The Pentabox Master Integrals with the Simplified Differential Equations approach

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INPP, NCSR "Demokritos"

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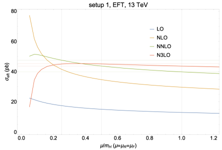


Figure 8: The dependence of the cross-section on a common renormalization and factorization scale $\mu = \mu_F = \mu_R$.

		$\Delta^{scale}_{EFT,k}$
LO	$(k=0)$	$\pm 14.8\%$
NLO	$(k=1)$	$\pm 16.6\%$
NNLO	$(k=2)$	$\pm 8.8\%$
N^3LO	$(k=3)$	$\pm 1.9\%$

Table 5: Scale variation of the cross-section as defined in eq. (3.11) for a common renormalization and factorization scale $\mu = \mu_F = \mu_R$.

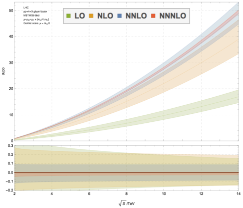
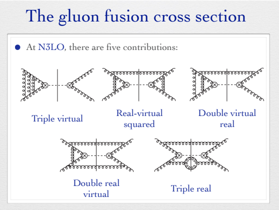


FIG. 3: The gluon fusion cross-section at all perturbative orders through N^3LO in the scale interval $[\frac{m_H}{4}, m_H]$ as a function of the center-of-mass energy $\sqrt{S}$.

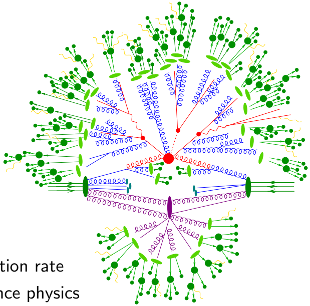


C. Anastasiou, C. Duhr, F. Dulat, E. Furlan, T. Gehrmann, F. Herzog, A. Lazopoulos and B. Mistlberger, arXiv:1602.00695

Factorization

Collins,Soper,Sterman'85–'89

- Calculate
  - Scattering probability
  - Gluon emission probability
- Measure
  - Long distance interactions
  - Particle decay rates



Divide et Impera

- Quantity of interest: Total interaction rate
- Convolution of short & long distance physics

$$\sigma_{p_1 p_2 \to X} = \sum_{i,j \in \{q,g\}} \int \mathrm{d}x_1 \mathrm{d}x_2 \underbrace{f_{p_1,i}(x_1,\mu_F^2) f_{p_2,j}(x_2,\mu_F^2)}_{\text{long distance physics}} \underbrace{\hat{\sigma}_{ij \to X}(x_1 x_2, \mu_F^2)}_{\text{short distance physics}}$$

QCD as a perturbative quantum field theory: Fixed-order calculations

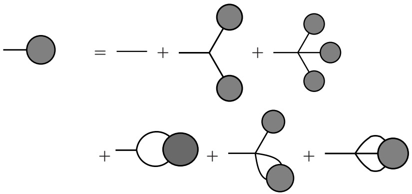
From Feynman graphs ...

$gg \to ng$	2	3	4	5	6	7	8	9
# FG	4	25	220	2,485	34,300	559,405	10,525,900	224,449,225

From Feynman graphs ...

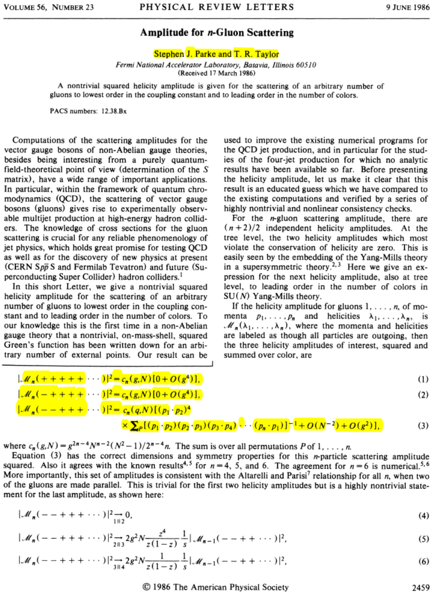
$gg \to ng$	2	3	4	5	6	7	8	9
# FG	4	25	220	2,485	34,300	559,405	10,525,900	224,449,225

to Dyson-Schwinger recursion! `Helac-Phegas`



$gg \to ng$	2	3	4	5	6	7	8	9
#	5	15	35	70	126	210	330	495

Taming the beast ...



VOLUME 56, NUMBER 23 PHYSICAL REVIEW LETTERS 9 JUNE 1986

Amplitude for n-Gluon Scattering

Stephen J. Parke and T. R. Taylor
Fermi National Accelerator Laboratory, Batavia, Illinois 60510
(Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

PACS numbers: 12.38.Bx

Computations of the scattering amplitudes for the vector gauge bosons of non-Abelian gauge theories, besides being interesting from a purely quantum-field-theoretical point of view (determination of the S matrix), have a wide range of important applications. In particular, within the framework of quantum chromodynamics (QCD), the scattering of vector gauge bosons (gluons) gives rise to experimentally observable multijet production at high-energy hadron colliders. The knowledge of cross sections for the gluon scattering is crucial for any reliable phenomenology of jet physics, which holds great promise for testing QCD as well as for the discovery of new physics at present (CERN $Sp\bar{p}S$ and Fermilab Tevatron) and future (Superconducting Super Collider) hadron colliders.[1]

In this short Letter, we give a nontrivial squared helicity amplitude for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors. To our knowledge this is the first time in a non-Abelian gauge theory that a nontrivial, on-mass-shell, squared Green's function has been written down for an arbitrary number of external points. Our result can be used to improve the existing numerical programs for the QCD jet production, and in particular for the studies of the four-jet production for which no analytic results have been available so far. Before presenting the helicity amplitude, let us make it clear that this result is an educated guess which we have compared to the existing computations and verified by a series of highly nontrivial and nonlinear consistency checks.

For the n-gluon scattering amplitude, there are $(n+2)/2$ independent helicity amplitudes. At the tree level, the two helicity amplitudes which most violate the conservation of helicity are zero. This is easily seen by the embedding of the Yang-Mills theory in a supersymmetric theory.[2,3] Here we give an expression for the next helicity amplitude, also at tree level, to leading order in the number of colors in $SU(N)$ Yang-Mills theory.

If the helicity amplitude for gluons $1, \ldots, n$, of momenta $p_1, \ldots, p_n$ and helicities $\lambda_1, \ldots, \lambda_n$, is $\mathcal{M}_n(\lambda_1, \ldots, \lambda_n)$, where the momenta and helicities are labeled as though all particles are outgoing, then the three helicity amplitudes of interest, squared and summed over color, are

$$|\mathcal{M}_n(+++++\cdots)|^2 = c_n(g,N)[0+O(g^4)], \tag{1}$$

$$|\mathcal{M}_n(-++++\cdots)|^2 = c_n(g,N)[0+O(g^4)], \tag{2}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 = c_n(q,N)[(p_1\cdot p_2)^4 \times \textstyle\sum_P[(p_1\cdot p_2)(p_2\cdot p_3)(p_3\cdot p_4)\cdots(p_n\cdot p_1)]^{-1}+O(N^{-2})+O(g^2)], \tag{3}$$

where $c_n(g,N) = g^{2n-4}N^{n-2}(N^2-1)/2^{n-4}n$. The sum is over all permutations P of $1, \ldots, n$.

Equation (3) has the correct dimensions and symmetry properties for this n-particle scattering amplitude squared. Also it agrees with the known results[4,5] for $n=4$, 5, and 6. The agreement for $n=6$ is numerical.[5,6] More importantly, this set of amplitudes is consistent with the Altarelli and Parisi[7] relationship for all n, when two of the gluons are made parallel. This is trivial for the first two helicity amplitudes but is a highly nontrivial statement for the last amplitude, as shown here:

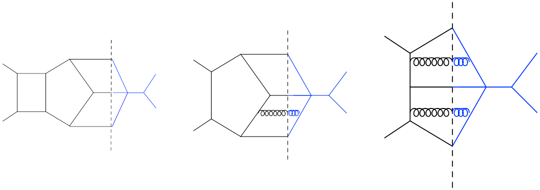
$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{1\parallel 2}{\longrightarrow} 0, \tag{4}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{2\parallel 3}{\longrightarrow} 2g^2N\frac{z^4}{z(1-z)}\frac{1}{s}|\mathcal{M}_{n-1}(--++\cdots)|^2, \tag{5}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{3\parallel 4}{\longrightarrow} 2g^2N\frac{1}{z(1-z)}\frac{1}{s}|\mathcal{M}_{n-1}(--++\cdots)|^2, \tag{6}$$

What do we need for an NNLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$



Perturbative QCD at NNLO

What do we need for an NNLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NNLO} \quad \to \quad & \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left| M_m^{(1)} \right|^2 \right) J_m(\Phi) & \quad VV \\
+ \quad & \int_{m+1} d\Phi_{m+1} \left(2Re \left(M_{m+1}^{(0)*} M_{m+1}^{(1)} \right) \right) J_{m+1}(\Phi) & \quad RV \\
+ \quad & \int_{m+2} d\Phi_{m+2} \left| M_{m+2}^{(0)} \right|^2 J_{m+2}(\Phi) & \quad RR
\end{aligned}$$

$RV + RR \to$ Antenna-S, Colorfull-S, STRIPPER, q_T, N-jetiness

A. Gehrmann-De Ridder, T. Gehrmann and M. Ritzmann, JHEP **1210** (2012) 047
P. Bolzoni, G. Somogyi and Z. Trocsanyi, JHEP **1101** (2011) 059
M. Czakon and D. Heymes, Nucl. Phys. B **890** (2014) 152
S. Catani and M. Grazzini, Phys. Rev. Lett. **98** (2007) 222002
R. Boughezal, C. Focke, X. Liu and F. Petriello, Phys. Rev. Lett. **115** (2015) no.6, 062002

OPP AT TWO LOOPS

coefficients of MI $\oplus$ spurious terms

$$\frac{N(q)}{\bar{D}_0\bar{D}_1\cdots\bar{D}_{m-1}} = \sum_{i_0<i_1<i_2<i_3}^{m-1}\frac{d(i_0i_1i_2i_3)+\tilde{d}(q;i_0i_1i_2i_3)}{\bar{D}_{i0}\bar{D}_{i1}\bar{D}_{i2}\bar{D}_{i2}}$$

$$+\sum_{i_0<i_1<i_2}^{m-1}\frac{c(i_0i_1i_2)+\tilde{c}(q;i_0i_1i_2)}{\bar{D}_{i0}\bar{D}_{i1}\bar{D}_{i2}}$$

$$+\sum_{i_0<i_1}^{m-1}\frac{b(i_0i_1)+\tilde{b}(q;i_0i_1)}{\bar{D}_{i0}\bar{D}_{i1}}$$

$$+\sum_{i_0}^{m-1}\frac{a(i_0)+\tilde{a}(q;i_0)}{\bar{D}_{i0}}$$

$+$ rational terms

G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763**, 147 (2007)

- Write the "OPP-type" equation at two loops

$$\frac{N\left(l_1, l_2; \{p_i\}\right)}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}\left(l_1, l_2; \{p_i\}\right)}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

$$\sum \frac{\Delta_{i_1 i_2 \ldots i_m}\left(l_1, l_2; \{p_i\}\right)}{D_{i_1} D_{i_2} \ldots D_{i_m}} \rightarrow \text{spurious} \oplus \text{ISP} - \text{irreducible integrals}$$

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ISP-irreducible integrals → use IBPI to Master Integrals

Libraries in the future: `QCD2LOOP`, `TwOLOop`

P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, Phys. Lett. B **718** (2012) 173

J. Gluza, K. Kajda and D. A. Kosower, Phys. Rev. D **83** (2011) 045012

H. Ita, arXiv:1510.05626 [hep-th].

C. G. Papadopoulos, R. H. P. Kleiss and I. Malamos, PoS Corfu **2012** (2013) 019.

IBPI: The current approach

- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \ldots D_N$, allows to express all scalar products $D_i = (\{k,l\} + p_i)^2 - M_i^2$
- $$F[a_1, \ldots, a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1} \ldots D_N^{a_N}}$$
$$\int d^d k d^d l \ \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1} \ldots D_N^{a_N}} \right) = 0$$
- IBP Laporta: FIRE, AIR, Reduze reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations
- Or numerical: SecDec, Weinzierl

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F. V. Tkachov, Phys. Lett. B **100** (1981) 65.

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C. Studerus, Comput. Phys. Commun. **181** (2010) 1293

A. V. Smirnov, Comput. Phys. Commun. **189** (2014) 182

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Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.

V. A. Smirnov, Phys. Lett. B **460** (1999) 397

T. Gehrmann and E. Remiddi, Nucl. Phys. B **580** (2000) 485 [hep-ph/9912329].

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

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S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk and T. Zirke, Comput. Phys. Commun. **196** (2015) 470

S. Becker, C. Reuschle and S. Weinzierl, JHEP **1012** (2010) 013

IBPI: The current approach

- Find a better IBP algorithm … Generating function technique, Baikov ?

P. A. Baikov, Nucl. Instrum. Meth. A **389** (1997) 347

V. A. Smirnov and M. Steinhauser, Nucl. Phys. B **672** (2003) 199

K. J. Larsen and Y. Zhang, Phys. Rev. D **93** (2016) no.4, 041701

$$F_{a_1\ldots a_N} = \sum_{i=\mathrm{masters}} c^{(i)}_{a_1\ldots a_N} G_i$$

- Baikov polynomial $\leftrightarrow$ LZ construction
- Sector $\leftrightarrow$ cut

$$\delta\left((k+p)^2 - m^2\right) \leftrightarrow \oint_{z=0} dz \frac{1}{z^{n=1}}$$

- Cut with higher powers in denominator

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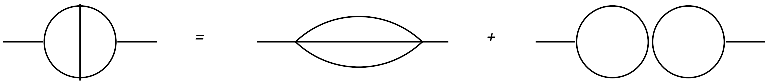

$$F_{11111} = \frac{(3d-10)(3d-8)}{(d-4)^2 (p^2)^2} F_{10011} + \frac{(3d-10)(3d-8)}{(d-4)^2 (p^2)^2} F_{01101} - 2\frac{(d-3)}{(d-4)\, p^2} F_{11110}$$

Differential Equations Approach

The integral is a function of external momenta, so one can set-up differential equations by differentiating and using IBP

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \ldots, a_n] \to \sum C_{a'_1,\ldots,a'_n} G[a'_1, \ldots, a'_n]$$

- Find the proper parametrization; Bring the system of equations in a form suitable to express the MI in terms of GPs

$$\partial_m f(\varepsilon, \{x_i\}) = \varepsilon A_m(\{x_i\}) f(\varepsilon, \{x_i\})$$
$$\partial_m A_n - \partial_n A_m = 0 \quad [A_m, A_n] = 0$$

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

- Boundary conditions: expansion by regions or regularity conditions.

B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012) 2139 [arXiv:1206.0546 [hep-ph]].

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- Iterated Integrals K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831
- Multiple Polylogarithms, Symbol algebra
- Goncharov Polylogarithms

$$\mathcal{G}\left(a_n,\ldots,a_1,x\right)=\int_0^x dt\,\frac{1}{t-a_n}\mathcal{G}\left(a_{n-1},\ldots,a_1,t\right)$$

with the special cases, $\mathcal{G}(x)=1$ and

$$\mathcal{G}\left(\underbrace{0,\ldots 0}_{n},x\right)=\frac{1}{n!}\log^n\left(x\right)$$

- Shuffle algebra

Differential Equations Approach

- Iterated Integrals
- Multiple Polylogarithms, Symbol algebra

A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605.

C. Duhr, H. Gangl and J. R. Rhodes, JHEP **1210** (2012) 075 [arXiv:1110.0458 [math-ph]].

C. Bogner and F. Brown

- Goncharov Polylogarithms

$$\mathcal{G}\left(a_n, \ldots, a_1, x\right) = \int_0^x dt \frac{1}{t - a_n} \mathcal{G}\left(a_{n-1}, \ldots, a_1, t\right)$$

with the special cases, $\mathcal{G}(x) = 1$ and

$$\mathcal{G}\left(\underbrace{0, \ldots 0}_{n}, x\right) = \frac{1}{n!} \log^n(x)$$

- Shuffle algebra

Differential Equations Approach

- Iterated Integrals
- Multiple Polylogarithms, Symbol algebra
- Goncharov Polylogarithms

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- Shuffle algebra

$$\mathcal{G}\left(a_1, a_2; x\right) \mathcal{G}\left(b_1; x\right) = \mathcal{G}\left(a_1, a_2, b_1; x\right) + \mathcal{G}\left(a_1, b_1, a_2; x\right) + \mathcal{G}\left(b_1, a_1, a_2; x\right)$$

THE SIMPLIFIED DIFFERENTIAL EQUATIONS APPROACH

C. G. Papadopoulos, JHEP **1407** (2014) 088

Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

- Introduce one parameter

$$G_{11\dots1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)\,(k + x\,p_1)^2\,(k + p_1 + p_2)^2 \dots (k + p_1 + p_2 + \dots + p_n)^2}$$

- Now the integral becomes a function of x, which allows to define a differential equation with respect to x, schematically given by

$$\frac{\partial}{\partial x} G_{11\dots1}(x) = -\frac{1}{x} G_{11\dots1}(x) + x p_1^2 G_{12\dots1} + \frac{1}{x} G_{02\dots1}$$

- and using IBPI we obtain, for instance for the one-loop 3 off-shell legs

$$\begin{aligned} m_1 x G_{121} + \tfrac{1}{x} G_{021} &= \left(\tfrac{1}{x-1} + \tfrac{1}{x - m_3/m_1}\right)\left(\tfrac{d-4}{2}\right) G_{111} \\ &+ \tfrac{d-3}{m_1 - m_3}\left(\tfrac{1}{x-1} - \tfrac{1}{x - m_3/m_1}\right)\left(\tfrac{G_{101} - G_{110}}{x}\right) \end{aligned}$$

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The Simplified Differential Equations Approach

- The integrating factor M is given by

$$M = x\,(1-x)^{\frac{4-d}{2}}\,(-m_3+m_1x)^{\frac{4-d}{2}}$$

- and the DE takes the form, $d = 4-2\varepsilon$,

$$\frac{\partial}{\partial x} M G_{111} = c_\Gamma \frac{1}{\varepsilon}\,(1-x)^{-1+\varepsilon}\,(-m_3+m_1x)^{-1+\varepsilon}\left(\left(-m_1x^2\right)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)$$

- Integrating factors $\epsilon = 0$ do not have branch points
- DE can be straightforwardly integrated order by order $\to$ GPs.

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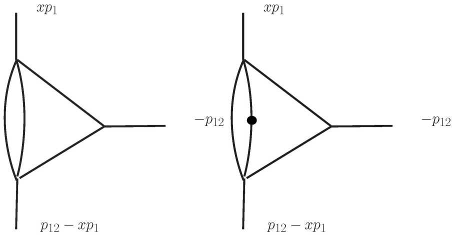
The Simplified Differential Equations Approach

The two-loop 3-off-shell-legs triangle



The Simplified Differential Equations Approach

We are interested in $G_{0101011}$. The DE involves also the MI $G_{0201011}$, so we have a system of two coupled DE, as follows:

$$\frac{\partial}{\partial x} f(x) = \frac{A_3(2-3\varepsilon)(1-x)^{-2\varepsilon} x^{-1+\varepsilon}(m_1 x - m_3)^{-2\varepsilon}}{2\varepsilon(2\varepsilon-1)} + \frac{m_1 \varepsilon (1-x)^{-2\varepsilon}(m_1 x - m_3)^{-2\varepsilon}}{2\varepsilon - 1} g(x)$$

$$\frac{\partial}{\partial x} g(x) = \frac{A_3(3\varepsilon-2)(3\varepsilon-1)(-m_1)^{-2\varepsilon}(1-x)^{2\varepsilon-1} x^{-3\varepsilon}(m_1 x - m_3)^{2\varepsilon-1}}{2\varepsilon^2} + (2\varepsilon-1)(3\varepsilon-1)(1-x)^{2\varepsilon-1}(m_1 x - m_3)^{2\varepsilon-1} f(x)$$

where $f(x) \equiv M_{0101011} G_{0101011}$ and $g(x) \equiv M_{0201011} G_{0201011}$, $M_{0201011} = (1-x)^{2\varepsilon} x^{\varepsilon+1} (m_1 x - m_3)^{2\varepsilon}$ and $M_{0101011} = x^{\varepsilon}$

- Solve sequentially in ε expansion
- Reproduce limit $\varepsilon \to 0$

The Simplified Differential Equations Approach

The *regularized* singularity at $x = 0$ is proportional to $x^{-1+\varepsilon}$ and can easily be integrated by the following decomposition

$$\int_0^x dt\ t^{-1+\varepsilon} F(t) = F(0) \int_0^x dt\ t^{-1+\varepsilon} + \int_0^x dt\ \frac{F(t)-F(0)}{t} t^{\varepsilon}$$
$$= F(0)\, \frac{x^{\varepsilon}}{\varepsilon} + \int_0^x dt\ \frac{F(t)-F(0)}{t} \left(1 + \varepsilon \log(t) + \frac{1}{2}\varepsilon^2 \log^2(t) + \ldots\right)$$

Reproduce correctly boundary term $x = 0$

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Reproduce correctly boundary term $x = 0$

Five-point one-loop integral with up to one off-shell leg at $\mathcal{O}(\varepsilon)$

Simplified differential equations approach

- simple parametrization of external momenta based on
  Triangle rule: Criterion for the $x-$parametrization

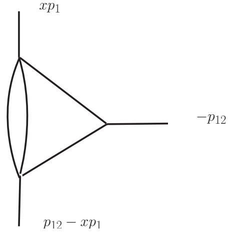

Figure : Required parametrization for off mass-shell triangles after possible pinching of internal line(s).

- DE in one parameter: addressing problems with many scales
- Boundary terms straightforwardly obtained by the DE itself, based on one-scale MI
- Expressions in terms of GP's straightforwardly obtained by expanding the DE in ε

Two-loop, four-point, two off-shell legs

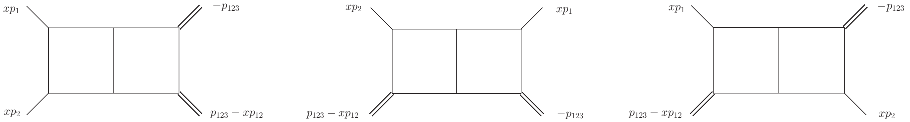

Figure : The parametrization of external momenta for the three planar double boxes of the families P_{12} (left), P_{13} (middle) and P_{23} (right) contributing to pair production at the LHC. All external momenta are incoming.

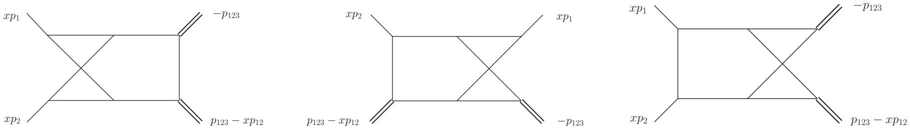

Figure : The parametrization of external momenta for the three non-planar double boxes of the families N_{12} (left), N_{13} (middle) and N_{34} (right) contributing to pair production at the LHC. All external momenta are incoming.

- original momentum assignment:

$$p(q_1)p'(q_2) \to V_1(-q_3)V_2(-q_4), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = M_4^2$$
$$S = (q_1+q_2)^2 \quad T = (q_1+q_3)^2$$

- underlying momentum assignment:
- induced parametrization:

Two-loop, four-point, two off-shell legs

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$$S = (q_1 + q_2)^2 \quad T = (q_1 + q_3)^2$$

- underlying momentum assignment:

$$q_1 = xp_1, \quad q_2 = xp_2, \quad q_3 = p_{123} - xp_{12}, \quad q_4 = -p_{123}, \quad p_i^2 = 0,$$
$$s_{12} := p_{12}^2, \qquad s_{23} := p_{23}^2, \qquad q := p_{123}^2,$$

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Two-loop, four-point, two off-shell legs

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$$s_{12} := p_{12}^2, \qquad s_{23} := p_{23}^2, \qquad q := p_{123}^2,$$

- induced parametrization:

$$S = s_{12}x^2, \quad T = q - (s_{12} + s_{23})x, \quad M_3^2 = (1 - x)(q - s_{12}x), \quad M_4^2 = q.$$

Two-loop, four-point, two off-shell legs

Planar topologies

$$G^{P_{12}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times \frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-xp_{12})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}},$$

$$G^{P_{13}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
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$$G^{P_{23}}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+p_{123}-xp_2)^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times \frac{1}{k_2^{2a_5}(k_2-p_1)^{2a_6}(k_2+xp_2-p_{123})^{2a_7}(k_2-p_{123})^{2a_8}(k_1+k_2)^{2a_9}},$$

Two-loop, four-point, two off-shell legs

Planar topologies

P_{12} : {010000011, 001010001, 001000011, 100000011, 101010010, 101010100, 101000110, 010010101, 101000011, 101000012, 100000111, 100000112, 001010011, 001010012, 010000111, 010010011, 101010110, 111000011, 101000111, 101010011, 011010011, 011010012, 110000111, 110000112, 010010111, 010010112, 111010011, 111000111, 111010111, 111m10111, 11101m111},

P_{13} : {000110001, 001000011, 001010001, 001101010, 001110010, 010000011, 010101010, 010110010, 001001011, 001010011, 001010012, 001011011, 001101001, 001101011, 001110001, 001110002, 001110011, 001111001, 001111011, 001211001, 010010011, 010110001, 010110011, 011010011, 011010021, 011110001, 011110011, 011111011, m11111011},

P_{23} : {001010001, 001010011, 010000011, 010000101, 010010011, 010010101, 010010111, 011000011, 011010001, 011010010, 011010011, 011010012, 011010100, 011010101, 011010111, 011020011, 012010011, 021010011, 100000011, 101000011, 101010010, 101010011, 101010100, 110000111, 111000011, 111010011, 111010111, 111m10111}.

Non-planar topologies

$$G^{N12}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
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$$G^{N13}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times \frac{1}{k_2^{2a_5}(k_2-xp_{12})^{2a_6}(k_2-p_{123})^{2a_7}(k_1+k_2+xp_1)^{2a_8}(k_1+k_2)^{2a_9}},$$

$$G^{N34}_{a_1\cdots a_9}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times \frac{1}{k_2^{2a_5}(k_2-xp_1)^{2a_6}(k_2-p_{123})^{2a_7}(k_1+k_2+xp_{12}-p_{123})^{2a_8}(k_1+k_2)^{2a_9}}.$$

Non-planar topologies

N_{12} : {100001010, 000110010, 000110001, 000101010, 000101001, 101010010, 100110010, 100101020, 100101010, 100101001, 001110010, 001110002, 001110001, 001101001, 101110020, 101110010, 101101002, 101101001, 100111020, 100111010, 100102011, 100101011, 001120011, 001111002, 001111001, 001110011, 000111011, 101011011, 100111011, 1m0111011, 001111011, 0m1111011, 101111011, 1m1111011, 1m1111m11},

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N_{34} : {001001010, 001010010, 010010010, 100000110, 100010010, 000010111, 010010110, 001010102, 001010101, 010010101, 001020011, 010000111, 001010011, 010010011, 101010020, 101010010, 101010100, 101000011, 110010120, 110010110, 010010112, 010010121, 010010111, 010020111, 020010111, 011010102, 001010111, 011010101, 110000211, 011020011, 110000111, 011010011, 111000101, 111010010, 101010101, 101010011, 111010110, 111010101, 101010111, 11m010111, 110m10111, 11001m111, 110010111, m11010111, 011m10111, 01101m111, 011010111, 111000111, 111010011, 111010111, 111m10111}.

The Simplified Differential Equations Approach

General setup

m: number of denominators

$$\partial_x G_{m+1} = H\left(\{s_{ij}\}, \epsilon; x\right) G_{m+1} + \sum_{m' \geq m_0}^{m} R\left(\{s_{ij}\}, \epsilon; x\right) G_{m'},$$

$m_0 = 3$ in the case of two loops

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$$M \sum_{m' \geq m_0}^{m} R\left(\{s_{ij}\}, \epsilon; x\right) G_{m'} =: \sum_i x^{-1+\beta_i \epsilon} \tilde{I}_{sin}^{(i)}(\{s_{ij}\}, \epsilon) + \tilde{I}_{reg}(\{s_{ij}\}, \epsilon; x).$$

The Simplified Differential Equations Approach

$$MG_{m+1} = C(\{s_{ij}\},\epsilon) + \sum_i \frac{x^{\beta_i \epsilon}}{\beta_i \epsilon} \tilde{I}^{(i)}_{sin}(\{s_{ij}\},\epsilon) + \int_0^x dx' \tilde{I}_{reg}(\{s_{ij}\},\epsilon;x'),$$

- Integrating factors M rational functions of x in the limit $\epsilon \to 0$
- *Sufficient condition* DE solvable in terms of GPs.
- All re-summed parts at $x \to 0 \to$ fully determined by the one-scale MI involved in the system
- Two-point integrals, two three-point integrals and double one-loop integrals $\to$ homogenous differential equations.
- $C(\{s_{ij}\},\epsilon) = 0$: *no independent calculation of boundary terms needed.*

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- *Sufficient condition* DE solvable in terms of GPs.
- All re-summed parts at $x \to 0 \to$ fully determined by the one-scale MI involved in the system
- Two-point integrals, two three-point integrals and double one-loop integrals $\to$ homogenous differential equations.
- $C(\{s_{ij}\},\epsilon) = 0$: *no independent calculation of boundary terms needed.*

THE SIMPLIFIED DIFFERENTIAL EQUATIONS APPROACH

When the DE are coupled

$$\partial_x \vec{G}_{m+1} = \mathbf{H}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m+1} + \sum_{m' \geq m_0}^{m} \mathbf{R}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m'},$$

- $\mathbf{M}_D$: $\partial_x \mathbf{M}_D = -\mathbf{M}_D \mathbf{H}_D$, where $\mathbf{H}_D$ is the diagonal part of $\mathbf{H}$.
- $\tilde{\mathbf{H}} =: \mathbf{M}_D \left(\mathbf{H} - \mathbf{H}_D\right) \mathbf{M}_D^{-1}$ of the reduced system of DE is then a *strictly triangular matrix* at order ϵ^0 and the system becomes effectively uncoupled.
- Problem: In very few specific cases, $\sim C\, x^{-2+\beta_i \epsilon}$ appears in the matrix $\tilde{\mathbf{H}}$,
- Solution: $x \to 1/x$ back to $x^{-1+\beta_i \epsilon}$ in the *inhomogeneous part* of the DE.

The Simplified Differential Equations Approach

When the DE are coupled

$$\partial_x \vec{G}_{m+1} = \mathbf{H}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m+1} + \sum_{m' \geq m_0}^{m} \mathbf{R}\left(\{s_{ij}\}, \epsilon; x\right) \vec{G}_{m'},$$

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The Simplified Differential Equations Approach

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Two-loop, four-point, two off-shell legs

GP-indices

$$I(P_{12}) = \left\{0, 1, \frac{q}{s_{12}}, \frac{s_{12}}{q}, \frac{q}{q-s_{23}}, 1-\frac{s_{23}}{q}, 1+\frac{s_{23}}{s_{12}}, \frac{s_{12}}{s_{12}+s_{23}}\right\},$$

$$I(P_{13}) = \left\{0, 1, \frac{q}{s_{12}}, \frac{s_{12}+s_{23}}{s_{12}}, \frac{q}{q-s_{23}}, \xi_-, \xi_+, \frac{q(q-s_{23})}{q^2-(q+s_{12})s_{23}}\right\},$$

$$I(P_{23}) = \left\{0, 1, \frac{q}{s_{12}}, 1+\frac{s_{23}}{s_{12}}, \frac{q}{q-s_{23}}, \frac{q}{s_{12}+s_{23}}, \frac{q-s_{23}}{s_{12}}\right\},$$

$$\xi_\pm = \frac{qs_{12} \pm \sqrt{qs_{12}s_{23}(-q+s_{12}+s_{23})}}{qs_{12}-s_{12}s_{23}}.$$

$$I(N_{12}) = I(P_{23}),$$

$$I(N_{34}) = I(P_{12}) \cup I(P_{23}) \cup \left\{\frac{s_{12}}{q-s_{23}}, \frac{s_{12}+s_{23}}{q}, \frac{q^2-qs_{23}-s_{12}s_{23}}{s_{12}(q-s_{23})}, \frac{s_{12}^2+qs_{23}+s_{12}s_{23}}{s_{12}(s_{12}+s_{23})}\right\},$$

$$I(N_{13}) = I(P_{23}) \cup \left\{\xi_-, \xi_+, 1+\frac{q}{s_{12}}+\frac{q}{-q+s_{23}}\right\}.$$

Example

$$G^{P_{13}}_{011111011}(x,s,\epsilon) = \frac{A_3(\epsilon)}{x^2 s_{12}(-q+x(q-s_{23}))^2}\left\{\frac{-1}{2\epsilon^4}+\frac{1}{\epsilon^3}\left(-GP\left(\frac{q}{s_{12}};x\right)+2\,GP\left(\frac{q}{q-s_{23}};x\right)\right.\right.$$
$$\left.+2\;GP(0;x)-GP(1;x)+\log\left(-s_{12}\right)+\frac{9}{4}\right)+\frac{1}{4\epsilon^2}\left(18\;GP\left(\frac{q}{s_{12}};x\right)-36\;GP\left(\frac{q}{q-s_{23}};x\right)\right.$$
$$\left.-8\;GP\left(0,\frac{q}{s_{12}};x\right)+16\;GP\left(0,\frac{q}{q-s_{23}};x\right)+8\;GP\left(\frac{s_{23}}{s_{12}}+1,\frac{q}{q-s_{23}};x\right)+\cdots\right)$$
$$+\frac{1}{\epsilon}\left(9\left(GP\left(0,\frac{q}{s_{12}};x\right)+GP(0,1;x)\right)-4\left(GP\left(0,0,\frac{q}{s_{12}};x\right)+GP(0,0,1;x)\right)+\cdots\right)$$
$$\left.+6\left(GP\left(0,0,1,\xi_-;x\right)+GP\left(0,0,1,\xi_+;x\right)\right)-2\,GP\left(0,0,\frac{q}{q-s_{23}},\frac{q\left(q-s_{23}\right)}{q^2-s_{23}\left(q+s_{12}\right)};x\right)+\cdots\right\}.$$

$$A_3(\epsilon) = -e^{2\gamma_E\epsilon}\frac{\Gamma(1-\epsilon)^3\Gamma(1+2\epsilon)}{\Gamma(3-3\epsilon)}.$$

C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1501** (2015) 072

C. G. Papadopoulos, D. Tommasini and C. Wever, arXiv:1511.09404 [hep-ph].

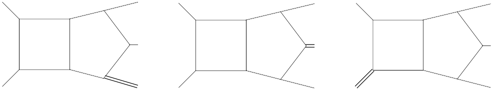

FIGURE : The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

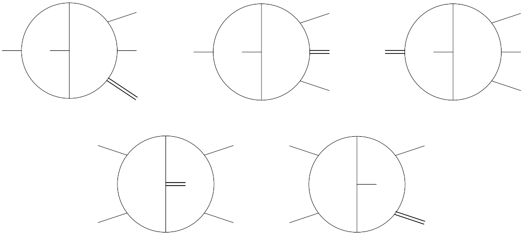

FIGURE : The five non-planar families with one external massive leg.

$$p(q_1)p'(q_2) \to V(q_3)j_1(q_4)j_2(q_5), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = q_5^2 = 0.$$

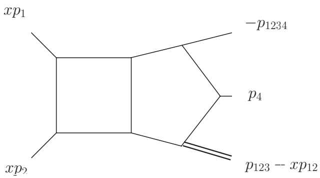

FIGURE : The parametrization of external momenta in terms of x for the planar pentabox of the family P_1. All external momenta are incoming.

$$s_{12} := p_{12}^2, \quad s_{23} := p_{23}^2, \quad s_{34} := p_{34}^2, \quad s_{45} := p_{45}^2 = p_{123}^2, \quad s_{51} := p_{15}^2 = p_{234}^2,$$

$$q_1^2 = q_2^2 = q_4^2 = q_5^2 = 0 \quad q_3^2 = (s_{45} - s_{12}x)(1-x)$$
$$q_{12}^2 = s_{12}x^2 \quad q_{23}^2 = s_{45}(1-x) + s_{23}x \quad q_{34}^2 = (s_{34} - s_{12}(1-x))\,x \quad q_{45}^2 = s_{45} \quad q_{51}^2 = s_{51}x$$

5BOX - ONE LEG OFF-SHELL: P1

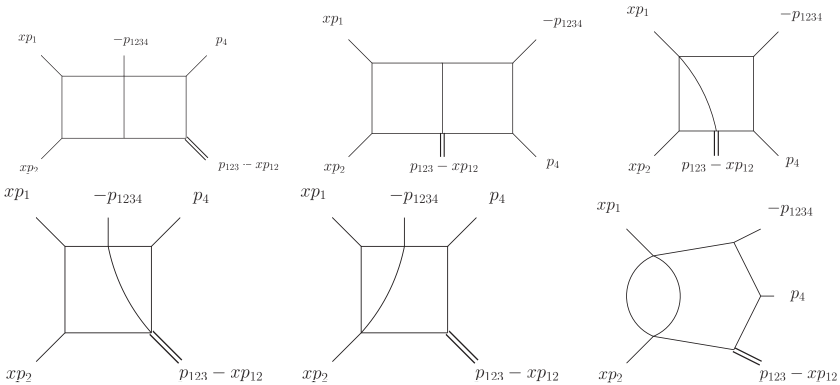

FIGURE : The five-point Feynman diagrams, besides the pentabox itself in Figure 4, that are contained in the family P_1. All external momenta are incoming.

5box - one leg off-shell: P1

$$G^{P_1}_{a_1\cdots a_{11}}(x,s,\epsilon) := e^{2\gamma_E \epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$

$$\times\frac{1}{(k_1+p_{1234})^{2a_5}k_2^{2a_6}(k_2-xp_1)^{2a_7}(k_2-xp_{12})^{2a_8}(k_2-p_{123})^{2a_9}(k_2-p_{1234})^{2a_{10}}(k_1+k_2)^{2a_{11}}},$$

P_1 : {10000000101, 01000000101, 00100000101, 10000001001, 01000000011, 00100000011, 10100001100, 10100001010, 10100101000, 01000101001, 10100100100, 10100000102, 10100000101, 10100000011, 10000001102, 10000001101, 10000001011, 01000100101, 01000001101, 01000001011, 00100100102, 00100100101, 11100000101, 11100000011, 11000001102, 11000001101, 11000001012, 11000001011, 11000000111, 10100000112, 10000001111, 01100100102, 01100100101, 01100100011, 01100000111, 01000101102, 01000101101, 01000101011, 01000100111, 01000001111, 00100100111, 10100101100, 10100100101, 10100001101, 10100001011, 10100000111, 111m0000111, 110000m1111, 11000001111, 10100101110, 10100100111, 10100001111, 011001m0111, 01100100111, 010m0101111, 01000101111, 11100100101, 11100001101, 11100001011, 11100000111, 111m0101101, 111001m1101, 11100101101, 1110m101011, 11100101011, 111m0100111, 11100100111, 111000m1111, 111m0001111, 11100001111, 111001m0111, 11100101111, 111001m1111, 111m0101111},

Choosing m= -1 or 2

$$\partial_x \mathbf{G} = \mathbf{M}\left(\{s_{ij}\}, \varepsilon, x\right)\mathbf{G}$$

$$\left(M_D\right)_{IJ} = \delta_{IJ}\, M_{II}\left(\varepsilon = 0\right), I, J = 1\ldots74$$

$\mathbf{G} \to \mathbf{S}^{-1}\mathbf{G}$, $\mathbf{S} = \exp\left(\int dx\, \mathbf{M}_D\right)$ and $\mathbf{M} \to \mathbf{S}^{-1}\left(\mathbf{M} - \mathbf{M}_D\right)\mathbf{S}$.

$$M_{IJ} = N_{IJ}\left(\varepsilon\right)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{\left(x - l_i\right)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

Letters (20):

$$0,\quad 1,\quad \frac{s_{45}}{s_{45}-s_{23}},\quad \frac{s_{45}}{s_{12}},\quad 1-\frac{s_{34}}{s_{12}},\quad 1+\frac{s_{23}}{s_{12}},$$

$$1-\frac{s_{34}-s_{51}}{s_{12}},\quad \frac{s_{45}-s_{23}}{s_{12}},\quad -\frac{s_{51}}{s_{12}},\quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}},\quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}-s_{45}-s_{51}\right)},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}\left(s_{23}-s_{45}-s_{51}\right)},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}+s_{34}-s_{51}\right)},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

$$\begin{aligned}
\Delta_1 &= \left(s_{12}(s_{51}-s_{23})+s_{23}s_{34}+s_{45}(s_{51}-s_{34})\right)^2+4s_{12}s_{45}s_{51}(s_{23}+s_{34}-s_{51})\\
\Delta_2 &= \left(s_{12}(-s_{23}+s_{45}+s_{51})+s_{23}s_{34}+s_{45}(s_{51}-s_{34})\right)^2-4s_{12}s_{45}s_{51}(-s_{23}+s_{45}+s_{51})\\
\Delta_3 &= -\left(s_{12}s_{34}s_{45}(s_{12}-s_{34}-s_{45})\right)
\end{aligned}$$

$$\partial_x \mathbf{G} = \mathbf{M}\left(\{s_{ij}\},\varepsilon,x\right)\mathbf{G}$$

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$$1-\frac{s_{34}-s_{51}}{s_{12}},\quad \frac{s_{45}-s_{23}}{s_{12}},\quad -\frac{s_{51}}{s_{12}},\quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}},\quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}-s_{45}-s_{51})},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}(s_{23}-s_{45}-s_{51})},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}+s_{34}-s_{51})},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

$$\begin{aligned}
\Delta_1 &= (s_{12}(s_{51}-s_{23})+s_{23}s_{34}+s_{45}(s_{51}-s_{34}))^2+4s_{12}s_{45}s_{51}(s_{23}+s_{34}-s_{51})\\
\Delta_2 &= (s_{12}(-s_{23}+s_{45}+s_{51})+s_{23}s_{34}+s_{45}(s_{51}-s_{34}))^2-4s_{12}s_{45}s_{51}(-s_{23}+s_{45}+s_{51})\\
\Delta_3 &= -(s_{12}s_{34}s_{45}(s_{12}-s_{34}-s_{45}))
\end{aligned}$$

$$\partial_x \mathbf{G} = \mathbf{M}\left(\{s_{ij}\}, \varepsilon, x\right)\mathbf{G}$$

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$$1-\frac{s_{34}-s_{51}}{s_{12}},\quad \frac{s_{45}-s_{23}}{s_{12}},\quad -\frac{s_{51}}{s_{12}},\quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}},\quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}-s_{45}-s_{51})},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}(s_{23}-s_{45}-s_{51})},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}+s_{34}-s_{51})},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

$$\Delta_1 = \left(s_{12}(s_{51}-s_{23})+s_{23}s_{34}+s_{45}(s_{51}-s_{34})\right)^2+4s_{12}s_{45}s_{51}(s_{23}+s_{34}-s_{51})$$
$$\Delta_2 = \left(s_{12}(-s_{23}+s_{45}+s_{51})+s_{23}s_{34}+s_{45}(s_{51}-s_{34})\right)^2-4s_{12}s_{45}s_{51}(-s_{23}+s_{45}+s_{51})$$
$$\Delta_3 = -\left(s_{12}s_{34}s_{45}(s_{12}-s_{34}-s_{45})\right)$$

5box P1 - DE

$$\partial_x \mathbf{G} = \mathbf{M}\left(\{s_{ij}\}, \varepsilon, x\right)\mathbf{G}$$

$$(M_D)_{IJ} = \delta_{IJ}\, M_{II}\,(\varepsilon = 0)\,, I, J = 1\ldots 74$$

$$\mathbf{G} \to \mathbf{S}^{-1}\mathbf{G},\ \mathbf{S} = \exp\left(\int dx\, \mathbf{M}_D\right) \text{ and } \mathbf{M} \to \mathbf{S}^{-1}\left(\mathbf{M} - \mathbf{M}_D\right)\mathbf{S}.$$

$$M_{IJ} = N_{IJ}(\varepsilon)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

Letters (20):

$$0,\quad 1,\quad \frac{s_{45}}{s_{45}-s_{23}},\quad \frac{s_{45}}{s_{12}},\quad 1-\frac{s_{34}}{s_{12}},\quad 1+\frac{s_{23}}{s_{12}},$$

$$1-\frac{s_{34}-s_{51}}{s_{12}},\quad \frac{s_{45}-s_{23}}{s_{12}},\quad -\frac{s_{51}}{s_{12}},\quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}},\quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}-s_{45}-s_{51})},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}(s_{23}-s_{45}-s_{51})},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}+s_{34}-s_{51})},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

$$\begin{aligned}
\Delta_1 &= (s_{12}(s_{51}-s_{23})+s_{23}s_{34}+s_{45}(s_{51}-s_{34}))^2+4s_{12}s_{45}s_{51}(s_{23}+s_{34}-s_{51})\\
\Delta_2 &= (s_{12}(-s_{23}+s_{45}+s_{51})+s_{23}s_{34}+s_{45}(s_{51}-s_{34}))^2-4s_{12}s_{45}s_{51}(-s_{23}+s_{45}+s_{51})\\
\Delta_3 &= -(s_{12}s_{34}s_{45}(s_{12}-s_{34}-s_{45}))
\end{aligned}$$

$$M_{IJ} = N_{IJ}(\varepsilon)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

$$\int_0^x dt \frac{1}{(t-a_n)^2}\mathcal{G}(a_{n-1},\ldots,a_1,t) \qquad \int_0^x dt\ t^m\ \mathcal{G}(a_{n-1},\ldots,a_1,t)$$

Fuchsian

$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon),\ G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

$$\mathbf{G} \to (\mathbf{I}-\mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M} - \partial_x\mathbf{K}_i - \mathbf{K}_i\mathbf{M})(\mathbf{I}-\mathbf{K}_i)^{-1} \quad i = 1,2,3$$

$$\partial_x \mathbf{G} = \left(\varepsilon\sum_{a=1}^{19}\frac{\mathbf{M}_a}{(x-l_a)}\right)\mathbf{G}$$

Fuchsian
$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon),\ G_I \to n_I(\varepsilon)\, G_I$

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M.A. Barkatou and E.Pflügel, Journal of Symbolic Computation, **44** (2009),1017

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5box P1 - solution

- Solution:

$$
\begin{aligned}
\mathbf{G} \quad = \quad & \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\
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+ \quad & \varepsilon\left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
+ \quad & \varepsilon^{2}\left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-1)}\right. \\
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\end{aligned}
$$

$\mathbf{b}_0^{(k)}$, $k = -2, \ldots, 2$ representing the x-independent boundary terms in the limit $x = 0$ at order ε^k

$$
\mathbf{G} \underset{x\to 0}{\sim} \sum_{k=-2}^{2} \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}
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$\mathcal{G}_{a,b,\ldots} = \mathcal{G}(l_a, l_b, \ldots; x)$ with $a, b, c, d = 1, \ldots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

$\mathbf{M}_a$ depend on kinematics, but eigenvalues not: $(x - l_a)^{-n_a \varepsilon}$, n_a positive integers, $x \to l_a$.

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- Resumed

$$G_{res} = \lim_{x \to 0} G = \sum_j c_j x^{i_0 + j\epsilon} + d_j x^{i_0 + 1 + j\epsilon} + \mathcal{O}(x^{i_0+2}),$$

- DE: using the above and equating terms $x^{i+j\epsilon}$, linear equations for c_i and d_i
- *bottom-up*: MI with homogeneous DE treated exactly
- MI needing special treatment (20)
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$$\{(10100000101), (10100000102), (11000001012), (11000001011), (01000101011), (10100100111), (10100001111), (111m0100111), (111000m1111), (11100001111), (111001m0111)\}.$$

  - Shifted boundary point (6)

$$\begin{array}{rl} \infty : & \{(10100000011), (10000001011), (11100000011), (01100100011), (10100100111)\} \\ (s_{12} - s_{34} + s_{51})/s_{12} : & \{(01000001011)\} \end{array}$$

  - Extraction from known integrals (3)

$$\begin{aligned} G_{11100001011}(x, s_{12}, s_{34}, s_{51}) &= G_{11100100101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ G_{11100101011}(x, s_{12}, s_{34}, s_{51}) &= G_{11100101101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ G_{111m0101011}(x, s_{12}, s_{34}, s_{51}) &= G_{111m0101101}(x' = 1, s'_{12}, s'_{23}, s'_{45}), \\ s'_{12} = x^2 s_{12}, \qquad s'_{23} = x s_{51}, \qquad & s'_{45} = -x s_{12} + x s_{34} + x^2 s_{12}. \end{aligned} \tag{1}$$

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Systematic approach: combining information from the expansion by regions technique (`asy2`) and the DE itself

Mellin-Barnes, `XSummer`

5BOX - ON-SHELL

All planar one-shell 5box by taking the limit $x \to 1$.

- $x = 1$ corresponds to l_2

$$\mathbf{G} = \sum_{n \geq -2} \varepsilon^n \sum_{i=0}^{n+2} \frac{1}{i!} \mathbf{c}_i^{(n)} \log^i (1-x)$$

- with $\mathbf{M}_2$ the residue matrix at $x = 1$ and
- $\mathbf{G}_{trunc} \equiv \mathbf{G}_{reg}(x = 1)$

$$\mathbf{G}_{x=1} = \left(\mathbf{I} + \frac{3}{2}\mathbf{M}_2 + \frac{1}{2}\mathbf{M}_2^2 \right) \mathbf{G}_{trunc}$$

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$$\mathbf{c}_i^{(n)} = \mathbf{M}_2 \mathbf{c}_{i-1}^{(n-1)} \quad i \geq 1$$

$$\mathbf{G}_{reg} = \sum_{n \geq -2} \varepsilon^n \mathbf{c}_0^{(n)}.$$

characteristic polynomial: $x^{61}(1+x)^9(2+x)^4$

$$\mathbf{G} = \mathbf{G}_{reg} + \frac{\left((1-x)^{-2\varepsilon} - 1\right)}{(-2\varepsilon)} \mathbf{X} + \frac{\left((1-x)^{-\varepsilon} - 1\right)}{(-\varepsilon)} \mathbf{Y}$$

$$\mathbf{X} = \sum_{n \geq -1} \varepsilon^n \mathbf{X}^{(n)} \quad \mathbf{Y} = \sum_{n \geq -1} \varepsilon^n \mathbf{Y}^{(n)}.$$

$$(-1)^n {\mathbf{M}_2}^n = {\mathbf{M}_2}^2 \left(2^{n-1} - 1\right) + \mathbf{M}_2 \left(2^{n-1} - 2\right), \quad n \geq 1.$$

minimal polynomial: $x(x+1)(x+2)$

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- Directly computed by using **GiNaC**
- All invariants negative Euclidean: perfect agreement with `SecDec`
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J. Vollinga and S. Weinzierl, Comput. Phys. Commun. **167** (2005) 177

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Discussion

1. SDE: proven reliable and efficient: evolving

$$\frac{\partial}{\partial s_i}\mathbf{G} = \mathbf{M}_i\mathbf{G}$$

$$s_i\left(x\right)$$

$$\frac{d}{dx}\mathbf{G} = \frac{\partial s_i}{\partial x}\frac{\partial}{\partial s_i}\mathbf{G} = \frac{\partial s_i}{\partial x}\mathbf{M}_i\mathbf{G} = \mathbf{M}'\mathbf{G}$$

2. IBP: better understanding
3. Complete massless MI with up to 8 denominators, at least 3 of-shell legs
4. Including internal masses
5. Feynman parametrization - MB vs DE: pros and cons
6. Integrand reduction at two loops: implementation

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Summary

1. Understanding QFT and provide precise calculations for analysis of experimental data
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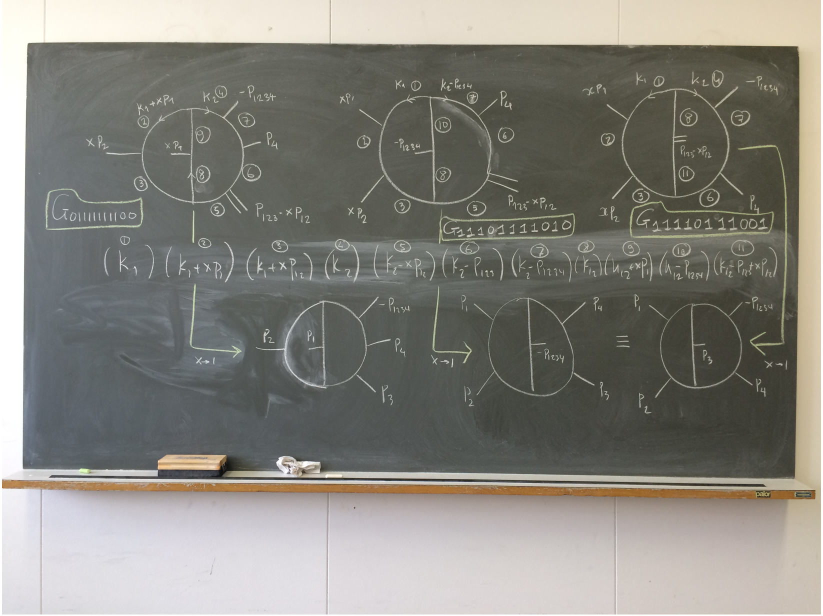
G0111111100
G11101111010
G11110111001
x→1
x→1
x→1

Backup Slides

Physical region

$$S > \left(\sqrt{M_3^2} + \sqrt{M_4^2}\right)^2, \quad T < 0, \quad U < 0,$$

$$M_3^2 > 0, \quad M_4^2 > 0, \quad q_\perp^2 = \frac{TU - M_3^2 M_4^2}{S} > 0,$$

$$x > 1, \quad \frac{q - s_{12}}{s_{23}} > 1, \quad x s_{12} > q, \quad q > 0.$$

$$x > 1, \quad \begin{cases} s_{23} < 0, & s_{12} + s_{23} > q, \quad q > 0 \\ s_{23} > 0, & s_{12} + s_{23} < q, \quad s_{12} > q/x. \end{cases}$$

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$$D \to D + i\epsilon$$

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- $\delta_{s_{ij}}$ and δ_x are determined as follows:

1) Input data:

$$G^{P12}_{001000011} \sim (-(-1+x)(-q+s_{12}x))^{1-2\epsilon} \sim (1-x)^{1-2\epsilon}(1-xs_{12}/q)^{1-2\epsilon}(-q)^{1-2\epsilon}$$

Two-loop, four-point, two off-shell legs

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C. Bogner and S. Weinzierl, Int. J. Mod. Phys. A **25** (2010) 2585 [arXiv:1002.3458 [hep-ph]].

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