac{3}{x} & -\frac{19}{x} & 0 & 0 & \frac{2}{1-x}+\frac{4}{x} & -\frac{1}{x} & 0 & 0 \\
\frac{6}{1+x}+\frac{6}{1-x}+\frac{16}{x} & \frac{76}{1+x}-\frac{152}{x} & 0 & 0 & \frac{12}{1+x}+\frac{12}{1-x}+\frac{32}{x} & -\frac{2}{1+x}-\frac{8}{x} & 0 & 0 \\
\frac{2}{19(1-x)}-\frac{4}{19x} & \frac{4}{(1-x)}+\frac{8}{x} & \frac{76}{x}-\frac{76}{1-x} & \frac{4}{19(1-x)}-\frac{16}{19x} & 0 & \frac{3}{19x} & \frac{2}{1-x}-\frac{2}{x} & 0 \\
\frac{8}{19x}-\frac{6}{19(1+x)} & \frac{4}{1+x}-\frac{4}{x} & 0 & 0 & \frac{4}{19(1-x)}+\frac{16}{19x}-\frac{8}{19(1+x)} & \frac{2}{19(1+x)}-\frac{3}{19x} & 0 & -\frac{2}{1+x}-\frac{2}{x}
\end{pmatrix}
$$

Summary

Fuchsia — a tool for reducing differential equations for master integrals to canonical form

- all main algorithms from Lee '14
- open-source and free (SageMath: Python, Maxima, GiNaC)

A little bit of optimization and we are ready for release!
Please, send us some examples of your systems.

Stay tuned we will release soon
Thank you!